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OSCILLATION AND NONOSCILLATION IN NONLINEAR THIRD ORDER DIFFERENCE EQUATIONS

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1. INTRODUCTION.

This paper is concerned with the oscillatory behavior of solutions of the third order nonlinear difference equation

$$\Delta(P_n\Delta^2V_n) + Q_nf(V_{n+1}, \Delta V_{n+1}, \Delta^2V_{n+1}) = 0, \ n = 1, 2, \ldots, \qquad (1.1)$$

where Δ is the forward difference operator i.e., $\Delta V_n = V_{n+1} - V_n$. It will be assumed throughout that the conditions below are satisfied:

(I) $P_n > 0$, $\Delta P_n \geq 0$ and $Q_n > 0$ for $n = 0, 1, 2, \ldots$

(II) $\sum^{\infty} \frac{1}{P_n} = \infty$

(III) $f: R^3 \to R$ is continuous and $xf(x, y, z) > 0$ for $x \neq 0$.

By a solution of (1.1) we mean a real sequence V satisfying equation (1.1) for $n = 1, 2, \ldots$. A solution V of (1.1) is called <u>nonoscillatory</u> if it is eventually positive or eventually negative. Otherwise, it is called <u>oscillatory</u>. The problem of determining oscillation criteria for certain second order nonlinear difference equations has been investigated by Hooker and Patula [1], and Szmanda [2]. The results of [2] were generalized by Li [3]. This paper examines a slightly more general equation than those studied in [2] and [3]. The authors began a study of similar equations in [4].

2. MAIN RESULTS.

LEMMA 2.1. Suppose V is a nonoscillatory solution of (1.1). Then, either

$$\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n = \operatorname{sgn} \Delta^2 V_n \tag{2.1}$$

for all n sufficiently large, or

$$\operatorname{sgn} V_n = \operatorname{sgn} \Delta^2 V_n \neq \operatorname{sgn} \Delta V_n \tag{2.2}$$

for all sufficiently large n, and $\lim_{n \to \infty} \Delta V_n = \lim_{n \to \infty} \Delta^2 V_n = 0$.

PROOF. Assume V is a nonoscillatory solution of (1.1), where $V_n > 0$ for all $n > N$, where N is a positive integer. The proof is similar if $V_n < 0$ for all $n > N$. Note that $\Delta(P_n \Delta^2 V_n) = -Q_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) < 0$, for each $n > N$. Thus $P_n \Delta^2 V_n$ is decreasing and is eventually sign definite. A positive integer $M > N$ exists for which ΔV_n and $\Delta^2 V_n$ are of one sign when $n > M$. The following cases must be considered:

(a) $V_n > 0$, $\Delta V_n > 0$, $\Delta^2 V_n > 0$, $n > M$,

(b) $V_n > 0$, $\Delta V_n < 0$, $\Delta^2 V_n > 0$, $n > M$,

(c) $V_n > 0$, $\Delta V_n < 0$, $\Delta^2 V_n < 0$, $n > M$,

(d) $V_n > 0$, $\Delta V_n > 0$, $\Delta^2 V_n < 0$, $n > M$.

Case (c) is impossible because if $\Delta V_n \Delta^2 V_n > 0$ for all sufficiently large n, then $\operatorname{sgn} V_n = \operatorname{sgn} \Delta V_n$ eventually. We show that (d) is also impossible. If (d) holds, then from above $P_n \Delta^2 V_n$ is negative and decreasing for all n sufficiently large. Let $k < 0$ be such that $P_n \Delta^2 V_n < k$ for all $n > M$. Then $\Delta^2 V_n < \frac{k}{P_n}$, $n > M$. Summing from M to R − 1 we obtain

$$\Delta V_R - \Delta V_M < k \sum_{n=m}^{R-1} \frac{1}{P_n} .$$

Letting $R \to \infty$, implies ΔV_R is eventually negative, but this contradicts (d), therefore (d) cannot hold. This completes the proof of the lemma.

We continue our study of (1.1) by considering a functional which plays a vital role in our investigation. Similar functionals have been used to study solutions of differential equations (Taylor [5]).

LEMMA 2.2. Let V be a solution of (1.1). Then

$$F[V_n] = F_n = 2V_n P_n \Delta^2 V_n - P_{n-1}(\Delta V_n)^2$$

is nonincreasing, in fact

$$\Delta F_n = -2Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) - P_n(\Delta^2 V_n)^2 - \Delta P_{n-1}(\Delta V_n)^2.$$

Since F_n is monotone for any nontrivial solution of (1.1) we have that F_n is of one sign for all n sufficiently large. Using this we will examine solutions of (1.1) where $F_n > 0$ for each n and those for which $F_m < 0$ for some positive integer m.

THEOREM 2.1. Let V be a nontrivial solution of (1.1) for which $F[V_n] > 0$. Then the following are true:

(i) $$\sum^{\infty} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) < \infty,$$

(ii) $$\sum^{\infty} P_n(\Delta^2 V_n)^2 < \infty, \text{ and}$$

(iii) $$\sum^{\infty} \Delta P_{n-1}(\Delta V_n)^2 < \infty.$$

PROOF. Since $F_n > 0$ for each n, differencing F_n and summing from 0 to k-1 we find

$$0 < F_k = F_0 - 2\sum_0^{k-1} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) - \sum_0^{k-1} P_n(\Delta^2 V_n)^2 - \sum_0^{k-1} \Delta P_{n-1}(\Delta V_n)^2.$$

Thus,

$$2\sum_0^{k-1} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) + \sum_0^{k-1} P_n(\Delta^2 V_n)^2 + \sum_0^{k-1} \Delta P_{n-1}(\Delta V_n)^2 < F_0.$$

Allowing k to tend to infinity establishes each of (i), (ii) and (iii) since F_0 is independent of k.

THEOREM 2.2. Suppose that $\frac{f(x,y,z)}{x} > r > 0$ for $x \neq 0$ and $\liminf Q_n > 0$. Let V be a solution of (1.1) for which $F[V_n] > 0$ for each n. Then

(iv) $$\sum^{\infty} V_n^2 < \infty,$$

(v) $$\lim_{n\to\infty} V_n = \lim_{n\to\infty} \Delta V_n = \lim_{n\to\infty} \Delta^2 V_n = 0.$$

PROOF. To prove (iv), observe that

$$V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) > rV_{n+1}^2.$$

Thus

$$Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) > \alpha r V_{n+1}^2$$

where $\alpha = \liminf Q_n$, so we have

$$\alpha r\sum_{0}^{\infty} V_{n+1}^2 \le \sum_{0}^{\infty} Q_n V_{n+1} f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}).$$

Now apply (i), of Theorem 2.1, and the proof of (iv) is complete.

The relations (v) follow as a consequence of (iv).

EXAMPLE 2.1. As an illustration of Theorem 2.2 consider equation (1.1) with $P_n = n$,

$$Q_n = \frac{2^{2n}(n-1)}{2^{2n+2}+1}, \quad f(x,y,z) = x^3 + x.$$

Then

$$\Delta(n\Delta^2 V_n) + \frac{2^{2n}(n-1)}{2^{2n+2}+1}(V_{n+1}^3 + V_{n+1}) = 0.$$

The sequence defined by 2^{-n} is a solution of this equation for which $F_n \to 0$ as $n \to \infty$.

THEOREM 2.3. If $\frac{f(x,y,z)}{x} \ge r > 0$, and $\sum^{\infty} Q_n = \infty$, then every nonoscillatory solution of (1.1) approaches zero as $n \to \infty$.

PROOF. Suppose V is an eventually positive solution of (1.1) that is bounded away from zero, i.e. $V_n \ge \beta > 0$ for all n sufficiently large. Because of Lemma 2.1, an integer M exists so that the relations (2.1) or (2.2) are satisfied by V for all $n \ge M$. Now $f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) \ge rV_{n+1}$ where r is a positive constant. From (1.1) we find

$$\Delta(P_n \Delta^2 V_n) = -Q_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}).$$

Thus,

$$\Delta(P_n \Delta^2 V_n) \le -rQ_n V_{n+1}. \tag{2.3}$$

Summing both sides of (2.3) from M to k-1 we find

$$P_k \Delta^2 V_k \le P_M \Delta^2 V_M - r\sum^{k-1} Q_n V_{n+1} \le P_M \Delta^2 V_M - r\beta \sum^{k-1} Q_n. \tag{2.4}$$

But as $k \to \infty$, the right hand side of (2.4) tends to $-\infty$, which in turn forces $P_k \Delta^2 V_k$ to tend to $-\infty$, and hence $\Delta^2 V_k < 0$ eventually, a contradiction of relations (2.1) and (2.2). A similar argument treats the case of an eventually negative solution. This completes the proof of the theorem.

COROLLARY 2.1. If $\frac{f(x,y,z)}{x} \ge r > 0$, for $x \ne 0$, and $\sum^{\infty} Q_n = \infty$, then every nonoscillatory solution of (1.1) satisfies the relations (2.2).

We are now in a position to show that oscillatory solutions exist under certain conditions.

THEOREM 2.4. Suppose $\frac{f(x,y,z)}{x} \ge r > 0$, for $x \ne 0$, $\sum^{\infty} Q_n = \infty$, and P_n is bounded. If V is a solution of (1.1) for which $F[V_n] < 0$ for some n, then V is oscillatory.

PROOF. Suppose V is a nonoscillatory solution of (1.1). We may suppose without any generality loss that $V_n > 0$ and $F[V_n] < 0$ for all $n \ge N$, since $F[V_n]$ is nonincreasing as $n \to \infty$. From Theorem 2.3, $V_n \to 0$, $\Delta V_n \to 0$ and $\Delta^2 V_n \to 0$ as $n \to \infty$.

This together with the boundedness of P_n implies that $F[V_n] \to 0$. This is clearly impossible since $F_n < 0$ and $\Delta F_n < 0$ for large n and the proof follows by contradiction.

Under certain conditions the bounded solutions of (1.1) behave rather nicely. Similar results appeared in [2] and [3].

THEOREM 2.5. Suppose $\sum^{\infty} nQ_n = \infty$ and $P_n \leq \beta$, β constant. Then every bounded solution of (1.1) is either oscillatory or tends to zero monotonically.

PROOF. By Lemma 2.1 a bounded nonoscillatory solution V satisfies

$$\operatorname{sgn} V_n = \operatorname{sgn} P_n \Delta^2 V_n \neq \operatorname{sgn} \Delta V_n$$

for all n sufficiently large. Assume that $V_n > 0$ eventually and suppose $\lim_{n \to \infty} \Delta V_n = A_0$ where $A_0 > 0$. Note also

$$\lim_{n \to \infty} \Delta V_n = \lim_{n \to \infty} P_n \Delta^2 V_n = \lim_{n \to \infty} \Delta^2 V_n = 0.$$

The fact that $P_n \Delta^2 V_n \to 0$ as $n \to \infty$ follows from the boundedness of V_n and (II). Consider the sequence $r_n = n(P_n \Delta^2 V_n)$. Note that

$$\Delta r_n = P_{n+1} \Delta^2 V_{n+1} - nQ_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}). \tag{2.5}$$

Since f is continuous $f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) \to f(A_0, 0, 0) > 0$ as $n \to \infty$, so there exists N such that

$$f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1}) > \tfrac{1}{2} f(A_0, 0, 0) \text{ for all } n > N.$$

Therefore from (2.5) we have

$$\Delta r_n < P_{n+1} \Delta^2 V_{n+1} - \tfrac{1}{2} n\, Q_n f(A_0, 0, 0)$$

$$\leq \beta\, \Delta^2 V_{n+1} - \frac{1}{2}\, n\, Q_n f(A_0, 0, 0).$$

Summing, from N to n

$$r_{n+1} < r_N + \beta \Delta V_{n+2} - \beta V_{N+1} - \tfrac{1}{2} f(A_0, 0, 0) \sum_{j=N}^{n} jQ_j.$$

As $n \to \infty$, $r_n \to -\infty$, a contradiction. Therefore $A_0 = 0$. This completes the proof of the theorem.

Finally we have

THEOREM 2.6. Suppose $\sum^{\infty} Q_n = \infty$, $f(\alpha x, \alpha y, \alpha z) = \alpha^{2m+1} f(x,y,z)$, $\alpha \neq 0$ and $f(x, y + h, z) \geq f(x,y,z)$ for $h > 0$. Then every solution of (1.) is either bounded or oscillatory.

PROOF. Suppose V_n is an unbounded nonoscillatory solution of (1.1). Without loss of generality we have

$$V_n > 0, \ \Delta V_n > 0, \ P_n\Delta^2 V_n > 0,$$

for all n sufficiently large. Consider the functional

$$q_n = \frac{P_n\Delta^2 V_n}{V_n}.$$

Differencing q_n we find

$$\begin{aligned}\Delta q_n &= \frac{-Q_n f(V_{n+1}, \Delta V_{n+1}, \Delta^2 V_{n+1})}{V_{n+1}} - \frac{P_n \Delta V_n \Delta^2 V_n}{V_n V_{n+1}}\\ &< - V_{n+1}^{2m} Q_n f(1, \frac{\Delta V_{n+1}}{V_{n+1}}, \frac{\Delta^2 V_{n+1}}{V_{n+1}})\\ &< - V_{n+1}^{2m} Q_n f(1, 0, \frac{\Delta^2 V_{n+1}}{V_{n+1}})\\ &< - \frac{V_N^{2m} Q_n f(1, 0, 0)}{2}.\end{aligned}$$

Summing we obtain

$$q_m < K_0 - \frac{V_N^{2m} f(1,0,0)}{2} \sum_N^{m-1} Q_n.$$

But this implies $q_m \to -\infty$ as $m \to \infty$, a contradiction since P_n, $\Delta^2 V_n$ and V_n are positive for all n sufficiently large.

EXAMPLE 2.2. It is possible for equations of the form of (1.1) to have unbounded oscillatory solutions. The sequence $V_n = (-2)^n$ is a solution of

$$\Delta^3 V_n + \frac{11}{18(4^{n+1})} (\Delta V_{n+1})^3 + 3V_{n+1} = 0.$$

Note that this example does not violate the conclusion of Theorem 2.6. Note also that (III) is not satisfied.

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