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ON THE STRUCTURE OF SUPPORT POINT SETS

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ABSTRACT. Let X be a metrizable compact convex subset of a locally convex space. Using Choquet's Theorem, we determine the structure of the support point set of X when X has countably many extreme points. We also characterize the support points of certain families of analytic functions.

KEY WORDS AND PHRASES: Support point, Extreme point, Choquet's Theorem.

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1. INTRODUCTION.

Let X be a subset of a locally convex space E . A continuous linear functional J on X is said to be <u>associated</u> with $f \in X$ if $\text{Re } J(f) = \max\{\text{Re } J(g): g \in X\}$ and Re J is non constant on X. In this case we call f a <u>support point</u> of X . The set of support points of X will be denoted by Supp X . The set of extreme points of a convex subset F of E will be denoted by Ext F .

Let $D = \{z: |z| < 1 , z \in C\}$ and equip the space A of functions analytic in D with the topology of uniform convergence on compact subsets of D . This topology is metrizable [1, p.1]. Every continuous linear functional J on A is induced by a sequence $\{b_n\}_{n=0}^{\infty}$ which satisfies $\limsup |b_n|^{1/n} < 1$ and $J(f) = \sum_{n=0}^{\infty} a_n b_n$ for $f(z) = \sum_{n=0}^{\infty} a_n z^n \in A$ [1, p.36]. Recently, the support points of many subclasses of A have been studied. For more details see [1] and [2].

In Section 2, we consider a metrizable compact convex set X in a locally convex space. Using Choquet's theorem we determine the structure of Supp X when Ext X is countable (Theorem 2.1).

In Section 3, we consider the classes: $P(p) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n \in A : \sum_{n=1}^{\infty} |a_n|^p \leq 1\}$, $1 \leq p < \infty$. In Theorem 3.4, we determine Supp P(p) . Indeed, it is shown that Supp X is in 1-1 correspondence with a proper subset of Supp Ball(ℓ_p) .

2. SUPPORT POINTS OF SETS WITH COUNTABLY MANY EXTREME POINTS.

Let E be a locally convex space, and suppose that X is a metrizable compact convex subset of E . A theorem by Choquet [3, p.19] says that if $x \in X$ then there exists a probability measure μ_x on X , supported by Ext X , such that $L(x) = \int_{\text{Ext } X} L \, d\mu_x$ for every L in E^* . In case Ext X is countable (possibly finite), we have the following:

CHOQUET'S THEOREM (Countable Case). Suppose Ext X = $\{f_n\}$ is countable. Then $X = \{\sum_n \lambda_n f_n: \lambda_n \geq 0$ for each n and $\sum_n \lambda_n = 1\}$.

PROOF. Let $f \in X$. By Choquet's Theorem, there exists a probability measure μ_f on X , supported by $\{f_n\}$, such that $L(f) = \int_{\{f_n\}} L \, d\mu_f$. Thus $L(f) = \sum_n \mu_f(f_n) L(f_n)$. Hence $L(f - \sum_n \mu_f(f_n) f_n) = 0.$

Since this is true for every L in E^* , we get $f = \sum_n \mu_f(f_n) f_n$, as required.

We proceed to the main result of this section.

THEOREM 2.1. Let X be a metrizable compact convex subset of a locally compact space E such that Ext X = $\{f_n\}$ is countable. For each positive integer n , set K_n equal to the closed convex hull of $\{f_i: i \neq n\}$. Then

(1) Supp X is contained in the union of those K_n which are proper subsets of X .

(2) $K_n \subseteq$ Supp X if and only if $f_n \notin$ closed affine hull of $\{f_i: i \neq n\}$.

PROOF. To prove (1), let $f \in$ Supp X . By Choquet's Theorem, we can write $f = \sum_i \lambda_i f_i$ with each $\lambda_i \geq 0$ and $\sum_i \lambda_i = 1$. Let ϕ be a continuous linear functional associated with f . Then $\text{Re } \phi(f) = \sum \lambda_i \text{ Re } \phi(f_i) \leq \sum_i \lambda_i \text{ Re } \phi(f) = \text{Re } \phi(f)$. Hence we must have $\text{Re } \phi(f_i) = \text{Re } \phi(f)$ whenever $\lambda_i > 0$. On the other hand, since $\text{Re } \phi$ is non-constant on X , we must have $\text{Re } \phi(f_i) \neq \text{Re } \phi(f)$ for some i . We conclude that $\lambda_i = 0$ for some i , as required.

To prove (2), suppose that f_n does not belong to the closed affine hull H of $\{f_i: i \neq n\}$ and fix $g \in K_n$. Then H - g is a closed real subspace of E not continuing f_n - g . A version of the Hahn-Banach theorem [4, page 59] gives a functional J in E^* whose real part ϕ vanishes on H - g while $\phi(f_n - g) = -1$. Set $\phi(f_{n+1}) = b$. Then $\phi(f_n) = b - 1$ while $\phi(f_i) = \phi(f_{n+1}) = b$ for every $i \neq n$. Thus, $\phi(g) = b$ for all $g \in K_n$. For any h in X , by Choquet's Theorem, we have $h = \sum_i \beta_i f_i$ with $\beta_i \geq 0$ and $\sum_i \beta_i = 1$. Thus $\phi(h) = \beta_n(b-1) + \sum_{i \neq n} \beta_i b = b - \beta_n \leq b$. This shows that $g \in$ Supp X .

Conversely, assume that $K_n \subseteq$ Supp X . For ease of notation we take n = 1 and assume Ext X = $\{f_n\}_{n=1}^{\infty}$ is infinite. By assumption, $f = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} f_i$ is a support point of X . Let ϕ be an associated linear functional in E^* and set $S = \{g \in E: \text{Re } \phi(g) = \text{Re } \phi(f)\}$. Note that S is a closed affine subspace of E . Since $\text{Re } \phi(f) = \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \text{Re } \phi(f_i) \leq \sum_{i=2}^{\infty} \frac{1}{2^{i-1}} \text{Re } \phi(f) = \text{Re } \phi(f)$, we have $\text{Re } \phi(f_i) = \text{Re } \phi(f)$ for all $i \geq 2$. Thus the closed affine hull of $\{f_i : i \neq 1\} \subseteq S$. On the other hand, in view of Choquet's Theorem, if $f_1 \in S$ then Re ϕ would be constant on X . Thus $f_1 \notin S$ and consequently, $f_1 \notin$ closed affine hull of $\{f_i : i \neq 1\}$.

EXAMPLES. (1) Let X be a triangle in R^2 with vertices f_1, f_2 and f_3 . These vertices are the extreme points of X and the affine hull of any two of them is a line, not containing the third. The theorem guarantees that Supp X = $\bigcup_{n=1}^{3} K_n$, which is indeed the boundary of X .

(2) Let X be a square in R^2 with vertices f_1, f_2, f_3 and f_4 . The affine hull of any three of the f_i's is all of R^2 . In particular, each $f_i \in$ affine hull of $\{f_j : j \neq i\}$. The theorem guarantees that no K_n is contained in Supp X . In fact, Supp X = the boundary of X has no interior.

(3) Let T be the family of all functions which are analytic and univalent in D , and take the form $f(z) = z - \sum_{n=2}^{\infty} a_n z^n$, $a_n \geq 0$. By [5], Ext T = $\{f_n\}_{n=1}^{\infty}$, where $f_1(z) = z$ and $f_n(z) = z - \frac{1}{n} z^n$ for $n > 1$. For $n > 1$, it is clear that f_n does not belong to the closed affine hull of the remaining $\{f_i\}$, so $\bigcup_{n=2}^{\infty} K_n \subseteq$ Supp X by the second part of the Theorem. Since f_1 is a limit point of the remaining f_i's , $K_1 = X$ and Supp X = $\bigcup_{n=2}^{\infty} K_n$ by the first part of the Theorem.

COROLLARY 2.2. Let X be as in Theorem 2.1. Then Supp X = $\bigcup_{\alpha} \overline{co}\,(E_\alpha)$, where each E_α is a subset of Ext X .

PROOF. Suppose $f \in$ Supp X and ϕ is an associated linear functional with f . Writing $f = \sum_i \lambda_i f_i$, we see that Re $\phi(f)$ = Re $\phi(f_i)$ whenever $\lambda_i \neq 0$. Take $E_\alpha = \{f_i | \lambda_i \neq 0\}$. Then $f \in \overline{co}\,(E_\alpha) \subseteq$ Supp X .

The theorem says these E_α are proper subsets of Ext X , i.e., they cannot be "too big". The next proposition implies that they can't all be singletons, i.e., "too small" .

PROPOSITION 2.3. Let X be a compact convex subset of a locally convex space. If X has more than two extreme points, then Supp X is uncountable.

PROOF. Without loss of generality we may assume that $0 \in X$. Let f_1 and f_2 be two independent elements of X, and let ϕ_1 and ϕ_2 be continuous and linear functionals such that $\phi_1(f_1) = \phi_2(f_2) = 1$ and $\phi_1(f_2) = \phi_2(f_1) = 0$. Define ψ: X $\to R^2$ by $\psi(f) = (\phi_1(f), \phi_2(f))$. Then $\psi(X)$ is a compact convex subset of R^2 with non empty interior. Since $\psi(X)$ has uncountably many boundary points, Supp(ψ(X)) is uncountable. Since ψ^{-1} takes support points to support points, we see that Supp X is uncountable too.

EXAMPLE. Take $f_n = e^{\frac{2\pi i}{n}}$ for n = 1, 2, ... and X = $\overline{co}\,\{f_n\}$ in $\mathbf{R}^2$. Then Supp X = $\bigcup_{n=1}^{\infty}$ co $\{f_n, f_{n+1}\}$. Here all the E_α's have cardinality two even though Ext X is infinite.

COROLLARY 2.4. Let X be as in Theorem 2.1. Then Ext X = Supp X if and only if X has two extreme points.

3. SUPPORT POINTS OF CERTAIN CLASSES OF ANALYTIC FUNCTIONS.

For $1 \le p < \infty$, define $P(p) = \{\sum_{n=1}^{\infty} a_n z^n \in A: \sum_{n=1}^{\infty} |a_n|^p \le 1\}$. It is easy to see that the classes $P(p)$ are compact convex subsets of A. These classes are closely related to $\mathrm{Ball}(\ell_p)$ and we will find that Supp $P(p)$ is in one-to-one correspondence with a proper subset of Supp $\mathrm{Ball}(\ell_p)$. As a corollary, we determine the support points of certain families of univalent functions. We use the notation a for the sequence $\{a_n\}_{n=1}^{\infty}$.

We begin with a simple observation.

PROPOSITION 3.1. Let X be the unit ball of a Banach space E. Then Supp $X = \{x \in X : ||x|| = 1\}$. If ϕ is associated with x, then $\phi(x) = ||\phi||$.

PROOF. That every vector of norm one belongs to Supp X is a consequence of the Hahn-Banach theorem. Suppose conversely that the real part of $\phi \in X^*$ achieves its maximum over X at x. Since X is closed under multiplication by scalars of absolute value at most one, we have $\mathrm{Re}\,\phi(x) = \sup_{y \in X} \mathrm{Re}\,\phi(y) = ||\phi||$. Thus $||\phi|| = \mathrm{Re}\,\phi(x) \le ||\phi||\,||x||$ and so $||x|| = 1$. Moreover $\mathrm{Re}\,\phi(x) = ||\phi||$ implies $\mathrm{Re}\,\phi(x) \ge |\phi(x)|$, so $\phi(x)$ is in fact real.

EXAMPLE. The family $P(p)$ "looks like" the unit ball of ℓ_p, but we cannot immediately apply Proposition 3.1 to find its support points. For example, the sequence $\{a_n\}_{n=1}^{\infty} = \{\sqrt{\frac{6}{\pi}}\,\frac{1}{n}\}_{n=1}^{\infty}$ belongs to the unit sphere of ℓ_2, but $\sum_{n=1}^{\infty} a_n z^n$ is not a support point of $P(2)$. The problem is that any non-constant linear functional $\{b_n\}_{n=1}^{\infty} \in \ell_2{}^*$ which assumes its maximum at $\{a_n\}_{n=1}^{\infty}$ must be a scalar multiple of $\{a_n\}_{n=1}^{\infty}$. So $\limsup \sqrt[n]{|b_n|} = 1$, which does not correspond to a continuous linear functional on A.

We find the support points of $P(p)$ by making the remarks in the preceeding example more precise.

PROPOSITION 3.2. Suppose $T : E \to F$ is a linear, injective, and continuous map between topological vector spaces E and F, and let X be a subset of E. Then $Tx \in$ Supp TX if and only if $x \in$ Supp X and some linear functional associated with x belongs to range T^*.

PROOF. Recall that $T^* : F^* \to E^*$ is defined by $T^*\psi = \psi \circ T$. Suppose $Tx \in$ Supp TX and choose $\psi \in F^*$ with $\mathrm{Re}\,\psi(Tx) = \max_{y \in X} \mathrm{Re}\psi(Ty)$. Set $\phi = \psi \circ T$; then $\psi \in$ range T^*, $\mathrm{Re}\,\phi(x) = \max_{y \in X} \mathrm{Re}\phi(y)$, and injectivity of T implies that $\mathrm{Re}\,\phi$ is not constant on X.

Conversely, let $\phi \in$ range T^* such that $\mathrm{Re}\,\phi(x) = \max_{y \in X} \mathrm{Re}\,\phi(y)$. Write $\phi = \psi \circ T$, $\psi \in F^*$. Then $\mathrm{Re}\,\psi(Tx) = \max_{y \in TX} \psi(y)$, and $\mathrm{Re}\,\psi$ cannot be constant on TX since $\mathrm{Re}\,\phi$ is not constant on X.

PROPOSITION 3.3. Let $a \in X = \mathrm{Ball}(\ell_p)$, $(1 < p < \infty)$, with $||a||_p = 1$, and $b \in \ell_q$. Then:

(1) If b is associated with a, then there exists $\beta \ne 0$ with $\beta|b_n|^q = |a_n|^p$ for all n.

(2) If $b_n = \begin{cases} \frac{a_n}{|a_n|}\,|a_n|^{p-1} & \text{if } a_n \ne 0 \\ 0 & \text{otherwise} \end{cases}$

then b is associated with a.

PROOF. (1) From Proposition 3.1, we learn that $b(a) = ||b||_q = ||b||_q||a||_p$. Thus we have Hölder equality, so there exists $\beta \neq 0$ with $\beta\, |b_n|^q = |a_n|^p$ for all n.

(2) $b(a) = \sum a_n b_n = \sum |a_n|\,|a_n|^{p-1} = \sum |a_n|^p = 1$, while $||b||_q = \sum_{n=1}^{\infty} |a_n|^{(p-1)q} = 1$, so this result follows from Holder's inequality.

The following is the main result of this section.

THEOREM 3.4. Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ be in $P(p)$. Then f is a support point of $P(p)$ if and only if

(1) f is analytic in $\bar{D}$ and $\sum_{n=1}^{\infty} |a_n|^p = 1$, for $1 < p < \infty$.

(2) $f(z) = \sum_{n=1}^{N} a_n z^n$, where N is some positive integer and $\sum_{n=1}^{N} |a_n| = 1$ for $p = 1$.

PROOF. Define $T : \ell_p \to A$ by $T(a) = \sum_{n=1}^{\infty} a_n z^n$. Clearly T maps $\mathrm{Ball}(\ell_p)$ onto $P(p)$ and T is injective. Moreover for any $r < 1$ and $a \in \ell_p$, $(1 < p < \infty)$, we have $\sup_{|z| \leq r} |T(a)(z)| \leq \sum_{n=1}^{\infty} |a_n| r^n \leq ||a||_p \left(\frac{1}{1-r^q}\right)^{1/q}$, by Hölder's inequality, so T is continuous. Similarly for $p = 1$.

If $\phi \in A^*$ is given by $\phi(\sum_{n=1}^{\infty} a_n z^n) = \sum_{n=1}^{\infty} a_n b_n$, then $(T^*\phi)(a) = \phi(Ta) = \sum_{n=1}^{\infty} a_n b_n$ for every $a \in \ell_p$. So $T^*\phi$ is the sequence $\{b_n\}_{n=1}^{\infty}$ considered as a member of $(\ell_p)^* = \ell_q$. Thus $\{b_n\}_{n=1}^{\infty} \in (\ell_p)^*$ is in the range of T^* if and only if $\limsup \sqrt[n]{|b_n|} < 1$.

(1) Suppose $f = Ta \in \mathrm{Supp}\, P(p)$. By Proposition 3.2, $a \in \mathrm{Supp\ Ball}\ (\ell_p)$. Thus by Proposition 3.1, we get $\sum_{n=1}^{\infty} |a_n|^p = 1$. If the functional associated with Ta is given by $\{b_n\}_{n=1}^{\infty}$, then $\limsup \sqrt[n]{|b_n|} < 1$. By Proposition 3.3, there exists $\beta \neq 0$ such that $|a_n|^p = \beta |b_n|^q$ for all n. Thus $\limsup \sqrt[n]{|a_n|} < 1$ and so f is analytic in $\bar{D}$.

Conversely, suppose that $f = T(a)$ is analytic in $\bar{D}$ with $\sum_{n=1}^{\infty} |a_n|^p = 1$. Then $a \in \mathrm{Supp\ Ball}\ (\ell_p)$ by Proposition 3.1, and one can choose the functional associated with a as in the formula of Proposition 3.3. Since the radius of convergence of the power series of f is greater than one, $\limsup \sqrt[n]{|a_n|} < 1$ so $\limsup \sqrt[n]{|b_n|} < 1$ and thus $b \in$ range T^*. Thus $f \in \mathrm{Supp}\, P(p)$ by Proposition 3.2.

(2) Suppose $f = Ta \in \mathrm{Supp}\, P(1)$ and b is a functional associated with a. Then $||a||_1 = 1$ and $b(a) = ||b||_\infty$ by Propositions 3.2 and 3.1. Thus equality must hold at all points of the chain $b(a) \leq \sum_{n=1}^{\infty} |a_n||b_n| \leq \sum_{n=1}^{\infty} |a_n|\, ||b||_\infty \leq ||b||_\infty$. In particular $|b_n| = ||b||_\infty$ whenever $a_n \neq 0$. Since $\limsup \sqrt[n]{|b_n|} < 1$, this means $a_n = 0$ for all but finitely many n, as required.

Conversely, suppose $Ta = f(z) = \sum_{n=1}^{N} a_n z_n$ and $\sum_{n=1}^{N} |a_n| = 1$. Then $a \in \mathrm{Supp\ Ball}\ (\ell_1)$.

$$\text{Define } b_n \equiv \begin{cases} \dfrac{a_n}{|a_n|} & \text{if } a_n \neq 0, \\ 0 & \text{otherwise}. \end{cases}$$

Then $\limsup \sqrt[n]{|b_n|} < 1$ and $\{b_n\}_{n=1}^{\infty} \in (\ell_p)^*$ is associated with a. By Proposition 3.2, f is a support point of $P(1)$, as required.

Let $Q(p) = \{f(z) = z + \sum_{n=2}^{\infty} a_n z^n \in A : \sum_{n=2}^{\infty} n|a_n|^p \leq 1\}$, $1 \leq p < \infty$. The class $Q(1)$ has been studied in [6]. We remark that each element of $Q(1)$ is univalent.

COROLLARY 3.5. A function $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is a support point of $Q(p)$ if and only if

(1) f is analytic in $\bar{D}$ and $\sum_{n=2}^{\infty} n|a_n|^p = 1$, if $1 < p < \infty$

(2) $f(z) = z + \sum_{n=2}^{N} a_n z^n$ and $\sum_{n=2}^{N} |a_n| = 1$, for some positive integer $N \geq 2$, if $p = 1$.

PROOF. One way to see this, is to replace ℓ_p by $\ell_p(\mu)$, where $\mu(n) = n$, $n = 2,3,\ldots$ in the proof of Theorem 3.4.

REMARK. One can define $P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : \sup|a_n| \leq 1\}$. One can show, using an argument similar to the proof of Theorem 3.4, that Supp $P(\infty) = \{f(z) = \sum_{n=1}^{\infty} a_n z^n : |a_n| = 1$ for some $n \geq 1\}$.

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