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A NOTE ON ANALYTIC MEASURES

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ABSTRACT: Let G be a compact Abelian group with character group X. Let S be a subset of X such that, for some real-valued homomorphism ψ on X, the set $S \cap \psi^{-1}(]-\infty, \psi(\chi)])$ is finite for all χ in X. Suppose that μ is a measure in M(G) such that $\hat{\mu}$ vanishes off of S, then μ is absolutely continuous with respect to the Haar measure on G.

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1. INTRODUCTION.

Let G denote a compact Abelian group with character group X. Suppose that ψ is a real-valued homomorphism on X, and let ϕ denote the adjoint homomorphism of ψ. Thus ϕ is the continuous homomorphism from $\mathbf{R}$ into G such that the identity $\chi \circ \phi(r) = \exp(i\psi(\chi)r)$ holds for all r in $\mathbf{R}$, and all χ in X. We denote by M(G) the linear space of all complex-valued regular Borel measures on G. In the terminology of de Leew and Glicksberg [1], a measure μ in M(G) is called *ϕ-analytic* if its Fourier transform $\hat{\mu}$ vanishes on $\{\chi \in X: \psi(\chi) < 0\}$.

Suppose that S is a nonvoid subset of X. Let $M_S(G)$ denote the closed linear subspace of M(G) consisting of the measures μ with $\hat{\mu}$ vanishing off of S. The set S will be called a B-*set* (B for Bochner) if there is a nonzero homomorphism ψ from X into $\mathbf{R}$ such that the set $S \cap \psi^{-1}(]-\infty, \psi(\chi)])$ is finite for all χ in X. The homomorphism ψ may depend on S, and may not be unique. For example, a sector with opening less than π in the lattice plane $\mathbf{Z}\times\mathbf{Z}$ is a B-set. The first orthant in $\mathbf{Z}^{\omega}$ (the weak direct product of countably many copies of $\mathbf{Z}$) is also a B-set. Once we have chosen a homomorphism ψ, we will refer to S as a B-set with respect to the homomorphism ψ.

A theorem due to Bochner [2], on $\mathbf{T}^2$, the two-dimensional torus, asserts that if $\mu \in M(\mathbf{T}^2)$ is such that $\hat{\mu}$ vanishes off of a sector of opening less than π, then μ is absolutely continuous. (The expression "absolutely continuous" will always mean absolutely continuous with respect to the Haar measure on the group in consideration.) A generalization of this result is given in de Leew and Glicksberg [1], Theorem (3.4).

It is easy to construct B-sets in $\mathbf{Z}\times\mathbf{Z}$ that are contained in no sector with opening less than π. For example, consider the set $S = \{(x,y) \in \mathbf{Z}\times\mathbf{Z}: \ y \geq \log(1+|x|)\}$. Using results from [1], we will show that the conclusion of Bochner's theorem holds for B-sets. We have the following theorem.

(1.1) THEOREM. Let S be a B-set in X. Suppose that μ is in $M_S(G)$, then μ is absolutely continuous.

Before proving the theorem we make a few observations. Suppose that S is a B-set, with respect to some homomorphism ψ. Clearly, there is a character χ_0 in S such that $\psi(\chi_0) \leq \psi(\chi)$ for all χ in S. Note that any translate of S by an element of X is also a B-set with respect to the same homomorphism ψ. Hence by shifting S by $-\chi_0$, if necessary, we may suppose that $\psi(\chi) \geq 0$ for all χ in S. In this case, given a measure μ in $M_S(G)$, we consider the measure $\overline{\chi_0}\mu$ which is in $M_{S-\chi_0}(G)$. The set $S-\chi_0$ is a B-set, with respect to the homomorphism ψ; and $\overline{\chi}_0\mu$ is absolutely continuous if and only if μ is.

If μ is in M(G), we write μ_a and μ_s to denote its absolutely continuous part and its singular part respectively.

(1.2) Lemma. Let S be a B-set in X. Suppose that μ is in $M_S(G)$, then μ_a and μ_s are in $M_S(G)$.

Proof. As we observed before the lemma, we may suppose that $\psi(S) \subseteq [0,\infty[$. Let ϕ denote the adjoint homomorphism of ψ, and let χ_1 be an arbitrary character in X\S, the complement of S in X. We want to show that

(1) $$\hat{\mu}_s(\chi_1) = \hat{\mu}_a(\chi_1) = 0.$$

First, note that if S is finite then $\mu = \mu_a$, and the lemma is obviously true. So suppose for the rest of the proof that S is infinite. Let χ_2 in X be such that $\psi(\chi_1) < \psi(\chi_2)$. Let $A = \{\chi \in X: \ \psi(\chi) < \psi(\chi_2)\} \cap \operatorname{supp}\hat{\mu}$. The set A is either void or finite. Define the measure σ in M(G) by,

$$\sigma = \mu - \sum_{\chi \in A} \hat{\mu}(\chi)\chi,$$

where the above sum is 0 if A is empty. We have

$$\hat{\sigma}(\chi) = \begin{cases} \hat{\mu}(\chi) & \text{if } \chi \notin A; \\ 0 & \text{if } \chi \in A. \end{cases}$$

Hence $\hat{\sigma}$ vanishes off of $\psi^{-1}([\psi(\chi_2), \infty[) \cap S$, which implies that σ is ϕ-analytic. It follows from [1], the Main Theorem , Proposition (2.3.2), and Theorem (5.1), that $\hat{\sigma}_a$ and $\hat{\sigma}_s$ vanish off of $\psi^{-1}([\psi(\chi_2), \infty[) \cap S$. Since $\mu_s = \sigma_s$, it follows that $\hat{\mu}_s$ vanishes off of $\psi^{-1}([\psi(\chi_2), \infty[) \cap S$. Therefore, $\hat{\mu}_s(\chi_1) = 0$, and the lemma follows. □

Proof of Theorem (1.1). According to Lemma (1.2), it is enough to show that $\hat{\mu}_s(\chi) = 0$ for all χ in S. The proof is by contradiction. Assume that $\hat{\mu}_s(\chi_0) \neq 0$ for some χ_0 in S. Let χ_1 in X be such that $\psi(\chi_1) > \psi(\chi_0)$. (Here also we are assuming that S is infinite and $\psi(S) \subseteq [0,\infty[$.) Let $A = \{\chi \in X: \ \psi(\chi) \leq \psi(\chi_1), \text{ and } \hat{\mu}_s(\chi) \neq 0\}$. Then A is contained in $\psi^{-1}(]-\infty, \psi(\chi_1)]) \cap S$; and so A is finite and χ_0 is in A. Define the measure ν in M(G) by

$$\nu = \mu_s - \sum_{\chi \in A} \hat{\mu}_s(\chi)\chi.$$

We have

$$\hat{\nu}(\chi) = \begin{cases} \hat{\mu}_s(\chi) & \text{if } \chi \notin A; \\ 0 & \text{if } \chi \in A. \end{cases}$$

Thus $\hat{\nu}$ vanishes off of $\psi^{-1}([\psi(\chi_1), \infty[) \cap S$, and hence it is ϕ-analytic. Applying Proposition (5.1), [1], we see that $\hat{\nu}_s$ and $\hat{\nu}_a$ vanish off of $\psi^{-1}([\psi(\chi_1), \infty[) \cap S$. Since $\nu_s = \mu_s$, it follows that $\hat{\mu}_s$ vanishes off of $\psi^{-1}([\psi(\chi_1), \infty[) \cap S$. This is plainly a contradiction since $\psi(\chi_0) < \psi(\chi_1)$, and by assumption $\hat{\mu}_s(\chi_0) \neq 0$. □

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