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ULTRAREGULAR INDUCTIVE LIMITS

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ABSTRACT. An inductive limit E =indlim E_n is ultraregular if it is regular and each set $B \subset E_n$, which is bounded in E, is also bounded in E_n. A necessary and sufficient condition for ultraregularity of E is given provided each E_n is an LF-space which is closed in E_{n+1}.

KEY WORDS AND PHRASES. Regular and ultraregular inductive limits, bounded set, fast completeness.
MATHEMATICS SUBJECT CLASSIFICATION. Primary 46 A 12, Secondary 46A06.

Let F be a locally convex space and $B \subset F$ an absolutely convex subset. We denote by F_B the linear hull of B and provide it with the topology generated by the Minkowski functional of B. If the toplogical space F_B is Banach then B is called a Banach disk. In [1] de Wilde calls the space F fast complete if every set B, bounded in F, is contained in a bounded Banach disk. Every sequentially complete space is fast complete.

A strict inductive limit of a sequence $F_1 \subset F_2 \subset \cdots$ of Fréchet spaces is called an LF-space.

If $S \subset X \cap Y$, where X and Y are locally convex spaces, then $c\ell_X S$ and $c\ell_Y S$ are the respective closures of S in X and Y. Throughout the paper $E_1 \subset E_2 \subset \cdots$ is a sequence of Hausdorff locally convex spaces with all inclusions id: $E_n \to E_{n+1}$ continuous. We denote indlim E_n by E.

In [2] Floret calls an inductive limit E regular, resp. α-regular, if every set bounded in E is bounded, resp. contained, in some E_n. An α-regular inductive limit is ultraregular, resp. weakly ultraregular, if each set $B \subset E_n$, which is bounded in E, is also bounded, resp. weakly bounded, in E_n.

In [3, §4, Prop. 4] Dieudonné and Schwartz proved that a strict inductive limit is ultraregular if each space E_n is closed in E_{n+1}. In the case the inductive limit is not strict some restrictions on topologies of the spaces E_n have to be imposed.

We introduce two properties:

(P) Every closed absolutely convex neighborhood in E_n is closed in E_{n+1}.

(Q) If U is a closed absolutely convex neighborhood in E_n and V is the closure of U in E_{n+1}, then $U = V \cap E_n$.

Evidently $P \Rightarrow Q$ and if each space E_n is closed in E_{n+1} then P is equivalent to Q. If the inductive limit E is strict then Q holds.

Lemma 1. (Q) holds iff each real $f \in E'_n$ has a real continuous linear extension to E_{n+1}.

Proof. Assume (Q) and take a real $f \in E'_n$, $f \neq 0$. The set $U = f^{-1}[-1,1]$ is a closed absolutely convex neighborhood in E_n. Let V be the corresponding set $V \subset E_{n+1}$ from (Q). If $E_n \subset V$ we would have $U = V \cap E_n = E_n$ and $f = 0$, which contradicts our assumption $f \neq 0$. Hence we can choose $x_0 \in E_n \backslash V$. Since V is absolutely convex and closed in E_{n+1}, there exists a real $g \in E'_{n+1}$ such that $V \subset g^{-1}(-\infty, 1]$ and $g(x_0) > 1$. Further $f^{-1}(0) \subset g^{-1}(0)$ which implies $f(x_0) \neq 0$. Without a loss of generality we may assume $f(x_0) = 1$. Then $f(x - f(x)x_0) = 0$ for $x \in E_n$ and $(x - f(x)x_0) \in f^{-1}(0) \subset g^{-1}(0)$. Hence, $g(x - f(x)x_0) = g(x) - f(x)g(x_0) = 0$ and the functional $(g(x_0))^{-1}g \in E'_{n+1}$ is the desired extension.

Assume that each real $f \in E'_n$ has a real extension $g \in E'_{n+1}$. Take a closed absolutely convex neighborhood $U \subset E_n$. By the Hahn-Banach theorem there exists $F \subset E'_n$ such that each $f \in F$ is real and $U = \cap\{f^{-1}(-\infty, 1];\ f \in F\}$. Let G be the set of all real extensions of all $f \in F$ to E_{n+1}. The set $V = \cap\{g^{-1}(-\infty, 1]; g \in G\}$ is closed and absolutely convex in E_{n+1}. Evidently $U \subset V \cap E_n$. Assume $U \neq V \cap E_n$. Then there is $y \in (V \cap E_n)\backslash U \subset E_n \backslash U$ and $f \in F$ such that $y \notin f^{-1}(-\infty, 1] = E_n \cap g^{-1}(-\infty, 1]$, where $g \in G$ is an extension of f. But then $y \notin E_n \cap V \subset E_n \cap g^{-1}(-\infty, 1]$, which is a contradiction. Hence $U = V \cap E_n$ and (Q) holds.

Lemma 2. $(P) \Rightarrow E$ α-regular.

Proof. Assume that E is not α-regular. Then there is a set B bounded in E which is not contained in any E_n. By taking a subsequence of $E_1, E_2, \cdots$, we may assume that for any $n \in N$ there exists $b_n \in (B \cap E_n)\backslash E_{n-1}, E_0 = \{0\}$.

Since $b_1 \neq 0$, there is a closed absolutely convex neighborhood U_1 in E_1 such that $b_1 \notin U_1$. Also $b_2 \notin E_1$. Hence $\frac{1}{2}b_2 \notin U_1$. By (P), U_1 is closed in E_2 and there exists an absolutely convex neighborhood V_1 in E_2 such that $(b_1 + V_1 + V_1) \cap U_1 = \emptyset$ and $(\frac{1}{2}b_2 + V_1 + V_1) \cap U_1 = \emptyset$. Then $U_2 = c\ell_{E_2}(U_1 + V_1)$ is a closed absolutely convex neighborhood in E_2 such that $b_1, \frac{1}{2}b_2 \notin U_2$. Again U_2 is closed in E_3 and $\frac{1}{3}b_3 \notin U_2$. Hence there is an absolutely convex neighborhood V_2in E_3 such that $(\frac{1}{k}b_k + V_2 + V_2) \cap U_2 = \emptyset$ for $k = 1, 2, 3$. The set $U_3 = c\ell_{E_3}(U_2 + V_2)$ is a closed absolutely convex neighborhood in E_3 for which $\frac{1}{k}b_k \notin U_3$, $k = 1, 2, 3, 4$.

Once all such $U_n, n \in N$, are constructed, $\cup\{U_n; n \in N\}$ is a neighborhood in E which does not absorb B, a contradiction.

Lemma 3. $(P) \Rightarrow E$ weakly ultraregular.

Proof. Assume (P) and E not weakly ultraregular. By Lemma 2, E is α-regular. Hence, there exists a set B bounded in E and $n \in N$ such that $B \subset E_n$ but B is not weakly bounded in E_n. Without a loss of generality we may assume $n = 1$.

Take a real $f_1 \in E_1'$ which is not bounded on B and choose a sequence $b_n \in B, n \in N$, such that $f_1(b_n) > n$. Since (P) implies (Q) there is a real extension $f_2 \in E_2'$ of f_1 and a real extension $f_3 \in E_3'$ of f_2, etc. Each set $U_n = f_n^{-1}(-\infty, 1]$, $n \in N$, is a closed absolutely convex neighborhood in E_n and $U_1 \subset U_2 \subset \cdots$. Hence $U = \cup\{U_n; n \in N\}$ is a 0-neighborhood in E. For any $n \in N$ we have $b_n \notin nU$, i.e. U does not absorb B which is a contradiction.

Theorem 1. Let (P) hold and each E_n be fast complete. Then E is ultraregular.

Proof. By Lemma 2, E is α-regular. Let $B \subset E$ be bounded. Then $B \subset E_n$ for some $n \in N$. By Lemma 3, B is weakly bounded in E_n. Since E_n is fast complete, B is also bounded with respect to the topology of E_n, see [4].

Lemma 4. Let each E_n be an inductive limit of metrizable spaces and E ultraregular. Then (Q) holds.

Proof. Take a real $f \in E_1', f \neq 0$. It suffices to show that f has a continuous linear extension to E_2. Put $F = (E_1, \text{top } E_2)$. Since the inclusion id: $E_1 \to F$ is continuous, each set bounded in E_1 is bounded in F. On the other hand if B is bounded in F it is bounded in E and $B \subset E_1$. Then B is bounded in E_1 by the ultraregularity of E. Hence the spaces E_1 and F have the same families of bounded sets.

The set $A = f^{-1}(-1, 1)$ absorbs all sets bounded in E_1, hence it absorbs all sets bounded in F. The space F, as an inductive limit of metrizable spaces, is bornological. This implies A is a 0-neighborhood in F. If $a < b, x_0 \in f^{-1}(a, b)$, and $d = \min(f(x_0) - a, b - f(x_0))$, then $d > 0$ and $x_0 + dA \subset f^{-1}(a, b)$. Thus $f^{-1}(a, b)$ is open in F and $f^{-1}(-\infty, 1] = F \setminus \cup\{f^{-1}(1, n); n \in N\}$ is closed in F. The set $M = c\ell_{E_2} f^{-1}(-\infty, 1]$ is closed absolutely convex in E_2 and $f^{-1}(\infty, 1] = M \cap F = M \cap E_1$.

Take $x_1 \in E_1$ for which $f(x_1) > 1$. Then $x_1 \notin M$ and there exists a real $g \in E_2'$ such that $M \subset g^{-1}(-\infty, 1]$ and $g(x_1) > 1$. If $x \in f^{-1}(0)$ then $f(kx) = 0$ for each integer k which implies $kx \in M$ and $x \in g^{-1}(0)$. Hence $f^{-1}(0) \subset g^{-1}(0)$ and there exists $c > 0$ such that $cg(x) = f(x)$ for $x \in E_1$. The functional cg is the sought linear continuous extension of f to E_2.

Theorem 2. Assume

1. Each E_n is closed in E_{n+1}.

2. Each E_n is an inductive limit of metrizable spaces.

3. E is ultraregular.

Then (P) holds.

Proof. By Lemma 4, assumptions 2 and 3 imply (Q) which combined with the assumption in 1 implies (P).

Theorem 3. Assume

1. Each E_n is closed in E_{n+1}.

2. Each E_n is $LF-$space.

Then E is ultraregular iff (P) holds.

Proof. The *if* part follows from Theorem 2. For the *only if* part we observe that each LF-space E_n satisfies the assumptions of the Dieudonné-Schwartz Theorem in [3]. Hence E_n is ultraregular and therefore also regular. Since regular inductive limit, not necessarily strict, of Fréchet spaces is fast complete, [5], each space E_n is fast complete. Then, by Theorem 1, (P) implies the ultraregularity of E.

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