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ON NORMAL LATTICES AND WALLMAN SPACES

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ABSTRACT. Let X be an abstract set and Ω a lattice of subsets of X. The notion of Ω being mildly normal or slightly normal is investigated. Also, the general Wallman space with an alternate topology is investigated, and for Ω not necessarily disjunctive, an analogue of the Wallman space is constructed.

KEY WORDS AND PHRASES. Normal lattice, 0-1 valued measures, Wallman space, disjunctive lattice, almost compact, almost countably compact.
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1. INTRODUCTION.

In the first part of this paper, we consider lattices which satisfy conditions weaker than normality; more precisely mildly normal and slightly normal lattices. We give examples of such lattices, and then investigate the preservation of these properties under lattice extension and restriction.

Next, we investigate spaces which are related to the general Wallman space. First, instead of considering the customary topology on the Wallman space, we introduce another topology and show how topological properties reflect strongly to the underlying lattice. Then we consider the case of a lattice which is not necessarily disjunctive and construct an associated Wallman type space. This work generalizes that of Liu (see Section 5).

We adhere to standard lattice terminology that can be found, for example, in [1], [2], [3], [4], [5]. However, in section 2, we summarize the principal lattice concepts and notations that will be used throughout the paper for the convenience of the reader. We then precede to the consideration of mildly normal and slightly normal lattices in section 3, and then to analogues of the general Wallman space in sections 4 and 5. We finally note that most of the results hold equally well for abstract lattices.

2. DEFINITIONS AND NOTATIONS.

a) Let X be an abstract set and Ω a lattice of subsets of X. We shall always assume, without loss of generality for our purposes, that ϕ, X ε Ω. The set whose general element L' is the complement of an element L of Ω is denoted by Ω'. Ω is said to be complement generated iff, for every L of Ω there exists a sequence $\{L_n\}_{n=1}^{\infty}$ in Ω such that $L = \cap_{n=1}^{\infty} L_n'$. Ω is separating if, for any two elements $x \neq y$ of X, there exists an element $L \in \Omega$ such that $x \in L$ and $y \notin L$. Ω is T_2 if, for any two elements $x \neq y$ of X, there exists $A,B \in \Omega$ such that $x \in A'$, $y \in B'$ and $A' \cap B' = \phi$. Ω is said to be disjunctive iff, for every $x \in X$ and $L \in \Omega$, if $x \notin L$ then there exists an $\hat{L} \in \Omega$, such that $x \in \hat{L}$ and $L \cap \hat{L} = \phi$. Ω is regular iff, for every $x \in X$ and every $L \in \Omega$, if $x \notin L$ then there exists $L_1, L_2 \in \Omega$ such that $x \in L_1'$, $L \subset L_2'$ and $L_1' \cap L_2' = \phi$. Ω is normal iff, for any $L_1, L_2 \in \Omega$, if $L_1 \cap L_2 = \phi$ then there exists $\hat{L}_1, \hat{L}_2 \in \Omega$ such that $L_1 \subset \hat{L}_1'$, $L_2 \subset \hat{L}_2'$ and $\hat{L}_1' \cap \hat{L}_2' = \phi$. Ω is lindelof iff, for every $L_\alpha \in \Omega$; $\alpha \in A$, if $\cap_\alpha L_\alpha = \phi$ then for a countable subcollection $\{L_{\alpha_i}\}$ of $\{L_\alpha\}$; $\cap_{i=1}^{\infty} L_{\alpha_i} = \phi$. Ω is compact iff, for every $L_\alpha \in \Omega$, $\alpha \in A$, if $\cap_\alpha L_\alpha = \phi$ then for some finite subcollection $\{L_{\alpha_i}\}$ of $\{L_\alpha\}$; $\cap_{i=1}^{n} L_{\alpha_i} = \phi$. Next, consider any two lattices Ω_1, Ω_2 of subsets of X. Ω_1 is said to semi-separate Ω_2 or for abbreviation (Ω_1 s.sΩ_2) iff, for every $L_1 \in \Omega_1$ and every $L_2 \in \Omega_2$ if $L_1 \cap L_2 = 0$ then there exists $\hat{L}_1 \in \Omega_1$ such that $L_2 \subset \hat{L}_1$ and $L_1 \cap \hat{L}_1 = \phi$. Ω_1 is said to separate Ω_2 if for any L_2, $\hat{L}_2 \in \Omega$, if $L_2 \cap \hat{L}_2 = \phi$ there exist L_1, $\hat{L}_1 \in \Omega_1$ such that $L_2 \subset L_1$, $\hat{L}_2 \subset \hat{L}_1$ and $L_1 \cap \hat{L}_1 = \phi$. We denote by $\tau\Omega$ the set whose general element is the intersection of arbitrary subsets of Ω. b) Let A be any algebra of subsets of X. A measure on A is defined to be a function, μ from A to R such that μ is bounded and finitely additive. The algebra of subsets of X generated by Ω is denoted by $A(\Omega)$. If $x \in X$, then μ_x is the measure concentrated at x so $\mu_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A \end{cases}$ where $A \notin A(\Omega)$. The set whose general element is a measure on $A(\Omega)$ is denoted $M(\Omega)$. Note that, since every element of $M(\Omega)$ is equal to the difference of nonnegative elements of $M(\Omega)$,, without loss of generality, we may work exclusively with nonnegative elements of $M(\Omega)$. Let $\mu \in M(\Omega)$, μ is Ω- regular if for any $A \in A(\Omega)$; $\mu(A) = \sup\{\mu(L); L \subset A,\ L \in \Omega\}$. The set whose general element is an element of $M(\Omega)$ which is Ω- regular is denoted by $M_R(\Omega)$. An element $\mu \in M(\Omega)$ is σ-smooth on Ω, if $L_n \in \Omega$, $n = 1,2,\ldots$ and $L_n \downarrow \phi$ then $\mu(L_n) \to 0$. The set whose general element is an element of $M(\Omega)$ which is σ-smooth on Ω is denoted by $M_\sigma(\Omega)$. We say that μ is σ-smooth on $A(\Omega)$ if $A_n \in A(\Omega)$, $n = 1,2\ldots$ and $A_n \downarrow \phi$ then $\mu(A_n) \to 0$. The set whose general element is an element of $M(\Omega)$ which σ-smooth on $A(\Omega)$ is denoted by $M^\sigma(\Omega)$. Note that if $\mu \in M_R(\Omega)$, then $\mu \in M_R^\sigma(\Omega)$ iff $\mu \in M_\sigma(\Omega)$. $I(\Omega)$, $I_R(\Omega)$, $I_\sigma(\Omega)$, $I^\sigma(\Omega)$, $I_R^\sigma(\Omega)$ are the subsets of the corresponding M's consisting of the non-trivial zero-one valued measures. For $\mu \in M(\Omega)$, the support of μ, $S(\mu) = \cap\{L \in \Omega,\ \mu(L) = \mu(X)\}$. L is replete iff, whenever $\mu \in I_R^\sigma(\Omega)$ then $S(\mu) \neq \phi$. A premeasure on Ω is defined to be a function π from Ω to $\{0,1\}$ such that $\pi(\phi) = 0$, $\pi(A) \leq \pi(B)$ for every $A \subset B$ where $A,B \in \Omega$ and if $\pi(A) = \pi(B) = 1$ then $\pi(A \cap B) = 1$. $\Pi(\Omega)$ denotes the set of all premeasures on Ω. c) As an immediate consequence of Zorn's Lemma, we have, for every $\mu \in I(\Omega)$, there exists an element $\nu \in I_R(\Omega)$ such that $\mu \leq \nu$ on Ω or simply ($\mu \leq \nu(\Omega)$). Also, for any two lattices Ω_1, Ω_2 of subsets of X, if $\Omega_1 \subset \Omega_2$, then for every $\mu \in I_R(\Omega_1)$, there exists a $\nu \in I_R(\Omega_2)$ such that $\nu|_{A(\Omega)} = \mu$ and such that a ν is unique if Ω_1 separates Ω_2. Moreover, Ω is normal iff, for $\mu \in I(\Omega)$, $\mu \leq \nu_1(\Omega)$, $\mu \leq \nu_2(\Omega)$ where $\nu_1, \nu_2 \in I_R(\Omega)$ then

$\nu_1 = \nu_2$. Ω is regular iff, for any $\mu_1, \mu_2 \in I(\Omega)$; $\mu_1 \le \mu_2(\Omega)$ then $S(\mu_1) = S(\mu_2)$. The result $\mu_x \in I_R(\Omega)$ iff Ω is disjunctive leads us to the Wallman Topology which is obtained by taking the totality of all $W(L) = \{\mu \in I_R(\Omega);\ \mu(L) = 1$ for $L \in \Omega\}$ as a base for the closed sets on $I_R(\Omega)$. For a disjunctive Ω, $I_R(\Omega)$ with the $\tau W(\Omega)$ of closed sets is a compact T_1 space and will be T_2 iff Ω is normal and is called the general Wallman space associated with X and Ω. Also, for a disjunctive Ω and $A, B \in A(\Omega)$, $W(A)$ is a lattice with respect to union and intersection. Moreover, $W(A') = (W(A))'$, $W(A(\Omega)) = A(W(\Omega))$, $W(A) = W(B)$ iff $A = B$ and $W(A) \subset W(B)$ iff $A \subset B$. Now, we note that, if Ω is disjunctive so is $W(\Omega)$, and in addition to each $\mu \in M(\Omega)$, there exists a $\hat{\mu} \in M(W(\Omega))$ defined by $\mu(A) = \hat{\mu}(W(A))$ for all $A \in A(\Omega)$ such that the map $\mu \to \hat{\mu}$ is one-to-one and onto; moreover $\mu \in M_R(\Omega)$ iff $\hat{\mu} \in M_R(W(\Omega))$.

3. ON NORMAL LATTICES.

In this section, we elaborate on the notion of a normal lattice, investigate lattices which satisfy weaker conditions, and discuss their interrelations under the extension and restriction properties. Throughout this section Ω, Ω_1, and Ω_2 will denote lattices of subsets of the set X. Note that, for $L_i \in \Omega$; $i=1,2,3$; Ω is normal iff $L_1 \subset L_2' \cup L_3'$, then $L_1 = A_1 \cup B_1$; $A_1 \subset L_2'$, $B_1 \subset L_3'$; $A_1, B_1 \in \Omega$.

THEOREM 3.1. Let Ω be normal, $\mu \in I(\Omega)$ and $G = \{L' \in \Omega';\hat{L} \subset L',\ \mu(\hat{L}) = 1,\ \hat{L} \in \Omega\}$ then, G is a prime Ω'-filter.

PROOF. Clearly, G is an Ω'- filter since $\phi \notin G$, $\mu(\hat{L}_1 \cap \hat{L}_2) = 1$ and if $L_1' \in G; L_1' \subset L_2'$ then $L_2' \in G$. In addition, if $L_1' \cup L_2' \in G$, then $\hat{L} \subset L_1' \cup L_2'$ and by the normality of Ω, $L = \hat{L}_1 \cup \hat{L}_2$ with $1 = \mu(\hat{L}) \le \mu(\hat{L}_1) + \mu(\hat{L}_2)$ then either $\mu(\hat{L}_1) = 1$ or $\mu(\hat{L}_2) = 1$ and so either $L_1' \in G$ or $L_2' \in G$. Thus, G is a prime Ω' - filter.

THEOREM 3.2. Suppose $\Omega_1 \subset \Omega_2$ and Ω_1 separates Ω_2. Then, Ω_1 is normal iff Ω_2 is normal.

PROOF. (i) Suppose Ω_1 is normal then by the separation. It is clear that Ω_2 is normal. (ii) Suppose Ω_2 is normal. Let $\mu \in I(\Omega_1)$, $\mu \le \nu_1, \nu_2(\Omega_1); \nu_1, \nu_2 \in I_R(\Omega_1)$. Extend ν_1, ν_2, to $\lambda_1, \lambda_2 \in I_R(\Omega_2)$ and μ to $\tau \in I(\Omega_2)$. Suppose there exists an $L_2 \in \Omega_2$; $\tau(\Omega_2) = 1$, but $\lambda_1(L_2) = 0$, then $\lambda_1(L_2') = 1$ and so there exists $\hat{L}_2 \in \Omega_2$; $\hat{L}_2 \subset L_2'$ $\lambda_1(\hat{L}_2) = 1$. By the separation, there exists $\hat{L}_1$, $L_1 \in \Omega$, $\hat{L}_1 \supset \hat{L}_2$, $L_1 \supset L_2$. Consequently, $\lambda_1(\hat{L}_1) = 1$ but $\tau(L_1) = 1$ then $\lambda_1(L_1) = 1$ which is a contradiction. Therefore, $\tau \le \lambda_1(\Omega_2)$ and similarly $\tau \le \lambda_2(\Omega_2)$. Thus, by the normality of Ω_2, $\lambda_1 = \lambda_2$ then $\nu_1 = \nu_2$ and so Ω_1 is normal.

DEFINITION 3.1. Ω is said to be mildly normal, if for all $\mu \in I_\sigma(\Omega)$, there exists a unique $\nu \in I_R(\Omega)$; $\mu \le \nu(\Omega)$.

DEFINITION 3.2. Ω is said to be almost countably compact, if for all $\mu \in I_R(\Omega')$, $\mu \in I_\sigma(\Omega)$.

DEFINITION 3.3. Ω is said to be prime replete, if for all $\mu \in I_\sigma(\Omega)$, $S(\mu) \neq \phi$.

THEOREM 3.3. If Ω is regular and prime replete, then Ω is mildly normal.

PROOF. If Ω is not mildly normal, then there exists $\mu \in I_\sigma(\Omega)$, $\nu_1 \neq \nu_2 \in I_R(\Omega)$ and $\mu \le \nu_1(\Omega)$, $\mu \le \nu_2(\Omega)$. Then, there exists L_1, $L_2 \in \Omega$; $L_1 \cap L_2 = \phi$. $\nu_1(L_1) = \nu_2(L_2) = 1$ and $\nu_1(L_2) = \nu_2(L_1) = 0$. But since Ω is regular, $S(\mu) = S(\nu_1) \subset L_1$ and $S(\mu) = S(\nu_2) \subset L_2$. Then, $S(\mu) \subset (L_1 \cap L_2) = \phi$ and so $S(\mu) = \phi$ which is a contradiction since Ω is prime replete. Thus, Ω must be mildly normal.

THEOREM 3.4. If Ω is regular and lindelof, then Ω is mildly normal.

PROOF. If Ω is not mildly normal, then by the proof of Theorem 3.3, $S(\mu)\subset(L_1\cap L_2) = \phi$. Moreover, $\phi = \bigcap_\alpha L_\alpha = S(\mu) = S(\nu_1) = S(\nu_2)$ with $\mu(L_\alpha) = 1$ and $\bigcup_\alpha L'_\alpha = X$. Since Ω is lindelof $\bigcup_{i=1}^{\infty} L'_{\alpha_i} = X$ and $\bigcap_{i=1}^{\infty} L_{\alpha_i} = \phi$, $L_{\alpha_i} \in \Omega$, $i=1,2,\ldots$ (may assume $L_{\alpha_i} \downarrow$), but $\mu(\alpha_i) = 1$ for all i which is a contradiction since $\mu \in I_\sigma(\Omega)$. Thus, Ω must be mildly normal.

THEOREM 3.5. If Ω is almost countably compact and mildly normal, then Ω is normal.

PROOF. Suppose Ω is almost countably compact, and $\mu \in I(\Omega)$. Then $\mu \le \nu \in I_R(\Omega')$ on Ω', and so $\nu \le \mu(\Omega)$; $\nu \in I_\sigma(\Omega)$. If $\zeta \in I_R(\Omega)$ and $\mu \le \zeta(\Omega)$, then $\nu \le \zeta(\Omega)$, $\nu \in I_\sigma(\Omega)$. Since Ω is mildly normal the rest of the proof is obvious.

THEOREM 3.6. Suppose $\Omega_1 \subset \Omega_2$ and Ω_1 separates Ω_2. Then, Ω_2 is mildly normal if Ω_1 is mildly normal.

PROOF. Suppose Ω_1 is mildly normal. Let $\mu \in I_\sigma(\Omega_2)$, $\mu \le \nu_1(\Omega_2)$, $\mu \le \nu_2(\Omega_2)$, ν_1, $\nu_2 \in I_R(\Omega_2)$, then $\mu \le \nu_1|(\Omega_1)$, $\mu \le \nu_2|(\Omega_1)$; $\nu_1|, \nu_2| \in I_R(\Omega_1)$ and $\mu \in I_\sigma(\Omega_2)$, hence $\nu_1| = \nu_2|$ since Ω_1 is mildly normal. Thus $\nu_1 = \nu_2$ since Ω_1 separates Ω_2 and so Ω_2 is mildly normal.

DEFINITION 3.4. Ω_2 is said to be Ω_1-countably bounded, if given $B_n \in \Omega_2$, $B_n \downarrow \phi$, there exists an $A_n \in \Omega_1$: $A_n \downarrow \phi$ and $B_n \subset A_n$.

In the next theorem, we will see when a partial converse of the theorem 3.6 is true.

THEOREM 3.7. Suppose Ω_1 separates Ω_2, and Ω_2 is Ω_1-countably bounded, then if Ω_2 is mildly normal so is Ω_1.

PROOF. Suppose Ω_2 is mildly normal and Ω_1 separates Ω_2. Let $\mu \in I_\sigma(\Omega_1)$ and $\mu \le \nu_1(\Omega_1)$, $\mu \le \nu_2(\Omega_1)$ where $\nu_1, \nu_2 \in I_R(\Omega_1)$. Extend μ to $\tau \in I(\Omega_2)$. By separation, ν_1 and ν_2 extend uniquely to ζ_1 and ζ_2 respectively, where ζ_1, $\zeta_2 \in I_R(\Omega_2)$, and $\tau \le \zeta_1, \zeta_2(L_2)$. Let $B_n \downarrow \phi$, $B_n \in \Omega_2$. Since Ω_2 is Ω_1-countably bounded, there exists $A_n \downarrow \phi$, $B_n \subset A_n$, $A_n \in \Omega_1$. Then $\tau(B_n) \le \tau(A_n) = \mu(A_n) = 0$ for some n. Then $\tau \in I_\sigma(\Omega_2)$. Since Ω_2 is mildly normal, $\zeta_1 = \zeta_2$ and hence $\nu_1 = \nu_2$ so Ω_1 is mildly normal.

DEFINITION 3.5. Ω is said to be countably paracompact if, whenever $A_n \downarrow \phi$, $A_n \in \Omega$, there exists $B_n \in \Omega$; $A_n \subset B'_n$ and $B'_n \downarrow \phi$.

DEFINITION 3.6. Ω is said to be slightly normal, if for all $\mu \in I_\sigma(\Omega')$, there exists a unique $\nu \in I_R(\Omega)$: $\mu \le \nu(\Omega)$

THEOREM 3.8. If Ω is regular and lindelof, then Ω is slightly normal.

PROOF. Suppose Ω is not slightly normal. Let $\mu \in I_\sigma(\Omega')$, $\mu \le \nu_1(\Omega)$, $\mu \le \nu_2(\Omega)$ where $\nu_1, \nu_2 \in I_R(\Omega)$; $\nu_1 \ne \nu_2$. Then $S(\mu) = \phi$. Let $\zeta(L) = \operatorname{Sup} \hat{\mu}(L'); \hat{L} \subset L$; $L \in \Omega$. Then, ζ is a premeasure, i.e. $\zeta \in \Pi(\Omega)$. Moreover, $S(\mu) = S(\zeta)$ since Ω is regular. Also $\zeta \in \Pi_\sigma(\Omega)$, and since Ω is lindelof, $S(\zeta) \ne \phi$ then $S(\mu) \ne \phi$ which is a contradiction. Thus, Ω must be slightly normal.

THEOREM 3.9. Suppose Ω_1 separates Ω_2. If Ω_1 is slightly normal, then Ω_2 is slightly normal.

PROOF. Let $\mu \in I_\sigma(\Omega'_2)$; $\mu \le \nu_1(\Omega_2)$, $\mu \le \nu_2(\Omega_2)$ where ν_1, $\nu_2 \in I_R(\Omega_2)$, then by separation $\mu| \le \nu_1(\Omega_1)$, $\mu| \le \nu_2|(\Omega_1)$; $\mu| \in I_\sigma(\Omega'_1)$ and $\nu_1|$, $\nu_2| \in I_R(\Omega_1)$. Therefore,

$\nu_1| = \nu_2|$ since Ω is slightly normal and so $\nu_1 = \nu_2$ by separation and hence Ω_2 is slightly normal.

LEMMA 3.1. If Ω is complement generated, then Ω is slightly normal.

PROOF. If Ω is complement generated, then $I_\sigma(\Omega') \subset I_R(\Omega)$, that is, if $\mu \in I_\sigma(\Omega')$ there exists a unique $\nu \in I_R(\Omega)$; $\mu \leq \nu(\Omega)$. Thus, Ω is slightly normal.

In the next theorem, we will see when a partial converse of theorem 3.9 is true.

THEOREM 3.10. Suppose Ω_1 separates Ω_2. If Ω_2 is mildly normal and countably paracompact then Ω_1 is slightly normal.

PROOF. Let $\mu \in I_\sigma(\Omega_1')$; $\mu \leq \nu_1(\Omega_1)$, $\mu \leq \nu_2(\Omega_1)$; ν_1, $\nu_2 \in I_R(\Omega_1)$. Extend μ to $\zeta \in I(\Omega_2)$. Consider $B_n \downarrow \phi$, $B_n \in \Omega_2$ then $B_n \subset A_n' \downarrow \phi$, $A_n \in \Omega_1$ since Ω_2 is countably paracompact and Ω_1 separates Ω_2 therefore Ω_2 is Ω_1 -countably paracompact and $\zeta(B_n) \leq \zeta(A_n') = \mu(A_n') \to 0$. Since $\mu \in I_\sigma(\Omega_1')$, then $\zeta \in I_\sigma(\Omega_2)$. Now extend ν_1, ν_2 to τ_1, $\tau_2 \in I_R(\Omega_2)$ then $\zeta \leq \tau_1(\Omega_2)$, $\zeta \leq \tau_2(\Omega_2)$ by separation and so $\tau_1 = \tau_2$ since Ω is mildly normal. Thus $\nu_1 = \nu_2$ and so Ω_1 is slightly normal.

REMARK 3.1. It is not difficult to give similar conditions (as in theorem 3.10) to obtain the other partial converse of theorem 3.9.

4. ON SPACES RELATED TO THE GENERAL WALLMAN SPACE.

We recall from section 2 that for an arbitrary lattice Ω of subsets of X, $I_R(\Omega)$ with the topology $\tau W(\Omega)$ of the closed set, is a compact T_1 space. Also, if Ω is disjunctive and separating then X can be embedded in $I_R(\Omega)$. Moreover, if Ω is disjunctive, so is $W(\Omega)$, and $I_R(\Omega)$ with the topology $\tau W(\Omega)$ is T_2 iff Ω is normal. In this section, we consider alternate topologies on $I_R(\Omega)$.

THEOREM 4.1. Consider $\delta = \tau W(\Omega')$ for a base of closed sets $W(L')$;$L' \in \Omega$. Then $I_R(\Omega)$ with the topology δ is T_2.

PROOF. If $\mu_1 \neq \mu_2$: $\mu_1, \mu_2 \in I_R(\Omega)$ then there exists $L_1, L_2 \in \Omega$;$L_1 \cap L_2 = \phi$, $\mu_1(L_1) = 1$ and $\mu_1(L_2) = 0$, $\mu_2(L_2) = 1$ and $\mu_2(L_1) = 0$. Therefore $\mu_1 \in W(L_1)$, $\mu_2 \notin W(L_1)$ and $\mu_2 \in W(L_2)$, $\mu_1 \notin W(L_2)$; $W(L_1)$ and $W(L_2)$ are open sets. Thus, $W(L_1) \cap W(L_2) = \phi$ and consequently, $I_R(\Omega)$ with δ is T_2.

DEFINITION 4.1. Ω is said to be almost compact, if for all $\mu \in I_R(\Omega')$, $S(\mu) \neq \phi$.

THEOREM 4.2. The lattice $W(\Omega')$ is almost compact.

PROOF. Let $\lambda \in I_R W((\Omega')') = I_R(W(\Omega))$ then $\lambda = \hat{\mu}$, $\mu \in I_R(\Omega)$ (section 2) and so $S(\hat{\mu}) = \cap W(L)$ with $1 = \hat{\mu}(W(L)) = \mu(L)$. Thus, $\mu \in W(L)$, $\mu \in S(\hat{\mu})$ on $W(\Omega)$ and so $I_R(\Omega)$ with $W(\Omega')$ is almost compact.

REMARK 4.1. We note (i) $S(\lambda) = S(\hat{\mu}) = \{\mu\}$. (ii) If Ω is disjunctive, it is then clear that for any $L \in \Omega$, $L = \cap L_\alpha'$, $L \subset L_\alpha'$; $L_\alpha \in \Omega$.

THEOREM 4.3. The sets of $W(\Omega)$ are clopen in the δ topology.

PROOF. Since $W(L)$ is disjunctive then by remark 4.2 (ii) for any $L \in \Omega$, $W(L) = \cap W(L_\alpha')$, $W(L) \subset W'(L_\alpha)$, $L_\alpha \in \Omega$ is closed in the δ-topology. But, $W(L)$ is also open in δ-topology since $W(L) = (W(L')')$. Thus, the sets of $W(\Omega)$ are closed in the δ-topology.

THEOREM 4.4. $I_R(\Omega)$ with the δ-topology is compact iff Ω is an algebra.

PROOF. (i) Suppose $I_R(\Omega)$ with δ is compact. Thus, $\tau W(\Omega')$ is a compact lattice and so is $W(\Omega')$. Let $\zeta \in I(W(\Omega'))$. Then $S(\zeta) \neq \phi$ on $W(\Omega')$, $\zeta = \hat{\lambda}$; $\lambda \in I(\Omega)$.

Let $\mu \in S(\zeta)$, $\mu \in I_R(\Omega)$ then $1 = \zeta(W(L'))$ iff $\hat{\lambda}(W(L')) = 1$ iff $\lambda(L') = 1$. Now $\lambda(L') = 1$. Then $\zeta(W(L')) = 1$ then $\mu \in W(L')$ then $\mu(L') = 1$. Thus, $\lambda < \mu(\Omega')$ and hence $\mu < \lambda(L)$. But $\mu \in I_R(L)$ then $\mu = \lambda$. Thus $\hat{\mu} = \hat{\lambda}$ and hence $I(W(\Omega)) = I(W(\Omega')) = I_R(W(\Omega))$. Thus, $\Omega = \Omega'$ [2]. (ii) The converse is clear.

THEOREM 4.5. $\delta = \tau W(\Omega)$ iff Ω is an algebra.

PROOF. (i) Since the sets $W(\Omega)$ are clopen by Theorem 4.3 then $W(\Omega) \in \delta$ and so $\tau W(\Omega) \subset \delta$. Now, if $\delta = \tau W(\Omega)$ then since $\tau W(\Omega)$ is compact so is δ and so Ω is an algebra by theorem 4.4. (ii) The converse is clear.

In the next theorem, we give another equivalent condition for Ω to be an algebra.

THEOREM 4.6. Ω is an algebra iff $W(\Omega')$ is a disjunctive lattice in $I_R(\Omega)$.

PROOF. (i) Suppose Ω is an algebra, then $I_R(\Omega) = I(\Omega)$. Let $\mu \in I_R(\Omega)$; $\mu \notin W(L')$, $L \in \Omega$. Then $\mu(L') = 0$, $1 = \mu(L) = \mu((L')')$. $L' \in \Omega$ and $W(L') \cap W((L'))' = \phi$. Thus, $W(\Omega')$ is disjunctive. (ii) Suppose $W(\Omega')$ is disjunctive. Let $\mu \in I(\Omega)$ then there exists a $\nu \in I_R(\Omega)$; $\mu < \nu(\Omega)$. For $\mu \neq \nu$, there exists $L \in \Omega$, $\mu(L) = 0$, $\nu(L) = 1$ then $\nu \notin W(L')$. Hence, by disjunctiveness $\nu \in W(\hat{L})'$, $L \in \Omega$ and $\phi = W(\hat{L})' \cap W(L)' = W(\hat{L}' \cap L') = W((\hat{L} \cup L)')$. Hence, $\hat{L} \cup L = X$, $L' \cap \hat{L}' = \phi$ and so $L' \subset \hat{L}$, but since $\mu(\hat{L}) = 1$ and $\nu \in W(\hat{L})'$, $\nu(\hat{L}) = 0$ which is a contradiction, since $\mu < \nu$. Thus, $I_R(\Omega) = I(\Omega)$, and so Ω is an algebra.

5. ON NON-DISJUNCTIVE LATTICES.

We next consider the case where Ω is not necessarily disjunctive. We begin, by introducing the notion of an Ω-convergent measure and some related results and then proceed to the construction of an analogue of the Wallman space.

DEFINITION 5.1. $\mu \in I(\Omega)$ is said to be Ω-convergent if there exists an $x \in X$ such that $\mu_x < \mu(\Omega)$.

THEOREM 5.1. μ is Ω-convergent iff $S(\mu) \neq \phi$ on Ω', for all $\mu \in I(\Omega)$.

PROOF. (i) Suppose μ is Ω-convergent. Then there exists $x \in X$; $\mu_x < \mu(\Omega)$ and so $\mu < \mu_x(\Omega')$. Moreover $x \in S(\mu_x) \subset S(\mu)$ on Ω'. Thus, $S(\mu) \neq \phi$ on Ω'. (ii) Suppose $S(\mu) \neq \phi$ on Ω' for $\mu \in I(\Omega)$. Let $x \in S(\mu)$ on Ω'. Then, $\mu < \mu_x(\Omega')$ and so $\mu_x < \mu(\Omega)$. Thus μ is Ω-convergent.

THEOREM 5.2. Suppose $\mu_1 < \mu_2(\Omega)$, for all μ_1, $\mu_2 \in I(\Omega)$. Then

a) If μ_1 is Ω-convergent so is Ω_2.

b) If Ω' is regular and μ_2 is Ω-convergent then μ_1 is Ω-convergent.

PROOF. a) Suppose μ_1 is Ω-convergent, then $\mu_x < \mu_1(\Omega)$ for some x, but $\mu_1 < \mu_2(\Omega)$ then $\mu_x < \mu_2(\Omega)$ and so μ_2 is Ω-convergent. b) Suppose μ_2 is Ω-convergent, then $\mu_x < \mu_2(\Omega)$ for some x and so $\mu_2 < \mu_x(\Omega')$ and $x \in S(\mu_2)$ on Ω' but $\mu_1 < \mu_2(\Omega)$ then $\mu_2 < \mu_1(\Omega')$ and since Ω' is regular $x \in S(\mu_2) = S(\mu_1)$ on Ω'. Thus, $S(\mu_1) \neq \phi$ on Ω' and so μ_1 is Ω-convergent.

THEOREM 5.3. Suppose Ω' is T_2 and μ is Ω-convergent where $\mu \in I(\Omega)$. Then, there exists a unique $x \in X$: $\mu_x < \mu(\Omega')$.

PROOF. If Ω' is T_2 and if $x \neq y$ then there exists L_1, $L_2 \in \Omega$; $x \in L_1$, $y \in L_2$; $L_1 \cap L_2 = \phi$ and $\mu_x(L_1) = 1$, $\mu_x(L_2) = 1$. Now, if $\mu_x < \mu(\Omega)$ and $\mu_y < \mu(\Omega)$ then $\mu(L_1) = \mu(L_2) = 1$ but $L_1 \cap L_2 = \phi$ which is a contradiction and so the desired result is true.

DEFINITION 5.2. Ω is said to be weakly compact if for all $\mu \in I_R(\Omega)$, μ is Ω-convergent.

DEFINITION 5.3. Ω is said to be almost compact if for any $\mu \in I_R(\Omega')$, $S(\mu) \neq \phi$ on Ω.

THEOREM 5.4. Ω is weakly compact iff Ω' is almost compact.

PROOF. (i) Let $\mu \in I_R(\Omega)$ then since Ω is weakly compact, there exists an $x \in X$ such that $\mu_x \leq \mu(\Omega)$ and $S(\mu) \neq \phi$ on Ω'. Thus, Ω' is almost compact. (ii) Let $\mu \in I_R(\Omega)$ then $S(\mu) \neq \phi$ on Ω' since Ω is almost compact. Let $x \in S(\mu)$ on Ω' then $\mu \leq \mu_x(\Omega')$, $\mu_x \leq \mu(\Omega)$ and so Ω is weakly compact.

Now, note that a topological space X is absolutely closed (generalized absolutely closed) iff the lattice Θ of open sets is $T_2(T_o)$ and weakly compact. Next, let Ω be a lattice of subsets of X and define $U(\Omega) = \{\cup L_\alpha;\ L_\alpha \in \Omega\}$.

THEOREM 5.5. a) If $\Omega_1 \subset \Omega_2$ and Ω_2 is weakly compact, then Ω_1 is weakly compact. b) Suppose $\Omega_1 \subset \Omega_2 \subset U(\Omega_1)$ and Ω_2 semi-separates Ω_2. Then, if Ω_1 is weakly compact, Ω_2 is weakly compact.

PROOF. a) Extend $\mu \in I_R(\Omega_1)$ to $\nu \in I_R(\Omega_2)$. Since Ω_2 is weakly compact, there exists an x; $\mu_x \leq \nu(\Omega_2)$. $\mu_x \leq \mu(\Omega_1)$ and so Ω_1 is weakly compact. b) Let $\nu \in I_R(\Omega_2)$. Then since $\Omega_1 s.s \Omega_2$, $\mu \in I_R(\Omega_1)$ where μ is the restriction of ν to $A(L_1)$. Thus, there exists an $x \in X$ such that $\mu_x \leq \mu(\Omega_1)$. Now, suppose $L_2 \in \Omega_2$ and $\mu_x(L_2) = 1$ then $x \in X$ $X \in L_2$, but $L_2 = \cup L_{1\alpha}, L_{1\alpha} \in \Omega$; then x is on some $L_{1\alpha}$ and so $\mu_x(L_{1\alpha}) = 1$ for some $L_{1\alpha}$ and moreover $\mu(L_{1\alpha}) = 1$ since $\mu_x \leq \mu(\Omega_1)$, but $L_{1\alpha} \subset L_2$ then $\nu(L_2) = 1$ and so $\mu_x \leq \nu(\Omega_2)$. Thus, Ω_2 is weakly compact.

REMARK 5.1. Let X be a topological space and Θ the collection of open sets. Then, by Theorem 5.4, Θ is weakly compact iff $F = \Theta'$ is almost compact.

Now consider X. Suppose Ω is non-disjunctive and define $\hat{I} = \{\mu_x : x \in X\} \cup \{\mu \in I_R(\Omega): \mu$ is not Ω-convergent$\}$ and $\hat{W}(A) = \{\mu \in \hat{I};\ \mu(A) = 1,\ A \in A(\Omega)\}$. We also assume Ω is T_o, so $x, y \in X$ and $x \neq y$ implies $\mu_x \neq \mu_y$.

THEOREM 5.6. For $A, B \in A(\Omega)$, we have: a) $A = B$ iff $\hat{W}(A) = \hat{W}(B)$, b) $\hat{W}(A \cup B) = \hat{W}(A) \cup \hat{W}(B)$, c) $\hat{W}(A \cap B) = \hat{W}(A) \cap \hat{W}(B)$, d) $\hat{W}(A') = (\hat{W}(A))'$, e) $\hat{W}(A(\Omega)) = A(\hat{W}(\Omega))$.

PROOF. a) (i) If $A = B$, then, clearly $\hat{W}(A) = \hat{W}(B)$. (ii) If $A \neq B$, then say $A \cap B' \neq \phi$, let $x \in A \cap B'$ then $\mu_x(A \cap B') = 1$; $\mu_x \in \hat{I}$ and so $\mu_x(A) = 1$, $\mu_x(B) = 0$ which implies that $\mu \in \hat{W}(A)$, $\mu \notin \hat{W}(B)$ and $\hat{W}(A) \neq \hat{W}(B)$. b), c), d) and e) are not difficult to show and are omitted.

Now consider $\mu \in I(\Omega)$ and define $\hat{\mu} \in I(\hat{W}(\Omega))$ to be $\hat{\mu}(\hat{W}(A)) = \mu(A)$, $A \in A(\Omega)$. Then, one can easily note that $\mu \to \hat{\mu}$ is 1-1 and onto from $I(\Omega)$ to $I(\hat{W}(\Omega))$, and moreover $\mu \in I_R(\Omega)$ iff $\hat{\mu} \in I_R(\hat{W}(\Omega))$.

THEOREM 5.7. $\hat{W}(\Omega)$ is weakly compact and T_o.

PROOF. a) Let $\hat{\mu} \in I_R(\hat{W}(\Omega))$ then $\mu \in I_R(\Omega)$. If μ is Ω-convergent then $\mu_x \leq \mu(\Omega)$ implies that $\hat{\mu}_x \leq \hat{\mu}(\hat{W}(\Omega))$. Note that for $A \in A(\Omega)$,

$$\mu_x(A) = \hat{\mu}_x(\hat{W}(A)) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$$ and $\mu_x \in \hat{W}(A)$ iff $\mu_x(A) = 1$. Thus, $\hat{\mu}_x$ is the measure

concentrated at μ_x and so $\hat{\mu}$ is $\hat{W}(\Omega)$-convergent. If μ is not Ω-convergent then $S(\hat{\mu}) = \cap \hat{W}(L)$ with $\hat{\mu}(\hat{W}(L)) = 1 = \mu(L)$, hence $\mu \in \hat{W}(L)$, $\mu \in I$, $\mu \in S(\hat{\mu})$ and $\hat{\mu}$ is the measure concentrated at μ, and consequently $\hat{W}(\Omega)$ is weakly compact. b) Let μ_1,

$\mu_2 \in \hat{I}$, $\mu_1 \neq \mu_2$ then there exists an $L \in \Omega$ such that say, $\mu_1(L) = 1$, $\mu_2(L) = 0$. Therefore $\mu_1 \in \hat{W}(L)$, $\mu_2 \in \hat{W}(L')$ and so $\hat{W}(L)$ is T_o.

THEOREM 5.8. If $\Omega = U(\Omega)$ then $\hat{W}(\Omega)$ separates $U(\hat{W}(\Omega))$.

PROOF. Suppose (i) $(\cup_\alpha \hat{W}(L_\alpha)) \cap (\cup_\beta \hat{W}(L_\beta')) = \phi$. Let $A = (\cup_\alpha L_\alpha) \in \Omega$, $B = (\cup_\beta L_\beta) \in \Omega$.

Since $\Omega = U(\Omega)$ then $(\cup_\alpha \hat{W}(L_\alpha)) \subset \hat{W}(A)$ and $(\cup_\beta \hat{W}(L_\beta) \subset W(B)$. If $A \cap B \neq \phi$ then $L_\alpha \cap L_\beta \neq \phi$ for some α, β. Let $x \in L_\alpha \cap L_\beta$ then $\mu_x \in \hat{I}$ and $\mu_x \in \hat{W}(L_\alpha)$ and $\mu_x \in \hat{W}(L_\beta)$ which contradicts (i). Thus, $A \cap B = \phi$, $\hat{W}(A) \cap \hat{W}(B) = \phi$, and the desired result is now clear.

Now, we note that if $\Omega = U(\Omega)$ then $\hat{I}$ with $\Theta = \hat{W}(\Omega)$ is generalized absolutely closed and is absolutely closed if Ω' is T_2. Thus, if we consider X and let $\Omega = U(\Omega) = \Theta_x$ and T_2, then $\hat{I}$, Θ is an absolute closure of X since one can easily observe that $\bar{X} = \hat{W}(X)$.

REMARK. An analogous construction can now be done for $\hat{I}^\sigma$, $\hat{W}^\sigma$. Where $\hat{I}^\sigma = \{\mu_x : x \in X\} \cup \{\mu \in I_R^\sigma(\Omega)$: μ is not Ω-convergent$\}$ and $\hat{W}^\sigma(A) = \{\mu \in I^\sigma$, $\mu(A) = 1$, $A \in A(\Omega)$ and one can show that $\hat{I}^\sigma(\Omega)$ is weakly replete. Ω is weakly replete if for any $\mu \in I_R^\sigma(\Omega)$, μ is Ω-convergent. We note that the constructions here generalize the work of Lui [6].

REFERENCES

1. NOEBELING, G., Gründlagen der analytischen topologie, Springer-Verlag, Berlin 1954.

2. FROLIK, Z., Prime Filters with the C.I.P., Comm. Math. Univ. Carolinae, 13 (1972), 533-575.

3. BACHMAN, G. and STRATIGOS, P., Criteria for σ-smoothness, , τ-smoothness, and tightness of Lattice Regular Measures, with Applications, Con. J. Math. 33(1981), 1498-1525.

4. GRASSI, P., On subspaces of replete and measure replete spaces. Con. Math. Bull. 27(1), (1984), 58-64.

5. SZETO, M., Measure Repleteness and Mapping Preservations, J. Indian Math. Soc. 43(1979), 35-52.

6. LIU, C.T., Absolutely Closed Spaces, Trans. Amer. Math. Soc. 130 (1968), 86-104.

7. BOURBAKI, N., Elements of Mathematics, General Topology, Part I, Addison-Wesley, Reading, Mass. 131.