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A NOTE ON RINGS WITH CERTAIN VARIABLE IDENTITIES

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ABSTRACT. It is proved that certain rings satisfying generalized-commutator constraints of the form $[x^m, y^n, y^n, \ldots, y^n] = 0$ with m and n depending on x and y, must have nil commutator ideal.

KEY WORDS AND PHRASES. Commutator ideal, periodic ring.
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1. INTRODUCTION.

Let $[x_1, x_2]$ denote $x_1 x_2 - x_2 x_1$, and for $k > 2$, let $[x_1, x_2, \ldots, x_k] = [[x_1, \ldots, x_{k-1}], x_k]$. For $x_1 = x$ and $x_2 = x_3 = \ldots = x_k = y$, denote $[x, y, \ldots, y]$ by $[x,y]_k$. A result of Herstein [1] and of Anan'in and Zyabko [2] asserts that if for any x and y in a ring R, there exist positive integers $m = m(x,y)$, $n = n(x,y)$ such that $x^m y^n = y^n x^m$, then the commutator ideal of R is nil. Recently, Herstein [3] proved that a ring R in which for any x, y, $z \in R$ there exists positive integers $m = m(x, y, z)$, $n = n(x, y, z)$, and $q = q(x, y, z)$ such that $[[x^m, y^n], z^q] = 0$ must have nil commutator ideal. More recently Klein, Nada and Bell [4] raised the following conjecture which arises naturally from the above mentioned work.

CONJECTURE. Let $k > 1$. If for each x, $y \in R$, there exists positive integers m and n such that $[x^m, y^n]_k = 0$, then the commutator ideal of R is nil.

In [4], Klein, Nada and Bell proved the conjecture for rings with identity 1. Given the complexity of [1] and [3], it would appear that no proof of this conjecture is in sight. Our objective is to prove the conjecture for certain classes of rings and to generalize a result of Herstein in [3] and some results in [4] and [5].

A ring R is called periodic if for each x in R, there exists distinct positive integers m and n for which $x^m = x^n$. In preparation for the proofs of our main theorems, we start with the following lemma which is known [5] and we omit its proof.

LEMMA 1. If R is a periodic ring, then for each x in R, there exists a positive integer k = k(x) such that x^k is idempotent.

2. MAIN RESULTS.

The following theorem shows that the conjecture is true for Artinian rings.

THEOREM 1. Let $k > 1$, and let R be an Artinian ring such that for each x, y in R, there exists positive integers m and n such that $[x, y]_k = 0$. Then the commutator ideal of R is nil.

PROOF. To prove that the commutator ideal of R is nil it is enough to show that if R has no nonzero nil ideals then it is commutative. So we suppose that R has no nonzero nil ideals. Since R is Artinian, the Jacobson radical J of R is nilpotent. So J = 0, and hence R is semisimple Artinian. This implies that R has an identity element and now, R is commutative by Theorem 3 of [4].

Next, we prove Theorem 2 which shows that the conjecture is true for periodic rings. This result generalizes a result of Bell in [5].

THEOREM 2. Let $k > 1$ and let R be a periodic ring such that for each x, y in R there exists positive integers m and n such that $[x^m , y^n]_k = 0$. Then the commutator ideal of R is nil.

PROOF. If k = 2, then the result follows by the theorem in [1]. So assume $k > 2$ and let x be any element of R and let e be any idempotent of R. By hypothesis, there exists integers m and n such that $[x^m , e^n]_k = 0$. This implies that $[x^m , e]_k = 0$, and hence

$$[x^m , e]_{k-1}e = e[x^m , e]_{k-1} .$$

Multiplying by e from the right and using the fact that $e[x^m , e]_{k-1} e = 0$ we obtain $[x^m, e]_{k-1} e = 0$. Hence $0 = ([x^m, e]_{k-2} e - e[x^m , e]_{k-2})e = [x^m , e]_{k-2} e$. Continuing this way we get $[x^m , e]e = 0$ which implies that $x^m e = ex^m e$. Similarily, we can get $ex^m = ex^m e$. This implies that

$$x^m e = ex^m , x \in R, \text{ e any idempotent and } m = m(x, e). \qquad (2.1)$$

Let y be any element of R. Since R is periodic, Lemma 1 implies that y^p is idempotent for some positive integer p = p(y). So (2.1) implies that for each x, y in R there exists positive integers m and p such that $x^m y^p = y^p x^m$. Now, the result follows by the well-known theorem in [1] or [2].

THEOREM 3. Let $k > 1$. If R is a prime ring having a nonzero idempotent element such that for each x, y in R there exists positive integers m and n such that $[x^m , y^n]_k = 0$. Then R is commutative.

PROOF. The argument used in Theorem 2 to reach statement (2.1) in the proof shows that a ring satisfying the generalized commutator constraint $[x^m , y^n]_k = 0$ must have its idempotent elements in the center. For let e_1 and e_2 be idempotent elements in R. (2.1) implies that $e_1 e_2 = e_2 e_1$ and hence the idempotents of R commute. This implies that the idempotents of R are central in R [6, Remark 2]. Let e be a nonzero idempotent of R. Then e is a nonzero central idempotent in the prime ring R. Hence e is an identity element of R since it can not be a zero divisor. Now R is commutative by Theorem 3 of [4].

The proof of Theorem 4 below was done by Kezlan in the proof of his main theorem in [7]. So we omit its proof here.

THEOREM 4. Let $k > 1$. If R is a prime ring with a nontrivial center such that for each x, y in R there exists positive integers m and n such that $[x^m, y^n]_k = 0$, then R is commutative.

The following result generalizes Theorem 1 of [4].

THEOREM 5. Let R be a ring and let M be a fixed positive integer. Suppose that for each $x, y \in R$ there exist positive integers $m = m(x, y) < M$ and $n = n(x, y)$ such that $[x^m, y^n, y^n]$ belongs to the center of R. Then the commutator ideal of R is nil.

PROOF. Again, we suppose that R has no nil ideals and hence R is a subdirect product of prime rings satisfying the above hypothesis of R. So we may assume that R is prime. Let Z be the center of R. If $Z = 0$, then for each $x, y \in R$, $[x^m, y^n, y^n] = 0$ where $m = m(x, y) < M$, and $n = n(x, y)$. This implies that R is commutative by Theorem 1 of [4]. So we may assume that R has a nontrivial center, and hence R is commutative by Theorem 4 above.

The following result generalizes Theorem 8 in [3].

THEOREM 6. Let R be a ring in which, for each $x, y, z \in R$, there exists positive integers $m = m(x, y, z)$ $n = n(x, y, z)$ and $q = q(x, y, z)$ such that $[x^m, y^n, z^q]$ belongs to the center of R. Then the commutator ideal of R is nil.

PROOF. Again, we may assume that R is a prime ring satisfying the above hypothesis. Let Z be the center of R. If $Z = 0$, then for each $x, y, z \in R$, $[[x^m, y^n], z^q] = 0$, where $m = m(x, y, z)$, $n = n(x, y, z)$ and $q = q(x, y, z)$. This implies that R is commutative by Theorem 8 of [3]. So we may assume that R has a nontrivial center. For any x, y in R, $[[x^m, y^n], y^q] \in Z$ where m, n, q are each functions of the variables x and y. So $[[[x^m, y^n], y^q], y] = 0$, which implies that $[[[x^m, y^{nq}], y^{nq}], y^{nq}] = 0$. Hence R is commutative by Theorem 4 above.

REMARK. The result in Theorem 6 can be generalized as follows. Let R be a ring such that for each $x, y, z \in R$, there exists positive integers $m = m(x, y, z)$, $n = n(x,y,z)$ and $q = q(x,y,z)$ such that $[x^m, y^n, z^q, r_1, r_2, \ldots, r_k] = 0$ for all elements $r_1, \ldots, r_k$ in R. Then the commutator ideal of R is nil. This can be done by induction on k and using the argument in Theorem 6. We omit the details of the proof.

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