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ON STABILITY AND BOUNDEDNESS OF SOLUTIONS OF A CERTAIN FOURTH-ORDER DELAY DIFFERENTIAL EQUATION

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ABSTRACT. Using a Razumikhin - type theorem, we deduce sufficient conditions that guarantee the uniform asymptotic stability and boundedness of solutions of a scalar real fourth-order delay differential equation. The Lyapunov function constructed for an ordinary fourth-order differential equation is seen to work for the delay system.

KEY WORDS AND PHRASES. Stability, boundedness, uniform asymptotic stability.
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1. INTRODUCTION.

The Razumikhin-type theorems give sufficient conditions that ensure the stability and boundedness of the solutions of a delay differenital equation in terms of the rate of change of a function along solutions. For the use of Lyapunov functionals to study stability and boundedness of solutions of delay differential equations of the first, second, third and the fourth orders refer to the papers of Chukwu [1], Sinha [2]; and to Driver [3], and [5]. On the other hand, using the Razumikhin approach, Hale [5], used Lyapunov functions to give sufficient conditions for stability and boundedness of a first-order and a second-order delay differential equations. Razumikhin in [6] utilized his theorems to determine stability regions of a second-order control system dscribed by a delay differential equation, and in another case in [6] investigated the stability problem of a third-order delay system of equations. Essentially, our main aim here is to use the Lyapunov function utilized by Ezeilo in [7] for ordinary differential equations to attempt to prescribe some sufficient conditions that guarantee the uniform asymptotic stability and the boundedness of the solutions of the fourth-order delay differential equation of the form

$$\ddddot{x}(t) + f(\dddot{x}(t))\dddot{x}(t) + \alpha_2 \ddot{x}(t) + \beta_2 \ddot{x}(t-h) + g(\dot{x}(t-h)) + \alpha_4 x(t) + \beta_4 x(t-h) = P(t) \tag{1.1}$$

where α_2, β_2, α_4 β_4 are constants and $h > 0$ is a constant. The function f, g, p are completely continuous depending on the arguments displayed explicitly; f, g, p are assumed also to satisfy enough additional smoothness conditions to ensure the solution of (1.1) through any initial data is continuous in the initial data and in time. We shall consider stability of the trivial solutions of (1.1) for the case $p \equiv 0$. Corresponding results are deduced for a real fourth-order delay differential equation with constant coefficients. As a consequence, a generalized Routh-Hurwitz condition for a delay fourth order linear equation is deduced when the delay is sufficiently small.

2. PRELIMINARIES.

Dots such as are in equation (1.1) denote differentiation with respect to t. E^n is an n-dimensional linear vector space over the reals with norm for any $x \in E^n$ written $|x|$. For $h > 0$, $C = C([-h,0], E^n)$ with the topology of uniform convergence. We designate the norm of an element ϕ by $||\phi||$ and defined by $||\phi|| = \sup_{-h \leq \theta \leq 0} |\phi(\theta)|$.

If $\sigma \in E$, $a > 0$ and $x \in C([\sigma - h, \sigma + a], E^n)$ then for any $t \in [\sigma, \sigma + a]$ we let $x_t \in C$ be defined by $x_t(\theta) = x(t+\theta)$, $-h \leq \theta \leq 0$. If D is a subset of $E \times E$, and $f: D \to E^n$ is given function, then

$$\dot{x}(t) = f(t, x_t) \tag{2.1}$$

is a retarded functional differential equation on D. Note that (1.1) is a special case of (2.1) and it also includes ordinary differential equations when $h = 0$.

DEFINITION 2.1. A function x is said to be a solution of (2.1) on $[\sigma + h, \sigma + a)$ if there are $\sigma \in E$ and $a > 0$ such that $x \in C(\sigma - h, \sigma + a], E^n)$, $(t, x_t) \in D$ and $x(t)$ satisfies (2.1) for $t \in [\sigma, \sigma + a]$. For given $\sigma \in E$, $\phi \in C$, we say $x(\sigma, \phi)$ is a solution of (2.1) with initial value ϕ at σ or simply through (σ, ϕ) if there is an $a > 0$ such that $x(\sigma, \phi)$ is a solution of equation (2.1) on $[\sigma - h, \sigma + a)$ and $x_\sigma(\sigma, \phi) = \phi$.

DEFINITION 2.2. Suppose $f(t,0) = 0$ for all $t \in E$, , then the solution $x = 0$ of (2.1) is said to be uniformly stable if for any $\sigma \in E$, $\varepsilon > 0$, there is $\delta = \delta(\varepsilon) > 0$ such that $||\phi|| < \delta$ implies $||x_t(\sigma, \phi)|| < \varepsilon$ for $t \geq \sigma$. The solution $x = 0$ of (2.1) is uniformly asymptotically stable if it is uniformly stable and there is a $b > 0$ such that for every $\eta > 0$, there is a $T(\eta)$ such that $||\phi|| \leq b$ implies $||x_t(\sigma, \phi)|| \leq \eta$ for $t \geq \sigma + T(\eta)$ for every $\sigma \in E$.

DEFINITION 2.3. The solutions $x(\sigma, \phi)$ of (2.1) are uniformly bounded if for any $\alpha > 0$ there is a $B = B(\alpha) > 0$ such that for all $\sigma \in E$, $\phi \in C$ and $||\phi|| \leq \alpha$, we have $||x_t(\sigma, \phi)|| \leq B$ for all $t \geq \sigma$.

The following theorems (due to Razumikhin and Krasovskii [8]) for stability of solutions of (2.1) are reproduced from [5]. First if $V: E \times C \to E$ is continuous and $x(\sigma, \phi)$ is the solution of (2.1) through (σ, ϕ), then we define

$V(t, \phi(0)) = \overline{\lim}_{r \to 0^+} \frac{1}{r} [V(t+r, x_{t+r}(t, \phi) - V(t, \phi(0))]$ where $x(t, \phi)$ is the solution of (2.1) through (t, ϕ).

PROPOSITION 2.1. (Razumikhin) Suppose $f: E \times C \to E^n$ takes $E \times$ (bounded sets of C) into bounded sets of E^n and consider (2.1). Suppose u, v, w: $[0, \infty) \to [0, \infty)$ are continuous nondecreasing functions, u(s), v(s) positive for $s > 0$, $u(0) = v(0) = 0$. If there is a continuous function $V: E \times E^n \to E$ such that

$$u(|x|) \leq V(t,x) \leq v(|x|),\ t \in E,\ x \in E^n, \tag{2.2}$$

$$\dot{V}(t, \phi(0)) \leq -w(|\phi(0)|), \tag{2.3}$$

if $V(t+\theta, \phi(\theta)) \leq V(t, \phi(0))$, $\theta \in [-h,0]$, then the solution $x = 0$ of (2.1) is uniformly stable.

PROPOSITION 2.2: (Krasovskii) Suppose all the conditions of proposition 2.1 are satisfied and in addition $w(s) > 0$ if $s > 0$. If there is a continuous nondecreasing function $J(s) > s$ for $s > 0$ such that condition (2.3) is strengthened to

$$\dot{V}(t, \phi(0)) \leq -w(|\phi(0)|) \text{ if } V(t+\theta, \phi(\theta)) < J(V(t, \phi(0)) \ \theta \in [-h,0], \tag{2.4}$$

then the solution $x = 0$ of (2.1) is uniformly asymptotically stable. If $u(s) \to \infty$ as $s \to \infty$, then the solution $x = 0$ is also a global attractor for (2.1) so that every solution $x(\sigma, \phi)$ of (2.1) satisifes $x_t(\sigma, \phi) \to 0$ as $t \to \infty$. We shall investigate (1.1) for $p \equiv 0$, $p \not\equiv 0$ respectively in the equivalent forms

$$\begin{aligned}
\dot{x}(t) &= y(t)\\
\dot{y}(t) &= z(t)\\
\dot{z}(t) &= w(t)\\
\dot{w}(t) &= -w(t)f(z(t)) - a_2 z(t) - g(y(t)) - a_4 x(t) +\\
&\beta_2 \int_{-h}^{0} w(t+\theta)d\theta + \beta_4 \int_{-h}^{0} y(t+\theta)d\theta + \int_{-h}^{0} g'(y(t+\theta))z(t+\theta)d\theta
\end{aligned} \tag{2.5}$$

and

$$\begin{aligned}
\dot{x}(t) &= y(t)\\
\dot{y}(t) &= z(t)\\
\dot{z}(t) &= w(t)\\
\dot{w}(t) &= -w(t)f(z(t)) - a_2 z(t) - g(y(t)) - a_4 x(t) +\\
&\beta_2 \int_{-h}^{0} w(t+\theta)d\theta + \beta_4 \int_{-h}^{0} y(t+\theta)d\theta + \int_{-h}^{0} g'(y(t+\theta))z(t+\theta)d\theta + p(t)
\end{aligned} \tag{2.6}$$

where $a_2 = \alpha_2 + \beta_2$, $a_4 = \alpha_4 + \beta_4$.

3. STATEMENT OF RESULT.

THEOREM 3.1. Assume that

(i) the constants

$a_2 > 0$, $a_4 > 0$ and $0 < a_1, a_3, c_0, M \leq \infty$

(ii) $f(\xi) \geq a_1 > 0$ for all ξ, and $g(\xi)/_{\xi} \geq a_3 > 0$ for all $\xi \neq 0$.

$$[a_1 a_2 - g'(\xi)]a_3 - a_1 a_4 f(z(t)) \geq c_0 > 0 \text{ for all } \xi,\ z(t). \tag{3.1}$$

(iii) $g(0) = 0 \quad |g'(\eta)| \leq M$ for all η, and

$g'(\xi) - g(\xi)/_{\xi} \leq \lambda_1$ for all $\xi \neq 0$ where λ_1 is such that

(iv) $[\frac{1}{z(t)} \int_0^{z(t)} f(\xi)d\xi] - f(z(t)) \leq \lambda_2$ for all $z(t) \neq 0$

where

$$\lambda_2 < \frac{2c_0}{a_1^2 a_3}. \tag{3.3}$$

Furthermore,

(v) if $q > 1$, $\beta = \max\ [\beta_2, \beta_4, M]$, $d = \max\ [1, d_1, d_2]$ where

$$d_1 = \varepsilon + 1/a_1;\ d_2 = \varepsilon + a_4/_{a_3} \tag{3.4}$$

and where $\varepsilon > 0$ is defined by

$$\varepsilon = \min \left[\frac{a_3}{4a_4 d_0} \left(\frac{2a_4 c_0}{a_1 a_3^2} - \lambda_1\right), \frac{a_1}{4d_0} \left(\frac{2c_0}{a_1^2 a_3} - \lambda_2\right), \frac{c_0}{2a_1 a_3 d_0} \right] \tag{3.5}$$

with c_0, $d_0 = d_0(a_1, a_2, a_3, a_4)$ positive constants, λ_1, λ_2 nonnegative constants, and with ρ defined by

$$\rho = \min\ [\frac{1}{3} a_3 \varepsilon, \frac{1}{3} a_1 \varepsilon, \frac{c_0}{6a_1 a_3}], \text{ then the condition } \beta dqh < \rho. \tag{3.6}$$

holds and the trivial solution of (2.6) is uniformly asymptotically stable. Observe that since $a_1 > 0$, $a_2 > 0$, $a_3 > 0$, $a_4 > 0$, $c_0 > 0$, $d_0 > 0$, by (3.2) and (3.3), ε is positive. Consider the special case of (1.1) namely

$$\dddot{x}(t) + a_1 \ddot{x}(t) + \alpha_2 \dot{x}(t) + \beta_2 \dot{x}(t-h) + a_3 \dot{x}(t-h) + \alpha_4 x(t) + \beta_4 x(t-h) = 0 \tag{3.7}$$

where $a_1, \alpha_2, \beta_2, a_3, \alpha_4, \beta_4$ are constants. Then condition (iii) and (iv) are fulfilled trivially with $\lambda_1 = \lambda_2 = 0$. Conditions (i) and (ii) reduce to

$a_1 > 0$, $a_2 = (\alpha_2 + \beta_2) > 0$, $a_3 > 0$, $a_4 = (\alpha_4 + \beta_4) > 0$, $(a_1 a_2 - a_3) a_3 - a_1 a_4^2 > c_0 > 0$.

If we use (3.4) we find that

$$a_2 - d_1 g'(\xi) - d_2 f(z(t)) = a_2 - \frac{a_3}{a_1} - \frac{a_1 a_4}{a_3} - (a_1 + a_3)\ \varepsilon > \frac{c_0}{a_1 a_3} - (a_1 + a_3)\ \varepsilon.$$

We can therefore choose $d_0 = (a_1 + a_3)$ so that $\varepsilon = \frac{c_0}{2a_1 a_3 (a_1 + a_3)}$.

Hypothesis (v) now becomes

$$\beta dqh < \min\ [\frac{c_0}{6a_1(a_1 + a_3)}, \frac{c_0}{6a_3(a_1 + a_3)}]$$

where $\beta = \max\ [\beta_2, \beta_4, a_3]$,

$d = \max\ [1, d_1, d_2]$, and

$$d_1 = \varepsilon + \frac{1}{a_1},\ d_2 = \varepsilon + \frac{a_4}{a_3}.$$

Therefore the sufficient conditions for all solution of (3.7) to be uniformly asymptotically stable are

(i) the Routh-Hurwitz Criteria

$$a_1 > 0,\ a_2 > 0,\ a_3 > 0$$
$$a_1 a_2 - a_3 > 0,\ a_4 > 0$$
$$(a_1 a_2 - a_3)\, a_3 - a_1^2\, a_4 \geq c_0 > 0$$

(ii) $q > 1$

$$\beta d q h < \min\left[\frac{c_0}{6a_1(a_1 + a_3)}, \frac{c_0}{6a_3(a_1 + a_3)}\right].$$

Hence all roots of the equation

$$\lambda^4 + a_1\lambda^3 + \alpha_2\lambda^2 + \beta_2 e^{-\lambda h}\lambda^2 + a_3\lambda e^{-\lambda h} + \alpha_4 + \beta_4\, e^{-\lambda h} = 0 \qquad (3.8)$$

will have negative real parts if conditions (i) and (ii) hold. If $p \neq 0$, we establish:

Theorem 3.2. If the conditions in the hypotheses (i) - (v) of theorem 3.1 hold and if further

$$|p(t)| \leq m \qquad (3.9)$$

for some $m > 0$ and for all $t \geq \sigma$, then the solutions of (2.6) are uniformly bounded.

4. THE FUNCTION $V = V(x(t), y(t), z(t), w(t))$

Define the Lyapunov function $V = V(x(t), y(t), z(t), w(t))$ by

$$\begin{aligned} 2V &= a_4 d_2\, x^2(t) + (a_2 d_2 - a_4 d_1) y^2(t) + 2\int_0^{y(t)} g(\eta)d\eta + \\ &+ (a_2 d_1 - d_2) z^2(t) + 2d_2\, y(t)w(t) + 2\int_0^{z(t)} \xi f(\xi)d\xi + \\ &+ d_1 w^2(t) + 2a_4 x(t)y(t) + 2a_4 d_1\, x(t)z(t) + 2z(t)w(t) \\ &+ 2d_2\, y(t)\int_0^{z(t)} f(\xi)d\xi + 2d_1 z(t)g(y(t)). \end{aligned} \qquad (4.1)$$

where $d_1 = \varepsilon + \frac{1}{a_1}$ and $d_2 = \varepsilon + \frac{a_4}{a_3}$ with ε defined by (3.5). The proofs of Theorems 3.1 and 3.2 rest on the function V defined by (4.1) and which was utilized by Ezeilo in [7].

LEMMA 4.1. Given the hypotheses (i) - (iv) of Theorem 3.1, there are continuous nondecreasing functions $u,v:[0,\infty)\to[0,\infty)$, $u(s)$, $v(s)$ positive for $s > 0$ with $u(0)=v(0)=0$, such that $u(|x|) \leq V(x(t), y(t), z(t), w(t)) \leq v(|x|)$.

PROOF. Take

$$\varepsilon = \varepsilon_1 \leq \min\left[\frac{a_3}{4a_4 d_0}\left(\frac{2a_4 c_0}{a_1 a_3^2} - \lambda_1\right), \frac{a_1}{4d_0}\left(\frac{2c_0}{a_1^2 a_3} - \lambda_2\right)\right]. \qquad (4.2)$$

Then, by the analysis in [7], $V(0,0,0,0) = 0$ and there exist constants $B_i > 0$ $(i = 1,2,3,4)$ depending on $\varepsilon, a_1, a_2, a_3, a_4, \lambda_1, \lambda_2$ and c_0 such that

$$V(x(t),y(t),z(t),w(t)) \geq B_5\,[x^2(t)+y^2(t)+z^2(t)+w^2(t)] \tag{4.3}$$

for all x(t), y(t), z(t), w(t) where $B_5 = \min B_i$ $(i = 1,2,3,4)$ provided ε is fixed by (4.2).

Now take $B_5\,[x^2(t) + y^2(t) + z^2(t) + w^2(t)]$ to be $u(|x|)$. It now remains to produce a $v(|x|)$. From relation (4.1)

$$\begin{aligned} 2V \quad &\leq a_4\, d_2\, x^2(t) + d_1 w^2(t) + 2a_4|x(t)y(t)| \\ &+ 2a_4 d_1|x(t)z(t)| + 2d_2\,|y(t)w(t)| + 2\,|z(t)w(t)| \\ &+ |a_2 d_2 - a_4 d_1)|\, y^2(t) + |a_2 d_1 - d_2)|\, z^2(t) \\ &+ \int_0^{y(t)} g(\eta)d\eta + 2\int_0^{z(t)} \xi f(\xi)d\xi + 2d_2 y(t)\int_0^{z(t)} f(\xi)d\xi \\ &+ 2d_1\, z(t)g(y(t)). \end{aligned} \tag{4.4}$$

Now from (3.1) of hypotheses (ii) Theorem 3.1, $g'(y(t)) < a_1 a_2$ so that $g(y(t)) < a_1 a_2\,|y(t)|$; and $f(z(t)) < a_2 a_3/a_4$. Therefore,

$$2\int_0^{y(t)} g(\eta)d\eta \leq a_1 a_2\, y^2(t), \quad 2\int_0^{z(t)} \xi f(\xi)d\xi \leq \frac{a_2 a_3}{a_4}\, z^2(t),$$

$$2d2\ y(t)\int_0^{z(t)} f(\xi)d\xi \leq 2d_2\,|y(t)|\,|z(t)|\,\frac{a_2 a_3}{a_4}, \text{ and}$$

$$2d_1\, z(t)\, g(y(t)) \leq 2d_1 a_1 a_2\,|z(t)|\,|y(t)|.$$

Substituting these estimates into (4.4) we have,

$$\begin{aligned} 2V &\leq a_4 d_2\, x^2(t) + d_1 w^2(t) + 2a_4\,|x(t)y(t)| + 2a_4 d_1\,|x(t)z(t)| \\ &+ 2d_2\,|y(t)w(t)| + 2\,|z(t)w(t)| + |(a_2 d_2 - a_4 d_1)|\, y^2(t) + \\ &+ |(a_2 d_1 - d_2)|\, z^2(t) + a_1 a_2\, y^2(t) + (a_2 a_3 / a_4)\, z^2(t) + \\ &+ (2a_2 a_3 d_2/a_4)\,|y(t)z(t)| + 2a_1 a_2 d_1\,|z(t)y(t)|. \end{aligned}$$

using the inequality

$2|ab| \leq a^2 + b^2$, we have

$$\begin{aligned} 2V \quad &\leq a_4 d_2\, x^2(t) + d_1 w^2(t) + a_1 a_2 y^2(t) + \ell y^2(t) + m z^2(t) \\ &+ a_4(x^2(t) + y^2(t)) + a_4 d_1(x^2(t) + z^2(t)) + d_2(y^2(t) + w^2(t)) \\ &+ (z^2(t) + w^2(t)) + a_1 a_2 d_1\,(z^2(t) + y^2(t)) + a_2 a_3/a_4\, z^2(t) \\ &+ \frac{a_2 a_3 d_2}{a_4}(y^2(t) + w^2(t)). \end{aligned} \tag{4.5}$$

where

$\ell = |a_2 d_2 - a_4 d_1|$ and $m = |a_2 d_1 - d_2|$. On gathering terms,

$$V < B_6 x^2(t) + B_7 y^2(t) + B_8 z^2(t) + B_9 w^2(t), \text{ where}$$

$$B_6 = (a_4 d_2 + a_4 + a_4 d_1),$$

$$B_7 = (a_1 a_2 + \ell + a_4 + d_2 + a_1 a_2 d_1 + a_2 a_3 d_2/a_4)$$

$$B_8 = (m + a_4 d_1 + a + a_1 a_2 d_1 + a_2 a_3/a_4 + a_2 a_3 d_2/a_4) \text{ and}$$

$$B_9 = (1 + d_1 + d_2).$$

Let $B_{10} = \max B_i$ (i = 6,7,8,9. Then

$$V(x(t), y(t), z(t), w(t)) < B_{10} [x^2(t) + y^2(t) + z^2(t) + w^2(t)] \quad (4.6)$$

Take $v(|x|) = B_{10} [x^2(t) + y^2(t) + y^2(t) + z^2(t) + w^2(t)]$. Clearly $u(0) = v(0) = 0$,

$$u(s) > 0, \ v(s) > 0 \text{ for } s = x^2(t) + y^2(t) + z^2(t) + w^2(t) > 0$$

This proves lemma 4.1.

LEMMA 4.2. Subject to hypotheses (i) - (iv) of Theorem 3.1, there are continuous nondecreasing functions $J(s) > s$ for $s > 0$ and a function $w(s)$ with $w(s) > 0$, $s \neq 0$ such that

$$V(t,\phi(0)) < -w(|\phi(0)|) \text{ if } V(t+\theta,\phi(\theta)) < J(V(t,\phi(0))), \ \theta \quad [-h,0].$$

PROOF OF LEMMA 4.2. The proof depends on hypotheses (v) and (vi) and on the three ineqalities arising from hypotheses (i) - (iv) of Theorem 3.1, namely:

$$d_1 - {}^1/_{f(z(t)} > \varepsilon, \quad (4.7)$$

$$d_2 - \frac{a_4 y(t)}{g(y(t))} > \varepsilon, \quad (4.8)$$

for all $z(t) \neq 0$, $y(t) \neq 0$

and

$$a_2 - d_1 g'(y(t)) - d_2 f(z(t) > \frac{c_0}{a_1 a_3} - \varepsilon d_0 \text{ for all } y(t), z(t) \quad (4.9)$$

where d_0 is a constant that depends only on a_1, a_2, a_3, a_4. Now, by (3.4), $d_1 - 1/a_1 = \varepsilon$ and since by hypothesis (ii) of Theorem 3.1, $f(z(t)) > a_1 > 0$, (4.7) follows. Also by (3.4), $d_2 - {}^{a_4}/_{a_3} = \varepsilon$ and since by hypothesis (ii) again $y/g(y) < {}^1/a_3$, (4.8) is immediate.

Using (3.4) we have

$$a_2 - d_1 g'(y(t)) - d_2 f(z(t)) = a_2 - (\varepsilon + 1/a_1) g'(y(t)) - (\varepsilon + \frac{a_4}{a_3}) f(z(t))$$

$$= \frac{1}{a_1 a_3} [a_1 a_2 - g'(y(t)) a_3 - a_1 a_4 f(z(t))] - \varepsilon[g'(y(t)) + f(z(t))].$$

Therefore by (3.1), $a_2 - d_1 g'(y(t)) - d_2 f(z(t)) > \frac{c_0}{a_1 a_3} - \varepsilon[g'(y(t)) + f(z(t))]$.

Since $g'(y(t)) < a_1 a_2$ and $f(z(t)) < a_2 a_3/a_4$ for all $y(t)$, $z(t)$,

$$a_2 - d_1 g'(y(t)) - d_2 f(z(t)) > \frac{c_0}{a_1 a_2} (a_1 a_3 + \frac{a_2 a_3}{a_4} \varepsilon) \text{ for all } y(t), z(t)$$

and this establishes (4.9). Now define a function G of y(t) by

$$G(y(t)) = \begin{cases} \frac{g(y(t))}{y(t)}, & \text{if } y(t) \neq 0 \\ g'(0), & \text{if } y(t) = 0. \end{cases} \tag{4.10}$$

Also, let

$$F(z(t)) = \int_0^{z(t)} f(\xi) d\xi \,. \tag{4.11}$$

Observe that the conditions g(0) = 0 and F(0) = 0 imply resepectively that

$$\begin{aligned} G(y(t)) &= g'(\theta_1 y(t)) \\ F(z(t)) &= z(t) f(\theta_2 z(t)) \end{aligned} \tag{4.12}$$

where $0 < \theta_1 < 1$ $(i = 1,2)$.
Given any solution (x,y,z,w) of (2.5)

$$\begin{aligned} 2\dot{V} &= y(t)[2a_4 d_2 x(t) + 2a_4 y(t) + 2a_4 d_1 z(t)] + z(t)[2a_4 \, x(t) \\ &+ 2d_2 w(t) + 2Ky(t) + 2d_1 z(t) g'(y(t)) + 2g(y(t)) \\ &+ 2d_2 \int_0^{z(t)} f(\xi) d\xi] + w(t) \, [2a_4 d_1 \, x(t) + 2w(t) + 2cz(t) \\ &+ 2d_1 \, g(y(t)) + 2z(t) f(z(t)) + 2d_2 y(t) f(z(t)))] \\ &+ [2w(t) d_1 + 2d_2 y(t) + 2z(t)] \, [-w(t) f(z(t)) - a_2 z(t) \\ &-g(y(t)) - a_4 \, x(t)] + [2w(t) d_1 + 2d_2 \, y(t) + 2z(t)] \\ &\cdot [\beta_2 \int_{-h}^{0} w(t+\theta) d\theta + \beta_4 \int_{-h}^{0} y(t+\theta) d\theta + \int_{-h}^{0} g'(y(t+\theta)) z(t+\theta) d\theta] \end{aligned}$$

where $K = (a_2 d_2 - a_1 d_1)$ and $c = (a_2 d_1 - d_2)$. On simplication, the above relation reduces to

$$\begin{aligned} 2\dot{V} &= 2a_4 y^2(t) + 2d_1 z^2(t) g'(y(t)) + 2d_2 \, z(t) \int_0^{z(t)} f(\xi) d\xi \\ &+ 2w^2(t) - 2d_1 w^2(t) f(z(t)) - 2d_2 y(t) g(y(t)) - 2a_3 z^2(t)) \\ &+ [2d_1 w(t) + 2d_2 y(t) + 2z(t)] \, [\beta_2 \int_{-h}^{0} w(t+\theta) d\theta \\ &+ \beta_4 \int_{-h}^{0} y(t+\theta) d\theta + \int_{-h}^{0} g'(t+\theta)) z(t+\theta) d\theta], \end{aligned}$$

and using (4.11)

$$\dot{V} = -[d_2\, y(t)g(y(t)) - a_4 y^2(t)] - [(a_2 - d_1 g'(y(t))z^2(t) - d_2 z(t)\, F(z(t))] - [d_1 f(z(t))-1]\, w^2(t) + [d_1 w(t) + d_2 y(t) + z(t)]\, [\beta_2 \int_{-h}^{0} w(t+\theta)d\theta + \beta_4 \int_{-h}^{0} y(t+\theta)d\theta + \int_{-h}^{0} g'(y(t+\theta))z(t+\theta)d\theta].$$

Now, with G defined by (4.10)

$$[d_1 y(t)g(y(t)) - a_4 y^2(t)] = y^2(t)G(y(t))\, [d_2 - \frac{a_4}{G(y(t))}] = T_1 \quad \text{say.}$$

Since $f(z(t)) \neq 0$, $[d_1 f(z(t)-1]\, w^2(t)$ can be rewritten as

$$f(z(t))\, [d_1 - 1/_{f(z(t))}]\, w^2(t) = T_3, \text{ say}$$

Denoting $[(a_2 - d_1 g'(y(t)))z^2(t) - d_2 z(t) - d_2 z(t)F(z(t))]$ by T_2, we have

$$\dot{V} = -T_1 - T_2 - T_3 + [d_1 w(t) + d_2 y(t) + z(t)]\, [\beta_2 \int_{-h}^{0} w(t+\theta)d\theta + \beta_4 \int_{-h}^{0} y(t+\theta)d\theta + \int_{-h}^{0} g'(y(t+\theta))z(t+\theta)d\theta]$$ and using

hypothesis (iii) of Theorem 3.1, we obtain the inequality

$$\dot{V} \le -T_1 - T_2 - T_3 + [d|w(t)|+|y(t)|+|z(t)|]\, [\beta_2 \int_{-h}^{0} |w(t+\theta)|d\theta + \beta_4 \int_{-h}^{0} |z(t+\theta)|d\theta] + M \int_{-h}^{0} |z(t+\theta)|d\theta], \tag{4.13}$$

where $d = \max\,(1, d_1, d_2)$.

Choose $J(s) = q^2 s$ for some $q > 1$. Then

$$J(V) = q^2 V,\ q > 1. \tag{4.14}$$

Also assume the following:

$$|x(t+\theta)| < qA|x(t)|,\ |y(t+\theta)| < qA|y(t)|,\ |z(t+\theta)| < qA|z(t)|$$

and

$$|w(t+\theta)| < qA|w(t)| \tag{4.15}$$

for $q > 1$, θ $[-h,0]$, where $A = (B_5 / B_{10})^{1/2}$.

Then the inequality (4.13) is strengthened to

$$\dot{V} < -T_1 - T_2 - T_3 + \beta dqA\,[|w(t)| + |y(t)| + |z(t)|]\,[\int_{-h}^{0} |w(t)|d\theta + \int_{-h}^{0} |y(t)|d\theta + \int_{-h}^{0} |z(t)|d\theta]$$

$$\leq -T_1 - T_2 - T_3 + \beta dqh\,(|w(t)|+|y(t)|+z(t)|)^2$$

since $A \leq 1$ and $\beta = \max\,[\beta_2, \beta_4, M]$.

Noting that by relation (4.8) and hypothesis (ii) of Theorem 3.1, $T_1 \geq a_3\,\varepsilon\, y^2(t)$, and also by hypothesis (ii) of the same Theorem $T_3 \geq a_1\,\varepsilon\, w^2(t)$ then by (4.9) and (4.12)

$$T_2 \geq (\frac{c_0}{a_1a_3} - \varepsilon\, d_0)\, z^2(t) \geq 1/2\, (\frac{c_0}{a_1a_2})z^2(t)$$

provided that

$$\varepsilon = \varepsilon_2 \leq 1/2\, (\frac{c_0}{a_1a_3d_0}), \tag{4.16}$$

we have subject to (4.15)

$$\dot{V} \leq -a_3\,\varepsilon\, y^2(t) - a_1\varepsilon w^2(t) - 1/2\,(\frac{c_0}{a_1a_3})z^2(t) + \beta dhq\,(|y(t)|+|z(t)|+w(t)|)^2.$$

Since $(|y(t)|+|z(t)| + |w(t)|)^2 \leq 3[y^2(t) + z^2(t) + w^2(t)]$,

$$V \quad -a_3\,\varepsilon\, y^2(t) - a_1\varepsilon w^2(t) - 1/2\,(\frac{c_0}{a_1a_3})z^2(t) + 3\beta dhq\,[y^2(t) + z^2(t) + w^2(t)].$$

On gathering terms and subject to (4.15),

$$V = -(a_3\,\varepsilon - 3\beta dqh)\, y^2(t) - (\frac{c_0}{2a_1a_3} - 3\beta dqh)\, z^2(t) - (a_1\,\varepsilon - 3\beta dqh)\, w^2(t),$$ provided ε_2 is fixed by (4.16).

Therefore for ε_2 fixed by (4.16) and by condition (3.6) of Theorem 3.1 there are constants $B_j > 0$ (j=11,12,13) such that subject to assumption (4.15)

$$V(t,\phi(0)) \leq -\,[B_{11}y^2(t) + B_{12}z^2(t) + B_{13}w^2(t)], \tag{4.17}$$

where $B_{11} = (a_3\,\varepsilon\, 3\beta dhq)$, $B_{12} = (\frac{c_0}{2a_1a_3} - 3\beta dhq)$ and $B_{13} = (a_1\,\varepsilon - 3\beta dhq)$.

Taking $B_{14} = \min B_j$ (j = 11,12,13), the inequality (4.17) is sharpened to $V(t,\phi(0)) \leq B_{14}\,[y^2(t) + z^2(t) + w^2(t)]$ if assumption (4.15) holds. Using the relations (4.1), (4.3) and (4.6) observe that

$$B_5[x^2(t) + y^2(t) + z^2(t) + w^2(t)] \leq V(t,\phi(0)) \leq B_{10}\,[x^2(t) + y^2(t) + z^2(t) + w^2(t)], \tag{4.18}$$

so that

$$B_5[x^2(t+\theta) + y^2(t+\theta) + z^2(t+\theta) + w^2(t+\theta) + w^2(t+\theta)] \leq V(t+\theta,\phi(\theta)) \leq B_{10}[x^2(t+\theta) + y^2(t+\theta) + z^2(t+\theta) + w^2(t+\theta)],\ \theta \in [-h,0] \tag{4.19}$$

Now if (4.15) holds, then

$$x^2(t+\theta) < q^2A^2x^2(t);\ y^2(t+\theta) < q^2A^2y^2(t);$$

$$z^2(t+\theta) < q^2A^2x^2(t) \text{ and } w^2(t+\theta) < q^2A^2y^2(t),$$

so that

$$B_5[x^2(t+\theta) + y^2(t+\theta) + z^2(t+\theta) + w^2\ (t+\theta)] < q^2B\ \ [x^2(t) + y^2(t) + z^2(t) + w^2(t)]. \tag{4.20}$$

If (4.20) holds then by (4.19)

$$V(t+\theta(\theta)) < B_5q^2[x^2(t) + y^2(t) + z^2(t) + w^2(t)]$$

and by (4.18) since

$$B_5q^2[x^2(t) + y^2(t) + z^2(t) + w^2(t)] \leq q^2\ V(t,\phi(0))$$

we have

$V(t+\theta,\ \phi(\theta)) < q^2V(t,\phi(0))$, and by definition (4.14)

$V(t+\theta,\ \phi(\theta)) < q^2J(V(t,\phi(0)))$. Thus, for ε_2 fixed by (4.16),

taking $w(|\phi(0)|) = B_{14}[y^2(t) + z^2(t) + w^2(t)]$, we have

$V(t,\phi(0) \leq -W(|\phi(0)|)$ if

$V(t+\theta,\ \phi(\theta)) < J(V(t,\phi(0)))$ where $\theta\quad [-h,0]$.

This proves the lemma.

5. PROOF OF THE MAIN THEOREMS.

LEMMA 5.1. Subject to the conditions of Theorem 3.2,

$$V_{(2.6)} \leq -D < 0$$

provided

$$y^2(t) + z^2(t) + w^2(t) \geq R > 0, \; D = D(m,d,B_0) > 0$$

PROOF OF LEMMA 5.1. Again, set $V(t) = V(x(t),y(t),z(t),w(t))$. Then given any solution (x,y,z,w) of (2.6), by the methods of lemma (4.2), we obtain

$$\leq - B_0[(y^2(t)+z^2(t)+w^2(t)] + d(|y(t)|+|z(t)+|w(t)|)|p(t)| \tag{5.1}$$

$$\leq - B_0(y^2(t) + z^2(t) + w^2(t)) + md(|y(t)|+|z(t)|+|w(t)|)$$

where

$$B_0 = \min B_j \qquad j = 11,12,13 \; .$$

Letting $q(t) = \max(|y(t)|, |z(t)|, |w(t)|)$, the inequality is sharpened to

$$V = -B_0(y^2(t) + z^2(t) + w^2(t)) + 3md\, q(t). \tag{5.2}$$

If $q(t) = |y(t)|$, then at least

$$V \quad -B_0(y^2(t) + z^2(t) + w^2(t) + 2md|y(t)|$$

$$\leq -B_0y^2(t) + 3md|y(t)|$$

$$\leq - \frac{1}{2} B_0y^2(t), \text{ provided } |y(t)| \geq \frac{6md}{B_0} = D_0 \; .$$

So,

$$V \quad \leq - \frac{1}{2} \; B_0D_0{}^2, \text{ provided } |y(t)| \geq D_0 = D_0(m,d,B_0).$$

Similar conclusions are true for

$$q(t) = |z(t)| \text{ and } q(t) = |w(t)| \; .$$

Hence

$$V \leq -D < 0 \tag{5.3}$$

provided

$$y^2(t) + z^2(t) + w^2(t) \geq R, \text{ for some}$$

$$D == D(B_0,m,d) > 0 \text{ and some } R > 0 \; .$$

PROOF OF THEOREM 3.1. By lemma 4.1, for $\varepsilon = \varepsilon_1$ fixed by (4.2) there are:

(i) continuous nondecreasing functions
$u, v: [0,\infty) \to [0,\infty)$ given by
$u(s) = B_5[x^2(t) + y^2(t) + z^2(t) + w^2(t)]$.
$v(s) = B_{10}[x^2(t) + y^2(t) + z^2(t) + w^2(t)]$ with the required properties,

(ii) a continuous function $V: E \times E^4 \to E$ defined by (4.1) such that

$$u(|x|) \leq V(t,x) \leq v(|x|), \quad t \in E, \ x \in E^n.$$

By lemma 4.2, for $\varepsilon = \varepsilon_2$ fixed by (4.16) there are:

(iii) a function $w: [0,\infty) \to [0,\infty)$ continuous and nondecreasing such that

$$w(s) = w(|\phi(0)|) > 0 \text{ if } s = |\phi(0)| > 0, \text{ and}$$

(iv) a continuous nondecreasing function $J(s) > s$ such that

$$\dot{V}(t,\phi(0)) \leq -w(|\phi(0)|) \text{ if } V(t+\theta,\phi(\theta)) < J(V(t,\phi(0))), \text{ for } \phi \in [-h,0].$$

Then, from (i), (ii), (iii) and (iv) of this section, taking $\varepsilon = \min(\varepsilon_1, \varepsilon_2)$, Theorem 3.1 follows from proposition 2.2. of section 2.

Also, since $B_5[x^2(t) + y^2(t) + z^2(t) + w^2(t)] \to \infty$ as $x^2(t) + y^2(t) + z^2(t) + w^2(t) \to \infty$, the solution $x = 0$ of (1.1) is a global attractor for (1.1) so that the solution (x,y,z,w) satisfies $x_t^2 + y_t^2 + z_t^2 + w_t^2 \to 0$ as $t \to \infty$.

PROOF OF THEOREM 3.2. Use is made of lemmas 4.1, and 5.1 and Theorem 2.1 on p. 105 of [5]. Noting that $u(|x|) = B_5(x^2(t) + y^2(t) + z^2(t) + w^2(t))$ and $|x| = x^2(t) + y^2(t) + z^2(t) + w^2(t)$ clearly, $u(|x|) \longrightarrow \infty$ as $|x| \longrightarrow \infty$, and since by lemma 5.1, for any solution of (2.6) there is some $D > 0$ satisfying (5.3), the uniorm boundedness requirements of Theorem 2.1 of [5] are met and hence our uniform boundedness result follows.

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