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SUPERHARMONIC FUNCTIONS AND BOUNDED POINT EVALUATIONS

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ABSTRACT. Let E be a compact subset of the complex plane $\mathbb{C}$. We denote by $R_o(E)$ the algebra consisting of the (restrictions to E of) rational functions with poles off E. Let m denote the 2-dimensional Lebesgue measure. Let $R^2(E)$ be the closure of $R_o(E)$ in $L^2(E,dm)$.

In this paper we consider points $x \in E$ such that "evaluation at x" extends from $R_o(E)$ to a continuous linear functional on $R^2(E)$. These points are bounded point evaluations on $R^2(E)$. Hedberg, Fernström and Polking used capacity to identify bounded point evaluations. We use their results to show that the existence of a bounded point evaluation $x \in E$ is equivalent to the existence of a superharmonic function u(y) that grows sufficiently fast as y approaches x through the complement of E.

KEY WORDS AND PHRASES. Rational functions, compact set, L^p -spaces, bounded point evaluation, superharmonic function, balayage, Borel measure, Green function, Green potential, fine topology, thin, potential theoretic capacity, polar set.
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1. INTRODUCTION

Subharmonic and superharmonic functions have been useful in solving the Dirichlet problem: Given an open set $S \subset \mathbb{C}$ with compact closure and a real-valued, continuous function h defined on S, find a function v harmonic in S and continuous on the closure of S such that

$$v(x) = h(x) \text{ for each } x \in \partial S.$$

O. Perron showed that for many sets S one can get a solution by taking the supremum of the family of subharmonic functions on S whose boundary values are not greater than h(x). A point $x \in \partial S$ is an irregular boundary point for S if and only if there is a superharmonic function u on a neighborhood D of x such that

$$u(x) < \lim_{\substack{y \to x \\ y \in (D\setminus S)\setminus\{x\}}} u(y) = +\infty.$$

We will be particularly interested in those superharmonic functions that are the Green potentials of measures supported on compact subsets of $\mathbb{C}$. Using these measures we will define a capacity that is equivalent to the Wiener capacity. Hedberg, Polking, and Fernström have shown that this capacity is helpful in identifying bounded point evaluations. For compact sets $E \subset \mathbb{C}$ we will relate the existence of a bounded point evaluation x on $R^2(E)$ to the existence of a superharmonic function in a neighborhood of x. We will prove that x is a bounded point evaluation on $R^2(E)$ if and only if there is a superharmonic function u such that $u(x) < \infty$, and

$$u(y) \geq |y|^{-2} \, |\log |y||^{-1} \qquad y \in (D\backslash E)\backslash\{x\}$$

where D is a neighborhood of x.

2. SUPERHARMONIC FUNCTIONS AND BALAYAGE.

One way to define a superharmonic function u is to say that u is superharmonic if and only if -u is subharmonic. To be more specific let $S \subset \mathbb{C}$ be open and let u(x) be a function defined for $x \in S$.

DEFINITION 2.1. A function u(x) is called superharmonic on S if for $x \in S$

(i) $u(x) \leq +\infty$, and $u(x) \not\equiv +\infty$,

(ii) u is lower semi-continuous, and

(iii) $u(x) \geq \frac{1}{2\pi} \int_0^{2\pi} u(x + re^{i\theta})\, d\theta$ whenever the disk with radius $r > 0$ and center at x is contained in S.

EXAMPLES. If f(z) is analytic in S and $\lambda > 0$, then $-\log|f|$ and $-|f|^{\lambda}$ are superharmonic in S.

Although superharmonic functions need not be continuous, one can define a new topology on $\mathbb{C}$ in which all superharmonic functions are continuous. The smallest such topology is called the fine topology. A set $E \subset \mathbb{C}$ is <u>thin</u> at x if x is not a fine limit point of E. The following theorem is part of Brelot's contribution to potential theory. For the proof see[1, p. 210].

THEOREM 2.1. A set E is thin at a limit point x of E if and only if there is a superharmonic function u on a neighborhood D of x such that

$$u(x) < \lim_{\substack{y \to x \\ y \in (D\backslash E)\backslash\{x\}}} u(y) = +\infty.$$

Later we will construct a montone increasing sequence $\{u_i\}$ of superharmonic functions on a set S that is open in the ordinary topology. By a lemma in [1, p.68] $\sup_{i \in I} u_i$ is either harmonic or identically ∞.

There is a way to associate with each non-negative superharmonic function u on S and each set $E \subset S$ another superharmonic function that dominates u on E and satisfies a special property. This function can be defined so that when E is compact it equals a potential with respect to the Green function of S. We begin by letting G_S be the Green function of S. Let u be a non-negative superharmonic function on S. $\mathcal{S}_S$ will denote the class of superharmonic functions on S. If $u \in \mathcal{S}_S$ is non-negative and E is any subset of S, let

$$\Phi_E^u = \{v \in \mathcal{S}_S : v \geq 0 \text{ on } S,\ v \geq u \text{ on } E\}.$$

Let $R_E^u = \inf\{v : v \in \Phi_E^u\}$.

The function R_E^u satisfies (i) and (iii) of the definition of superharmonic. R_E^u may not be lower semi-continuous. By defining

$$\hat{R}_E^u(x) = \liminf_{y \to x} R_E^u(y)$$

we get a function that is superharmonic on S. $\hat{R}_E^u(x)$ is called the balayage of u relative to E in S. When $E \subset S$ is compact, the following fact about $\hat{R}_E^u(x)$ will be useful [1, p.135]: $\hat{R}_E^u(x)$ is a Green potential, i.e. there is a Borel measure μ on S such that

$$R_E^u(x) = \int G_S(x,y)\,d\mu(y).$$

3. POTENTIAL THEORETIC CAPACITY.

Let $S \subset \mathbb{C}$ be an open set having a Green function G_S. Let $E \subset \mathbb{C}$ be compact and let u be the function identically 1 on S. Then by [1, p.138] $\hat{R}_E^1(x)$ is a superharmonic function on S that is the potential of a measure with support in ∂E.

DEFINITION 3.1. The measure μ_E for which $\hat{R}_E^1(x) = G_S\mu_E$ is called the capacitary distribution of E.

DEFINITION 3.2. The capacity of E (relative to the set S) is defined to be $C(E) = \mu_E(E)$ with $C(\phi) = 0$.

The C capacity is equal to the Wiener capacity which we will denote by C_2. For more information on why $C(E) = C_2(E)$ see [1, Lemma 7.19] and [2, Chap. II]. Also, in [3, p. 160] it is shown that if E is a continuum with diameter d, there are positive constants K_1 and K_2 depending only on the distance from E to ∂S, such that

$$K_1 / (\log 1/d)^{1/2} < C_2(E) < K_2 / (\log 1/d)^{1/2}.$$

There is a C_2 capacity series that converges at the points where the complement of a set $E \subset \mathbb{C}$ is thin. To state this as a theorem we will need still more notation. Let $j < k$ be positive integers. Define

$$A(j,k) = \{z \in \mathbb{C}: 2^{-k} < |z| < 2^{-j}\}$$

$$\text{and } A[j,k] = \{z \in \mathbb{C}: 2^{-k} \le |z| \le 2^{-j}\}$$

Now let $A_n = A[n, n+1]$. The next theorem is due to Wiener [2]. It is a statement about thinness at an arbitrary point $x \in \mathbb{C}$. . We assume after a possible translation that x=0. The set E need not be compact.

THEOREM 3.1. (Wiener) Let $E \subset \mathbb{C}$. Then the complement of E is thin at 0 if and only if

$$\sum_{n=1}^{\infty} nC_2(A_n \setminus E) < \infty.$$

Fernstrom and Polking used another C_2 series to identify bounded point evaluations [4]. Let $E \subset \mathbb{C}$ be compact and let $R_0(E)$ denote the algebra of rational functions with poles off E. Let m be 2-dimensional Lebesgue measure. $R^2(E)$ will denote the closure of $R_0(E)$ in the norm $L^2(dm)$.

DEFINITION 3.3. A point $x \in E$ is a bounded point evaluation (BPE) on $R^2(E)$ if there is a constant C such that

$$|f(x)| \le C\{\int_E |f|^2 dm\}^{1/2} \quad \text{for all } f \in R_0(E).$$

The next theorem applies to an arbitrary point $x \in \mathbb{C}$. We may assume after a possible translation that x=0.

THEOREM 3.2. (Hedberg, Ferström, and Polking) The point $0 \in E$ is a BPE on $R^2(E)$ if and only if
$$\sum_{n=1}^{\infty} 2^{2n} C_2(A\setminus E) < \infty .$$

The existence of a BPE at $0 \in E$ is a local property; hence it is no restriction to assume that $E \subset \{z: |z| < 1/2\} = D$. The Green function for D is
$$G_D(0,z) = -\log 2|z| \quad \text{for } z \in D\setminus \{0\}.$$

We will need several lemmas to prove our theorem. These are modified versions of lemmas which can be found with their proofs in [1, Chap. 10].

LEMMA 3.1. There is a constant b independent of j such that
$$\frac{\log 2|y-z|}{\log 2|y|} < b$$
whenever $y \in \mathbb{C}\setminus\{\zeta: 2^{-j-2} < |\zeta| < 2^{-j+1}\}$ and $z \in \{\zeta: 2^{-j-1} < |\zeta| < 2^{-j}\}$ for each positive integer $j > 3$.

PROOF. We will consider two cases.

Case 1. $|y| > 2^{-j+1}$, > 3. The absolute value of $\log 2|y-z|$ is no greater than $(j-1)\log 2$. The absolute value of $\log 2|y|$ is no less than $(j-2)\log 2$. Thus the quotient does not exceed $(j-1)/(j-2)$.

Case 2. $|y| < 2^{-j-2}$, $j > 3$. Then $\log 2|y-z|$ does not exceed $(j+1)\log 2$ in absolute value. Moreover, $\log 2|y|$ is greater in absolute value than $(j+1)\log 2$. Thus the quotient does not exceed 1. Any $b > 1$ will satisfy the statement of the lemma.

LEMMA 3.2. If S is an open set having a Green function G_S, and U is a nonempty open set having a compact closure $\bar{U} \subset S$, there is a measure μ on ∂U such that
$\mu(\partial U) = C_2(U)$, and $G_S\mu = 1$ on U.

PROOF. Let $\{U_j\}$ be an increasing sequence of open sets with compact closures $K_j = \bar{U}_j \subset U$ such that $U_j \uparrow U$. Each set K_j has a capacitary distribution which we denote by μ_j. Now $C_2(U) = \lim_{j\to\infty} C_2(K_j) = \lim_{j\to\infty} \mu_j(K_j)$. Since $C_2(\bar{U}) < \infty$, the measures μ_j are uniformly bounded. There must be a subsequence of the sequence μ_j which we can assume to be the sequence itself and a measure μ such that
$$\int_{\bar{U}} f\,d\mu_j \to \int_{\bar{U}} f\,d\mu$$
for every function f continuous on $\bar{U}$.

We claim that μ has support in ∂U. If not, there is a compact set $S \subset U$, $S \cap \partial U = \phi$, such that $\mu(S) > 0$. To get a contradiction, take a non-negative function f continuous on $\overline{U_1}$ equal to 1 on S and equal to 0 on ∂U_j for j sufficiently large.

Then $\int_{\bar{U}} f d\mu > \mu(S) > 0$. Since each μ_j is supported in ∂U_j, $\int_{\bar{U}} f\, d\mu_j = 0$ for sufficiently large j. This is contradiction.

If $x \in U$, then $x \in U_j$ for all j sufficiently large. Using continuous functions with compact support to approximate G_S, we see that $G_S\mu_j(x) \to G_S\mu(x)$ as $j \to \infty$. By the definition of a capacitary distribution $G_S\mu_j(x) = 1$ for sufficiently large j; hence $G_S\mu(x) = 1$. The proof is complete.

DEFINITION 3.4. A set $Z \subset \mathbb{C}$ is a polar set if there is an open set $U \supset Z$ and a function u superharmonic on U such that $\{z: u(z) = +\infty\} \supset Z$.

The next two lemmas will be useful in proving that a certain C_2 capacity series converges.

LEMMA 3.3. Let ν be a measure having support $F \subset D$. If $G_D > \alpha$ on F except possibly for a polar subset of F, then $\nu(F) > \alpha C_2(F)$.

For the proof see [1, p. 219].

LEMMA 3.4. If ν is a finite measure on D such that $v = G_D\nu$ is finite at 0, there is a constant α depending only on ν, such that

$$\int_{D\setminus A(j-1,\, j+2)} G_D(y,z)\,d\nu(x) < \alpha$$

for all $y \in D \cap A[j, j+1]$.

PROOF. Since $G_D(y,z) < -\log|y-z|$, we may prove the lemma by proving the inequality with G_D replaced by $-\log|y-z|$. By Lemma 3.1. there is a constant b, independent of $j > 3$, such that

$$\begin{aligned}\int_{D\setminus A(j-1,\, j+2)} -\log|y-z|\,d\nu(y) &= \int_{D\setminus A(j-1,\, j+2)} [-\log 2|y-z| + \log 2]\,d\nu(y)\\ &< -b\int_{D\setminus A(j-1,\, j+2)} \log 2|y|\,d\nu(y) + (\log 2)\nu(D)\\ &< -b\int_D \log 2|y|\,d\nu(y) + (\log 2)\nu(D)\end{aligned}$$

for all $z \in D \cap A[j, j+1]$. Since we have assumed that $-\int_D \log 2|y|\,d\nu(y)$ is finite, we can take $\alpha = -b\int_D \log 2|y|\,d\nu(y) + (\log 2)\nu(D)$.

4. THE MAIN THEOREM.

Let $E \subset \mathbb{C}$ be compact. The property of being a BPE on $R^2(E)$ is local property and is invariant under translation. In stating our theorem about an arbitrary point $x \in E$, we may therefore assume that $E \subset \{z: |z| < (1/2)\} = D$ and that $x = 0$. We will combine Theorem 3.2. with ideas of Wiener and Brelot to prove:

THEOREM 4.1. The point $0 \in E$ is a BPE on $R^2(E)$ if and only if there is a function u superharmonic in D such that $u(0) < \infty$, and

$$\underset{y \in D\setminus E}{u(y) > |y|^{-2}\,|\log|y||^{-1}}\,.$$

PROOF. Suppose that $0 \in E$ is a BPE on $R^2(E)$.
Then by Theorem 3.2.

$$\sum_{n=1}^{\infty} 2^{2n} C_2(A_n\setminus E) < \infty.$$

Let $\{\varepsilon_n\}$ be a sequence of positive numbers such that

$$\sum_{n=1}^{\infty} \varepsilon_n 2^{2n} < \infty\,.$$

For each $n > 1$ let U_n be a nonempty open subset of D containing A_n E such that $\bar{U}_n \subset D$, and the following conditions hold:

(i) $\quad U_n \cap \{\zeta: |\zeta| < \frac{1}{2^{n+2}}\} = \phi$, and

(ii) $\quad C_2(U_n) < C_2(A_n\setminus E) + \varepsilon_n$.

Then $\sum_{n=2}^{\infty} 2^{2n} C_2(U_n) < \infty$.

We will obtain the required function u as the limit of a sequence of superharmonic functions. Let G denote the Green function for D. By Lemma 3.2. there is a measure μ_n with support in ∂U_n such that

$$\mu_n(\partial U_n) = C_2(U_n), \text{ and } G\mu_n = 1 \text{ on } U_n.$$

We have

$$G\mu_n(0) = \int_{\partial U_n} G(0,z)d\mu_n(z) = \int_{\partial U_n} -\log 2|z|d\mu_n(z)$$

$$< (n+1)(\log 2)\ \mu_n(\partial U_n)$$

$$= (n+1)(\log 2)C_2(U_n).$$

For $m > 1$, define $u_m = \sum_{n=2}^{m} 2^{2n+2} ((n+1)\log 2)^{-1} G\mu_n$. By a remark in Section 2

the u_m tend to a function u that is superharmonic in D and satisfies $u(0) < \infty$. Since $G\mu_n = 1$ on $U_n \supset A_n \setminus E$,

$$u > 2^{2k+2} ((k+1)\log 2)^{-1} \text{ on } \bigcup_{n \geq k} A_n \setminus E \subset \mathbb{C} \setminus E$$

for each $k > 2$. Thus

$$\lim_{y \in D \setminus E} u(y) > |y|^{-2}|\log|y||^{-1} .$$

Now suppose there is a function u superharmonic in D such that $u(0) < \infty$, and

$$\lim_{y \in D \setminus E} u(y) > |y|^{-2}|\log|y||^{-1}$$

The function u is lower semi-continuous on D; hence we may assume by taking a smaller D if necessary that u is positive on D. Moreover, the Riesz Decomposition Theorem implies that $u = G_\mu + h$ where μ is a measure supported on D, and h is harmonic in D. Since h is bounded, we may assume that $u = G\mu$. By Theorem 3.2. it suffices to show that

$$\sum 2^{2n} C_2(A_n \setminus E) < \infty .$$

For $n > 2$ consider the open sets

$$U_n = \{z: u(z) > \frac{2^{2n}}{n\log 2}\} \supset A_n \setminus E$$

and the sets $V_n = U_n \cap (A_n \setminus E)$. Since $V_n \supset A_n \setminus E$, it suffices to show that

$$\sum_{n=2}^{\infty} 2^{2n} C_2(V_n) < \infty .$$

Let K_n be a compact subset of V_n such that

$$C_2(K_n) > C_2(V_n) - \epsilon_n .$$

Then it is sufficient to prove that

$$\sum_{n=2}^{\infty} 2^{2n} C_2(K_n) < \infty .$$

One way to prove that this series converges is to prove that

$$\sum_{n=1}^{\infty} 2^{8n+2\ell} C_2(K_{4n+\ell}) < \infty$$

for $\ell = 0$, 1, 2, and 3. We will do this for $\ell = 0$; the 3 other cases are similar.

Let K be the compact set defined by $K = \bigcup_{n \geq 2} K_{4n} \cup \{0\}$. Let $w = \hat{R}_K^u$. Since $u(0) \geq \hat{R}_K^u(0) = w(0)$, $w(0) < \infty$. Now w is the Green potential of a measure ν with support in K[1, p. 135]. We note that $\nu(D \setminus \cup K_{4n}) = 0$ because $w(0) < \infty$. For each $n \geq 2$

$$w = \int_{K_{4n}} G(\cdot,z)d\nu(z) + \int_{\bigcup_{m \neq n} K_{4m}} G(\cdot,z)d\nu(z)$$

provided we can show that the sets K_{4n}, $n \geq 2$, are disjoint.

Since $V_{4n} \subset A_{4n}$,

$$K_{4n} \subset V_{4n} \subset \{z: 2^{-4n-2} < |z| < 2^{-4n+1}\}$$

$$= \{z: 2^{-4n-3} < 2^{-1} |z| < 2^{-4n}\}$$

Suppose that $m \neq n$. Then

$$K_{4m} \subset \{z: 2^{-4m-3} < 2^{-1}|z| < 2^{-4m}\}$$

If $m=n+k$ with $k > 0$, then

$$2^{-4m} = 2^{-4n-4k} < 2^{-4n-3} \text{ , and}$$

$$K_{4m} \subset D \setminus \{z: 2^{-4n-3} < 2^{-1}|z| < 2^{-4n}\}.$$

If $m=n-k$ with $k > 0$, then

$$2^{-4m-3} = 2^{-4n+4k-3} > 2^{-4n} \text{ , and}$$

$$K_{4m} \subset D \setminus \{z: 2^{-4n-3} < 2^{-1}|z| < 2^{-4n}\}.$$

In either case $K_{4m} \subset D \setminus A(4n-1, 4n+2)$. The sets K_{4n}, $n \geq 2$, are disjoint.

Since $\cup K_{4m} \subset D \setminus A(4n-1, 4n+2)$, Lemma 3.4. implies there is a constant β depending only on ν such that

$$\int_{\bigcup_{m \neq n} K_{4m}} G(y,z)d\nu(z) \leq \beta$$

for all $y \in D \cap A_{4n}$.

Thus

$$w(y) \leq \beta + \int_{K_{4n}} G(y,z)d\nu(z)$$

for all $y \in D \cap A_{4n}$. The functions w and u are equal on K except perhaps for a polar set $Z \subset K$. Thus

$$\underset{y \in K\setminus Z}{w(y)} \geq |y|^{-2}\,|\log|y||^{-1}.$$

Choose an integer N_o such that

$$\frac{2^{8n}}{4n\log 2} > \beta \quad \text{for } n \geq N_o .$$

Then $\int_{K_{4n}} G(y,z)d\nu(z) \geq \frac{2^{8n}}{4n\log 2} - \beta > 0$ for all $y \in K_{4n}\setminus Z$, and $n \geq N_o$. By

LEMMA 3.3.

$$\nu(K_{4n}) \geq \left(\frac{2^{8n}}{4n\log 2} - \beta\right)C_2(K_{4n}) \quad \text{for all } n \geq N_o .$$

Hence
$$\sum_{n \geq N_o} n\nu(K_{4n}) \geq \sum_{n \geq N_o} \left(\frac{2^{8n}}{4n\log 2} - \beta\right)nC_2(K_{4n}).$$

The series $\sum nC_2(K_{4n})$ converges because the hypothesis on u implies that the complement of E is thin at 0, and Theorem 3.1. applies. It remains only to show that the series

$$\sum n\nu(K_{4n}) \text{ converges.}$$

Now

$$-\int \log 2|z|d\nu(z) \geq \sum_{n=2}^{\infty} \int_{K_{4n}} -\log 2|z|d\nu(z)$$
$$\geq \sum_{n=2}^{\infty} (4n-1)(\log 2)\nu(K_{4n}).$$

Note that

$$\infty > w(0) = \int G(0,z)d\nu(z) = -\int \log 2|z|d\nu(z).$$

Thus the series $\sum_{n=2}^{\infty} (4n-1)\log 2\,\nu(K_{4n})$ converges, and so does the series $\sum_{n=1}^{\infty} n\nu(K_{4n})$. This completes the proof.

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