Internat. J. Math. & Math. Sci.
VOL. 11 NO. 4 (1988) 769-780

PYTHAGOREAN TRIANGLES OF EQUAL AREAS

MALVINA BAICA

Department of Mathematics and Computer Science
The University of Wisconsin
Whitewater, Wisconsin 53190
U. S. A.

(Received January 28, 1987 and in revised form April 13, 1987)

ABSTRACT. The main intent in this paper is to find triples of Rational Pythagorean Triangles (abbr. RPT) having equal areas. A new method of solving $a^2 + ab + b^2 = c^2$ is to set $a = y - 1$, $b = y + 1$, $y \in N - \{0,1\}$ and get Pell's equation $c^2 - 3y^2 = 1$. To solve $a^2 - ab + b^2 = c^2$, we set $a = \frac{1}{2}(y + 1)$, $b = y - 1$, $y \geq 2$, $y \in N$ and get a corresponding Pell's equation. The infinite number of solutions in Pell's equation gives rise to an infinity of solutions to $a^2 \pm ab + b^2 = c^2$. From this fact the following theorems are proved.

Theorem 1 Let $c^2 = a^2 + ab + b^2$, $a + b > c > b > a > 0$, then the three RPT-s formed by (c,a), (c,b), (a+b,c) have the same area $S = abc\,(b + a)$ and there are infinitely many such triples of RPT.

Theorem 2 Let $c^2 = a^2 - ab + b^2$, $b > c > a > 0$, then the three RPT-s formed by (b,c), (c,a), (c,b-a) have the same area $S = abc\,(b-a)$ and there are infinitely many such triples of RPT.

KEY WORDS AND PHRASES. Rational Pythagorean Triangles (abbr. RPT), Perimeter of the RPT, Pell's (Euler's) equation, Fibonacci sequence.
1980 AMS SUBJECT CLASSIFICATION CODE. 10B99

1. DEFINITIONS AND PREVIOUS RESULTS. In one of his papers Bernstein [1] returned to the Grecian classical mathematics, and investigated primitive rational Pythagorean Triangles concerning mainly k-tuples of them having equal perimeters 2P.

In this paper we deal with rational integral right triangles having equal areas and give the following

DEFINITION A rational triangle with sides a,b,c which are represented by a triple (a,b,c) of natural numbers will be called a Rational Pythagorean Triangle, if and only if

$$\left.\begin{array}{l} \text{there exists } (u,v),(u,v) \in N^2 - \{(0,0)\}, \\ u > v, \text{ such that} \\ a = u^2-v^2,\ b = 2uv,\ c = u^2+v^2, \\ a,b,c \in N - \{0\} = 1,2,\ldots \end{array}\right\} \qquad (1.1)$$

We abbreviate Rational Pythagorean Triangles by RPT and we write RPT(u,v) for RPT formed by (u,v). We also write S(u,v) for the area and P(u,v) for half perimeter of the RPT(u,v)

$$D = RPT(u,v): S(u,v) = \frac{1}{2}ab = uv(u^2 - v^2) \qquad (1.2)$$

$$P(u,v) = \frac{1}{2}(a+b+c) = u(u+v)$$

The main intent in this paper is to find triplets of RPT-s having equal areas.

The first who asked this question was the great Diophantus [3] and Dickson [2] enlarged the topic.

Let D be a triangle with integral sides and $\hat{C} = 120°$ one of its angles. Then if c is the side opposite $\hat{C}$ and a,b the two adjacent sides of $\hat{C}$, we have, by $c^2 = a^2 + b^2 - 2ab \cos \hat{C}$

$$\left.\begin{array}{l} a^2 + ab + b^2 = c^2 \\ a + b > c > b > a;\ a,b,c \in N - \{0\} \end{array}\right\} \qquad (1.3)$$

and if $\hat{C} = 60°$ we have

$$\left.\begin{array}{l} a^2 - ab + b^2 = c^2 \\ b > c > a > 0;\ a,b,c \in N - \{0\}. \end{array}\right\} \qquad (1.4)$$

The totality of solutions to $a^2 \pm ab + b^2 = c^2$ is given in parameter form by Hasse [6]. <u>The new idea in this paper rests in the fact that (1.3) and (1.4) are connected with the areas of the triangles</u>. In order to find a formula to derive explicitly the infinity of RPT-s of equal area, since we cannot use Hasse's [6] parametric form, we will give a new method to prove that the equations $a^2 \pm ab + b^2 = c^2$ have infinitely many solutions and state some of them explicitly.

The new method will bring us to the solution of a Pell's equation. The infinite number of solutions of Pell's equation [4] will give rise to an infinity of solutions to $a^2 \pm ab + b^2 = c^2$.

2. PELL'S (EULER'S) EQUATION: $u_n^2 - 3v_n^2 = 1;\ n = 0,1,\ldots$. In the sequel we shall permanently have to make use of Pell's equation

$$u_n^2 - 3v_n^2 = 1,\ n = 0,1,\ldots \qquad (2.1)$$

This could be solved by continued fraction with $\sqrt{3}+1 = [\overline{2,1}]$, but we use a simpler method since a solution of (2.1) is easily found. Neglecting $(u_0,v_0) = (1,0)$ we have

$$(u_1,v_1) = (2,1), \tag{2.2}$$

hence using [5]

$$u_n + v_n\sqrt{3} = (2+\sqrt{3})^n, \quad n = 0,1,\ldots . \tag{2.3}$$

From (2.3) we deduce

$$\left.\begin{aligned} u_n &= \sum_{i=0}^{[\frac{n}{2}]} \binom{n}{2i} 2^{n-2i}3^i; \quad n=0,1,\ldots \\ v_n &= \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n}{2i+1} 2^{n-1-2i}3^i; \quad n=1,2,\ldots \end{aligned}\right\} \tag{2.4}$$

From (2.4) we obtain

$$\left.\begin{aligned} u_{2m} &= \sum_{i=0}^{m} \binom{2m}{2i} 2^{2m-2i}3^i; \quad m=1,2,\ldots \\ v_{2m} &= \sum_{i=0}^{m-1} \binom{2m}{2i+1} 2^{2m-1-2i}3^i; \quad m=1,2,\ldots \\ (u_0,v_0) &= (1,0) \end{aligned}\right\} \tag{2.5}$$

$$\left.\begin{aligned} u_{2m+1} &= \sum_{i=0}^{m} \binom{2m+1}{2i} 2^{2m+1-2i}3^i; \quad m=0,1,\ldots \\ v_{2m+1} &= \sum_{i=0}^{m} \binom{2m+1}{2i+1} 2^{2m-2i}3^i; \quad m=0,1,\ldots \end{aligned}\right\} \tag{2.6}$$

In u_{2m} all summands are even but the last which is 3^m; in v_{2m} all summands are even, hence

$$\left.\begin{aligned} u_{2m} &= 2F+1; \\ v_{2m} &= 2G; \qquad F,G \in N \end{aligned}\right\} \tag{2.7}$$

In u_{2m+1} all summands are even; in v_{2m+1} all summands are even but the last which is 3^m; hence

$$\begin{aligned} u_{2m+1} &= 2S; \\ v_{2m+1} &= 2T+1; \quad S,T \in N. \end{aligned} \tag{2.7a}$$

We have the initial values

$$\left.\begin{aligned} &(u_0,v_0) = (1,0);\ (u_1,v_1) = (2,1);\ (u_2,v_2) = (7,4) \\ &(u_3,v_3) = (26,15);\ (u_4,v_4) = (97,56);\ (u_5,v_5) = (362,209) \\ &(u_6,v_6) = (1351,780);\ (u_7,v_7) = (5092,2911). \end{aligned}\right\} \tag{2.8}$$

3. A SOLUTION OF (1.3) $a^2+ab+b^2=c^2$. Here we shall give infinitely many solutions of (1.3) in explicit form, though they won't constitute all solutions of (1.3). Let

$$\left.\begin{array}{l} a^2+ab+b^2 = c^2;\ a+b>c>b>a>0, \\ a,b,c \in N - \{0\}. \end{array}\right\} \qquad (3.1)$$

We set

$$a=y-1;\ b=y+1;\quad y\in N-\{0\}. \qquad (3.2)$$

Substituting the values of a,b from (3.2) in (3.1) we obtain

$$(y-1)^2+(y^2-1)+(y+1)^2 = c^2$$

or $\quad c^2-3y^2 = 1.$

Now we set $c = u_n$, $y = v_n$, $n=1,2,\ldots$ and we get $u_n{}^2-3v_n{}^2 = 1$ or

$$u_n{}^2 - 3v_n{}^2 = 1 \quad \text{which is (2.1).}$$

The infinite number of solutions u_n,v_n in Pell's equation (2.1), give rise to an infinity of solutions to $a^2+ab+b^2 = c^2$.

THEOREM 1. Let $c^2 = a^2+ab+b^2$, $a+b>c>b>a>0$ then the three RPT-s formed by (c,a), (c,b), (a+b,c) have the same area S = abc(b+a) and there are infinitely many such triples of RPT-s.

Proof. To get the example given by Diophantus [3, p. 172] as a particular case of our formulas we consider $(a,b,c)=1$. Since we prefer $(a,b,c)=1$, though this is not absolutely necessary, we have to set v_n even in (2.1), so that a,b are both odd. If $v_n=v_{2m}$ we obtain

$$a=v_{2m}-1;\ b=v_{2m}+1;\ c=u_{2m};\ m=1,2,\ldots \qquad (3.3)$$

Using (1.2) and $a^2+ab+b^2 = c^2$ it is easy to show that

$$S(c,a)=S(c,b)=S(a+b,c)=abc(b+a).$$

$$D_1 = RPT(c,a)=RPT(u_{2m},v_{2m}-1);$$

$$D_2 = RPT(c,b)=RPT(u_{2m},v_{2m}+1);$$

$$D_3 = RPT(a+b,c)=RPT(2v_{2m},u_{2m}),$$

and following S = abc(b+a) the common area is

$$S = 2\,u_{2m}v_{2m}(v_{2m}^2-1). \qquad (3.4)$$

That there are infinitely many such triples of RPT-s follow from the infinity of solutions of (2.1).

Choosing m = 1, we have from (2.8)

$$\left.\begin{array}{l} (u_2,v_2) = (7,4);\ u_2 = 7,\ v_2 = 4 \\ c = 7;\ a = 3;\ b = 5 \\ S = 3\cdot5\cdot7\cdot8 = 840 \end{array}\right\} \qquad (3.5)$$

and this is exactly the example given by Diophantus.

If we choose m = 2, we have

$$\left.\begin{array}{l}(u_4,v_4) = (97,56);\ u_4 = 97;\ v_4 = 56;\\ c = 97;\ a = 55;\ b = 57.\\ S = 55\cdot 57\cdot 97\cdot 112 = 34{,}058{,}640.\end{array}\right\} \qquad (3.6)$$

For RPT(97,55) using (1.2) we have

$$S(97,55) = 97\cdot 55(97^2-55^2) = \\ = 55\cdot 57\cdot 97\cdot 112 = S \text{ in } (3.6).$$

4. SOLUTION OF (1.4) $a^2 - ab + b^2 = c^2$. In this chapter we state more triples of triangles RPT having the same area. We shall first prove

Let

$$a^2 - ab + b^2 = c^2;\quad a,b,c \in N - \{0\};\ \text{then} \qquad (4.1)$$
$$b > c > a > 0;\quad (a,b,c) = 1$$

In (4.1) we set arbitrarily $b > a$; then we have

$$\left.\begin{array}{l} b^2 - c^2 = a(b-a) > 0;\ b > c;\\ c^2 - a^2 = b(b-a) > 0;\ c > a;\end{array}\right\} \qquad (4.2)$$

Thus $b > c > a > 0$, as stated in (4.1). We shall now prove

THEOREM 2. Let $c^2 = a^2 - ab + b^2$, $b > c > a > 0$. Then the three RPT-s formed by (b,c), (c,a), (c,b-a) have the same area S, viz.

$$S = abc(b-a) \qquad (4.3)$$

and there are infinitely many such triples RPT.

Proof. If we set in (1.4) $a \longrightarrow -a$, we obtain from equation (1.3), the equation (4.1). But this is only an algebraic formality, since the RPT formed by (c,-a) makes sense, though from $S = abc(a+b)$ we obtain (4.4) by substituting -a for a. We also have

$$c^2 = (a-b)^2 + ab,\quad c > a-b,\ b-a. \qquad (4.4)$$

Now

$$D_1 = RPT(b,c);\ S(b,c) = \tfrac{1}{2}\cdot 2bc(b^2-c^2) = \tfrac{1}{2}xy$$
$$= \tfrac{1}{2}\cdot 2bca(b-a) = abc(b-a);$$
$$D_2 = RPT(c,a);\ S(c,a) = \tfrac{1}{2}\cdot 2ca(c^2-a^2) = \tfrac{1}{2}xy$$
$$= \tfrac{1}{2}\cdot 2cab(b-a) = abc(b-a);$$
$$D_3 = RPT(c,b-a);\ S(c,b-a) = \tfrac{1}{2}\cdot 2c(b-a)[c^2-(b-a)^2] = \tfrac{1}{2}xy$$
$$= c(b-a)ab = abc(b-a)$$
$$S = S(b,c) = S(c,a) = S(c,b-a) = abc(b-a).$$

We still have to prove that (4.1) has infinitely many solutions. We shall give two methods to find these solutions, though these may not be all infinitely many solutions of (4.1). This was done by Hasse [6] with algebraic number theory which we shall avoid here, giving simple methods to solve (4.1) in explicit form.

In (4.1) we set

$$b = a + v, \quad v = b-a > 0; \tag{4.5}$$

we obtain, setting $a = b-v$,

$$(b-v)^2 - b(b-v) + b^2 = c^2$$

$$b^2 - bv + v^2 - c^2 = 0$$

$$v^2 - bv - (c^2-b^2) = 0. \tag{4.6}$$

From (4.6) we obtain

$$v = \frac{1}{2}\left(b \pm \sqrt{b^2 + 4(c^2 - b^2)}\right),$$

$$v = \frac{1}{2}\left(b \pm \sqrt{4c^2 - 3b^2}\right). \tag{4.7}$$

We have, since setting $4c^2 - 3b^2 = x^2$ contradicts our condition $(a,b,c) = 1$

$$4c^2 - 3b^2 = 1. \tag{4.8}$$

Thus we have arrived at Pell's equation, setting

$$2c = u_n, \quad b = v_n. \tag{4.9}$$

Thus u_n is even, and we must take $u_n = u_{2m+1}$ and from (2.7a) we obtain

$$\left.\begin{array}{l} b = v_{2m+1}; \quad c = \frac{1}{2} u_{2m+1}; \\ a = b-v = b - \frac{1}{2}(b \pm 1). \end{array}\right\} \tag{4.10}$$

Since we obtain two values for a, we may have obtained <u>six</u> RPT-s with equal area. We shall investigate this later. We first take $v = \frac{1}{2}(b+1)$

$$\left.\begin{array}{l} a = \frac{1}{2}(b-1) = \frac{1}{2}(v_{2m+1} - 1) \\ b = v_{2m+1}; \; c = \frac{1}{2} u_{2m+1}; \\ b-a = \frac{1}{2}(v_{2m+1} + 1). \end{array}\right\} \tag{4.11}$$

Hence, from (4.3), or forming

$$D_1 = RPT(b,c), \; D_2 = RPT(c,a), \; D_3 = RPT(c,b-a)$$

$$S = \frac{1}{2}(v_{2m+1} - 1)\cdot v_{2m+1} \cdot \frac{1}{2} u_{2m+1} \cdot \frac{1}{2}(v_{2m+1} + 1)$$

$$S = \frac{1}{8} u_{2m+1} v_{2m+1} (v_{2m+1}^2 - 1). \tag{4.12}$$

We take, for an example $m = 1$,

$$(u_3, v_3) = (26,15)$$

$$u_{2m+1} = 26; \quad v_{2m+1} = 15$$

$$S = \frac{1}{8} \cdot 26 \cdot 15(15^2 - 1) = 10{,}920.$$

The reader should note that since

$$v_{2m+1} = 2S + 1, \; v_{2m+1}^2 - 1 \equiv 0 \pmod 8.$$

Hence, in (4.12) S is an integer. We now take $v = \frac{1}{2}(b-1)$ and obtain from (4.10)

$$\left.\begin{aligned} b &= v_{2m+1}; \quad c = \tfrac{1}{2} u_{2m+1}; \\ a &= b-v = \tfrac{1}{2}(b+1) = \tfrac{1}{2}(v_{2m+1}+1) \\ b-a &= \tfrac{1}{2}(v_{2m+1}-1), \end{aligned}\right\} \tag{4.13}$$

and from $D_4 = RPT(b,c)$, $D_5 = RPT(c,a)$, $D_6 = RPT(c,b-a)$. Previously, we had for $v = \frac{1}{2}(b+1)$,

$$a = \tfrac{1}{2}(v_{2m}-1); \quad b = v_{2m+1}, \quad c = \tfrac{1}{2} u_{2m+1}$$

$$b - a = \tfrac{1}{2}(v_{2m}+1).$$

Comparing (4.11) with (4.13) we see that

$$D_2 = D_6$$

$$D_3 = D_5$$

$$D_1 = D_4$$

and thus get the same triple of RTS-s in both cases.

5. SECOND SOLUTION OF $c^2 = a^2 - ab + b^2$. We give a second method of finding infinitely many solutions of (4.1). We set here

$$a = \tfrac{1}{2}(y+1); \quad b = y-1; \quad y \geq 2; \quad y \in N. \tag{5.1}$$

Substituting these values in (4.1), we obtain

$$[\tfrac{1}{2}(y+1)]^2 - \tfrac{1}{2}(y^2-1) + (y-1)^2 = c^2$$

$$y^2 + 2y+1 - 2y^2 + 2 + 4y^2 - 8y + 4 = 4c^2$$

$$3y^2 - 6y + 7 = 4c^2$$

$$3(y-1)^2 + 4 = 4c^2$$

$$4c^2 - 3(y-1)^2 = 4. \tag{5.2}$$

Since from (2.7) y-1 has to be even we can cancel (5.2) by 4 and we obtain

$$y-1 = 2v_{2m}, \quad y = 2v_{2m}+1; \tag{5.3}$$

and with $c = u_{2m}$ we obtain

$$u_{2m}^2 - 3v_{2m}^2 = 1$$

$$a = \tfrac{1}{2}(2v_{2m} + 2),$$

$$a = v_{2m}+1; \quad b = 2v_{2m}, \quad c = u_{2m} \tag{5.4}$$

$$S = abc(b-a) = (v_{2m}+1)2v_{2m} \cdot u_{2m}(v_{2m}-1)$$

$$S = 2u_{2m}v_{2m}(v_{2m}^2 - 1). \tag{5.5}$$

Comparing (5.5) with (4.12) we may think about the different expressions for area S; though one is expressed by u_{2m+1}, v_{2m+1}, the other by u_{2m},

v_{2m}, it is so because we are dealing here with different areas, depending on the values of a,b,c, which are different in each case. For an example for (5.5) we choose m = 2,

$$(u_4, v_4) = (97, 56),$$
$$S = 2\cdot 97\cdot 56\cdot(56^2-1) = 34{,}058{,}640.$$

If we set

$$\left.\begin{array}{l} a = 2m-1,\ b = 2v_{2m},\ c = u_n,\ b-a = v_{2m}+1, \\ u_{2m}^2 - 3v_{2m}^2 = 1 \end{array}\right\} \quad (5.6)$$

then $a^2 - ab + b^2 = c^2$, $S = 2u_{2m}v_{2m}(v_{2m}^2-1)$. But we obtain nothing new since RPT(c,a), RPT(c,b-a) are interchanged with RPT(c,b-a), RPT(c,a).

The author is asking whether two successive Fibonacci numbers could be solutions a,b of $a^2 \pm ab + b^2 = c^2$. The Fibonacci sequence, see [8] with $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$, n = 1,2,... goes 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ... Now $(F_4, F_5) = (3,5)$ is a solution of the Diophantine example $3^2 + 3\cdot 5 + 5^2 = 7^2$, and $(F_5, F_6) = (5,8)$ is a solution of

$$5^2 - 5\cdot 8 + 8^2 = 7^2.$$

Here $S = abc(b-a) = 5\cdot 8\cdot 7\cdot 3 = 840$ whether there are more pairs of adjacent Fibonacci numbers serving as solutions of $a^2 - ab + b^2 = c^2$ could not be decided generally here.

6. <u>PERIMETERS AND AREAS</u>. As known, P, half the perimeter of a RPT is a divisor of its area S, since

$$P = u(u+v), \quad S = u\cdot v(u^2-v^2) \qquad (6.1)$$
$$S \div P = v(u-v)$$

where (u,v) forms the RPT. The question arises whether there are other connections between these two elements. Here we could only prove

<u>THEOREM 3</u>. Let $a^2 - ab + b^2 = c^2$, $b > c > a > 0$, $a,b,c \in N - \{0\}$. Then the sum of the three perimeters formed by a,b,c, viz. RPT(b,c), RPT(c,a), RPT(c,b-a) is the sum of two squares.

<u>Proof</u>. We have

$$D_1 = RPT(b,c): \quad 2P_1 = x_1 + y_1 + z_1 = b^2 - c^2 + 2bc + b^2 + c^2$$
$$2P_1 = 2b(b+c);$$
$$D_2 = RPT(c,a): \quad 2P_2 = x_2 + y_2 + z_2 = c^2 - a^2 + 2ca + c^2 + a^2$$
$$= 2c(c+a);$$
$$D_3 = RPT(c,b-a): \quad 2P_3 = x_3 + y_3 + z_3 = c^2 - (b-a)^2 + 2c(b-a) + (b-a)^2 + c^2$$
$$= 2c^2 + 2c(b-a) = 2c(c+b-a).$$
$$P_1 + P_2 + P_3 = b(b+c) + c(c+a) + c(c+b) - ac$$
$$= 2c^2 + 2bc + b^2 = (b+c)^2 + c^2$$

7. ENTIRELY NEW RPT-s. In a very little known paper Hillyer [7] has given a most surprising infinity of triples of Rational Pythagorean triangles in an explicit form, having equal areas. These are formed by

$$\left.\begin{aligned} D_1 &= RPT(u,v) = RPT(a^2+ab+b^2,\ b^2-a^2) \\ D_2 &= RPT(u,v) = RPT(a^2+ab+b^2,\ 2ab+a^2) \\ D_3 &= RPT(u,v) = RPT(2ab+b^2,\ a^2+ab+b^2) \\ & b > a > 0;\ b,a \in Q^+ \end{aligned}\right\} \qquad (7.1)$$

As we see, Hillyer was not concerned about RPT-s; his Pythagorean triangles had just to have rational sides. We shall operate with RPT-s only and set

$$a,b \in N-\{0\};\quad b>a>0. \qquad (7.2)$$

We first investigate, whether the condition

$$u > v > 0 \qquad (7.3)$$

is fulfilled for D_1, D_2, D_3. We have

$$D_1\colon u-v = a^2+ab+b^2-(b^2-a^2) = 2b^2+ab > 0;\ u>v>0.$$

$$D_2\colon u-v = a^2+ab+b^2-(2ab+a^2) = b^2-ab = b(b-a) > 0,$$

since $b > a$ hence $u > v > 0$

$$D_3\colon u-v = 2ab+b^2-(a^2+ab+b^2) = ab-a^2 = a(b-a) > 0,$$

since $b-a > 0$, $u > v > 0$.

We shall now find the areas of D_1, D_2, D_3, and have

$$\begin{aligned} D_1\colon S_1 &= (a^2+ab+b^2)(b^2-a^2)[(a^2+ab+b^2)^2-(b^2-a^2)^2] \\ &= (a^2+ab+b^2)(b^2-a^2)(a^2+ab+b^2+b^2-a^2)\cdot(a^2+ab+b^2-b^2+a^2) \\ &= (a^2+ab+b^2)(b^2-a^2)(ab+2b^2)(ab+2a^2), \end{aligned}$$

$$S_1 = ab(a^2+ab+b^2)(b^2-a^2)(a+2b)(b+2a). \qquad (7.4)$$

$$\begin{aligned} D_2\colon S_2 &= (a^2+ab+b^2)(2ab+a^2)[(a^2+ab+b^2)^2-(2ab+a^2)^2] \\ &= (a^2+ab+b^2)a(a+2b)(a^2+ab+b^2+2ab+a^2)\cdot(a^2+ab+b^2-2ab-a^2) \\ &= (a^2+ab+b^2)a(a+2b)(2a^2+3ab+b^2)\cdot(b^2-ab). \end{aligned}$$

Now

$$2a^2+3ab+b^2 = (2a+b)(a+b),$$

hence

$$S_2 = ab(a^2+ab+b^2)(b^2-a^2)(a+2b)(b+2a) \qquad (7.5)$$

$$\begin{aligned} D_3\colon S_3 &= (2ab+b^2)(a^2+ab+b^2)[(2ab+b^2)^2-(a^2+ab+b^2)^2] \\ &= b(2a+b)(a^2+ab+b^2)(2b^2+3ab+b^2)(ab-a^2). \end{aligned}$$

Now

$$2b^2+3ab+b^2 = (2b-a)(b+a).$$

Hence

$$S_3 = ab(a^2+ab+b^2)(b^2-a^2)(a+2b)(b+2a). \tag{7.6}$$

Thus $S_1 = S_2 = S_3 = \sum$,

$$\sum = ab(a^2+ab+b^2)(b^2-a^2)(a+2b)(b+2a). \tag{7.7}$$

We now return to the equation of Diophantus from (1.3)

$$\left.\begin{array}{l} a^2+ab+b^2 = c^2 \\ a+b > c > b > a > 0 \\ a,b,c \in N - \{0\}. \end{array}\right\} \tag{1.3}$$

We found that the three triples RPT(c,b), RPT(c,a), RPT(a+b,c) have equal areas, viz.

$$S = abc(b+a).$$

If in (7.7) we demand that $a^2+ab+b^2 = c^2$, solvable in natural number $a+b > c > b > a > 0$ we obtain

$$\sum = abc^2(b^2-a^2)(a+2b)(b+2a), \tag{7.8}$$

and from (7.8), and $S = abc(a+b)$,

$$\sum \div S = c(b-a)(a+2b)(b+2a). \tag{7.9}$$

Now the quotient $\sum \div S$ is a natural number, and many authors have asked and solved the question of the ratio of the areas of two RPT-s.

8. THE MAIN RESULT. We now form three RPT-s, having equal areas. They are entirely new and unknown. We investigate

$$\left.\begin{array}{l} D_1 = RPT(u,v) = RPT(a^2-ab+b^2,\ b^2-a^2); \\ D_2 = RPT(u,v) = RPT(a^2-ab+b^2,\ 2ab-b^2); \\ D_3 = RPT(u,v) = RPT(2ab-a^2,\ a^2-ab+b^2) \\ 2a > b > a > 0;\ b,a \in Q^+ \end{array}\right\} \tag{8.1}$$

We first check for D_1, D_2, D_3,

$$u > v > 0,\ u-v > 0.$$

The reader note the condition

$$2a > b.$$

Later when we shall operate with RPT-s and the equation

$$a^2-ab+b^2 = c^2$$

we shall see that solutions of this equation are possible with $2a > b$.

We have, $(u,v) \in Q^+)$

$$D_1:\ u-v = a^2 - ab + b^2 - (b^2 - a^2)$$
$$= 2a^2 - ab = a(2a-b) > 0$$
$$v = b^2 - a^2 > 0.$$

$$D_2:\quad u-v = a^2 - ab + b^2 - (2ab - b^2)$$
$$= a^2 - 3ab + 2b^2$$
$$= (2b-a)(b-a) > 0$$
$$v = 2ab - b^2 = b(2a-b) > 0.$$

$$D_3:\quad u-v = 2ab - a^2 - (a^2 - ab + b^2) = 3ab - 2a^2 - b^2$$
$$= (2a-b)(b-a) > 0.$$
$$u = 2ab - a^2 = a(2b-a) > 0$$
$$v = a^2 - ab + b^2 = (a-b)^2 + ab > 0.$$

We shall now find the areas formed by D_1, D_2, D_3, and have

$$D_1: S_1 = (a^2-ab+b^2)(b^2-a^2)[(a^2-ab+b^2)^2-(b^2-a^2)]$$
$$= (a^2-ab+b^2)(b^2-a^2)(a^2-ab+b^2-b^2+a^2)\cdot(a^2-ab+b^2+b^2-a^2)$$
$$= (a^2-ab+b^2)(b^2-a^2)(2a^2-ab)(2b^2-ab)$$
$$= (a^2-ab+b^2)(b^2-a^2)a(2a-b)(2b-a).$$
$$S_1 = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a). \tag{8.2}$$

$$D_2: S_2 = (a^2-ab+b^2)(2ab-b^2)[(a^2-ab-b^2)-(2ab-a)^2]$$
$$= (a^2-ab+b^2)b(2a-b)(a^2-ab+b^2-2ab+b^2)(ab+a^2)$$
$$= (a^2-ab+b^2)b(2a-b)(a^2-3ab+2b^2)a(b+a)$$
$$= (a^2-ab+b^2)b(2a-b)(2b-a)(b-a)a(b+a)$$
$$= ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a).$$
$$S_2 = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a). \tag{8.3}$$

$$D_3: S_3 = (2ab-a^2)(a^2-ab+b^2)[(2ab-a^2)^2-(a^2-ab+b^2)^2]$$
$$= (2ab-a^2)(a^2-ab+b^2)(2ab-a^2-a^2+ab-b^2)\cdot(2ab-a^2+a^2-ab+b^2)$$
$$= a(2b-a)(a^2-ab+b^2)(3ab-2a^2-b^2)\cdot(ab+b^2)$$
$$= ab(a^2-ab+b^2)(2b-a)(2a-b)(b-a)\cdot(a+b)$$
$$= ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a).$$
$$S_3 = ab(a^2-ab+b^2)(b^2-a^2)(2a-b)(2b-a). \tag{8.4}$$

Thus we have obtained the wanted result

$$S_1 = S_2 = S_3 = \Sigma'.$$

We now return to the equation

$$\left.\begin{array}{l} a^2 - ab + b^2 = c^2, \quad b > a > 0 \\ b > c > a > 0;\ b,c,a \in N - \{0\} \end{array}\right\} \tag{4.1}$$

and recall from (5.4) that there is a solution with

$$a = \frac{b}{2} + 1; \quad 2a > b \tag{8.5}$$

as we needed. With equation (4.1), Σ' takes the form

$$\left.\begin{array}{l}\sum' = abc^2(b^2 - a^2)(2a-b)(2b-a)\\ 2a > b > c > a;\quad a,b,c \in N - \{0\}.\end{array}\right\} \quad (8.6)$$

Now with $a^2 - ab + b^2 = c^2$, and (4.3), viz.

$$S = abc(b-a)$$

and obtain thus the quotient

$$\sum' \div S = c(b+a)(2a-b)(2b-a). \quad (8.7)$$

As an example we shall take

$$a^2 - ab + b^2 = c^2$$

$$(a,b,c) = (8,15,13).$$

Here $2a = 16 > 15 = b$; we obtain

$$\sum' = 8\cdot 15\cdot 169\cdot 161\cdot 1\cdot 22,$$

$$\sum' = 2^4\cdot 3\cdot 5\cdot 7\cdot 11\cdot 13^2\cdot 23 = 71{,}831{,}760.$$

$$S = abc(b-a) = 8\cdot 15\cdot 13\cdot 7$$

$$= 2^3\cdot 3\cdot 5\cdot 7\cdot 13 = 10{,}920.$$

$$\sum' \div S = 2\cdot 11\cdot 13\cdot 23 = 6{,}578$$

$$71{,}831{,}760 \div 10{,}920 = 6{,}578.$$

ACKNOWLEDGEMENT. Dedicated to the memory of Leon Bernstein. Dr. Bernstein was the author's professor and advisor at Illinois Institute of Technology, Chicago.

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