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ON CONVEX FUNCTIONS OF ORDER α AND TYPE β

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ABSTRACT. Owa [1] gave three subordination theorems for convex functions of order α and starlike functions of order α. Unfortunately, none of the theorems is correct. In this paper, similar problems are discussed for a generalized class and sharp results are given.

KEY WORDS AND PHRASES. Subordination, Hadamard product, convex functions of order α and type β.
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1. INTRODUCTION.

Let $f(z)$ and $g(z)$ be analytic in the unit disk $D = \{z: |z| < 1\}$. $f(z)$ is said to be subordinate to $g(z)$, dentoed by $f(z) \prec g(z)$, if there exists a function $w(z)$ analytic and satisfying $|w(z)| \leq |z|$ in D such that $f(z) = g(w(z))$. valent in D, then $f(z) \prec g(z)$ is equivalent to $f(0) = g(0)$ and $f(D) \subset g(D)$.

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Let $S^*(\alpha,\beta)$ be the family of starlike functions of order α and type β. That is, it consists of analytic functions $f(z) = z + a_2z^2 + \ldots$ satisfying

$$|zf'(z)/f(z) - 1| < |(2\beta - 1)zf'(z)/f(z) + 1 - 2\beta\alpha| \qquad (z \in D), \tag{1.1}$$

where $0 \leq \alpha < 1$ and $0 < \beta \leq 1$. This class was first introduced by Juneja and Mogra [2]. It is clear that $S^*(\alpha,1) = S^*(\alpha)$, the usual class of starlike functions of order α.

Similarly, we define the following general class.

DEFINITION. An analytic function $f(z) = z + a_2z^2 + \ldots$ is called convex of order α and type β if and only if

$$|zf''(z)/f'(z)| < |(2\beta - 1)zf''(z)/f'(z) + 2\beta(1 - \alpha)| \qquad (z \in D). \tag{1.2}$$

The class of these functions is denoted by $K(\alpha,\beta)$.

$K(\alpha,1)$ is the well known class $K(\alpha)$, which consists of convex functions of order α. It is easily seen that $f(z) \in K(\alpha,\beta)$ if and only if $zf'(z) \in S^*(\alpha,\beta)$.

We got the following theorem in [3].

THEOREM A. Let $f(z) \in S^*(\alpha,\beta)$, then we have

$$zf'(z)/f(z) \prec (1 + (1 - 2\beta\alpha)z)/(1 + (1 - 2\beta)z), \tag{1.3}$$

$$f(z)/z \prec (1+(1-2\beta)z)^{2\beta(1-\alpha)/(1-2\beta)} \qquad (\beta \neq \tfrac{1}{2}),$$

$$f(z)/z \prec e^{(1-\alpha)z} \qquad (\beta = \tfrac{1}{2}).$$

All of the results are best possible.

In section 2 of this paper, we give a counterexample of Owa's results and point out the mistakes in [1]. Then we discuss similar problems for the class $K(\alpha,\beta)$ and get sharp subordination and convolution theorems. And we give a characterization for convex functions of order α and type β in section 3. Finally, we obtain some inequalities by using the subordination results.

2. A COUNTEREXAMPLE.

Theorem 1 in [1] is equivalent to that if $f(z) \in K(\alpha)$, then

$$f'(z) \prec e^{-4(1-\alpha)/(1-z)} = F(z), \tag{2.1}$$

and if $f'(re^{i\theta})$ lies for some $r \neq 0$ on the boundary of $F(|z| < r)$ if and only if

$$f(z) = \int_0^z e^{4(1-\alpha)/(1-\varepsilon t)}dt \qquad (|\varepsilon| = 1). \tag{2.2}$$

It is well known that

$$f(z) = \int_0^z (1-t)^{-2(1-\alpha)}\,dt \in K(\alpha).$$

(2.1) implies that

$$(1-z)^{-2(1-\alpha)} \prec e^{-4(1-\alpha)/(1-z)},$$

or equivalently,

$$\log(1-z) \prec 2/(1-z) \tag{2.3}$$

where log is to be the branch which vanishes at the point one. But clearly, (2.3) does not hold.

The mistake arises from that

$$zf''(z)/f'(z) \prec 4(1-\alpha)z/(1-z)^2 \tag{2.4}$$

implies

$$\log f'(z) \prec -4(1-\alpha)/(1-z).$$

In fact, from (2.4) we can only get

$$\log f'(z) \prec 4(1-\alpha)z/(1-z).$$

And (2.2) was got from $f(0) = f'(0) - 1 = 0$ and

$$zf''(z)/f'(z) = 4(1-\alpha)\,\varepsilon z\,/(1-\varepsilon z)^2.$$

So it should be

$$f(z) = \int_0^z \exp[4(1-\alpha)\,\varepsilon t/(1-\varepsilon t)]dt.$$

Furthermore, this function does not belong to $K(\alpha)$.

There are similar mistakes in the theorem 2 [1]. And the family of functions which satisfy the conditions in theorem 3 [1] is empty since $\mathrm{Re}\{zf'(z)\} = 0$ at $z = 0$. Therefore this theorem is meaningless. From the proof of the theorem 3, maybe the condition should be $\mathrm{Re}\{zf'(z)\} < \alpha$, not $\mathrm{Re}\{zf'(z)\} > \alpha$. If so, the following proof goes wrong again. There are also several places needed to be corrected. We omit them here.

3. SUBORDINATION AND CONVOLUTION THEOREMS.

The leading element of $K(\alpha,\beta)$ is

$$k(\alpha,\beta,z) = \begin{cases} (1 - 2\beta\alpha)^{-1}\{(1+(1-2\beta)z)^{(1-2\beta\alpha)/(1-2\beta)}-1\} & (\beta \neq \frac{1}{2},\ \alpha \neq \frac{1}{2}/\beta) \\ (1 - 2\beta)^{-1}\log(1 + (1 - 2\beta)z) & (\beta \neq \frac{1}{2},\ \alpha = \frac{1}{2}/\beta) \\ (e^{(1-\alpha)z}-1)/(1 - \alpha) & (\beta = \frac{1}{2}) \end{cases} \tag{3.1}$$

It is not difficult to prove that for an analytic function $f(z) = z+a_2z^2+\ldots$, (1.3) implies that $f(z) \in S^*(\alpha,\beta)$. From Theorem A and the correspondence between $K(\alpha,\beta)$ and $S^*(\alpha,\beta)$, we have the following

THEOREM 1. $f(z) \in K(\alpha,\beta)$ if and only if $f(z) = z + a_2z^2 + \ldots$ is analytic in D and

$$zf''(z)/f'(z) \prec 2\beta(1 - \alpha)z/(1 + (1 - 2\beta)z). \tag{3.2}$$

Moreover, let $f(z) \in K(\alpha,\beta)$, then we have sharp subordinations

$$f'(z) \prec (1 + (1 - 2\beta)z)^{2\beta(1-\alpha)/(1-2\beta)} \qquad (\beta \neq \tfrac{1}{2}),$$

$$f'(z) \prec e^{(1-\alpha)z} \qquad (\beta = \tfrac{1}{2}).$$

The first result of Theorem 1 is equivalent to that an analytic function $f(z) = z+a_2z^2 + \ldots \in K(\alpha,\beta)$ if and only if

$$1 + zf''(z)/f'(z) \in Q(\alpha,\beta) \qquad (z \in D), \tag{3.3}$$

where

$$Q(\alpha,\beta) = \begin{cases} \{w;\ |w - \alpha - (1-\alpha)/2(1-\beta)| < \frac{1}{2}(1-\alpha)/(1-\beta)\} & (\beta < 1) \\ \{w;\ \mathrm{Re}\,w > \alpha\} & (\beta = 1). \end{cases}$$

COROLLARY 1. $K(\alpha,\beta_1) \subset K(\alpha,\beta_2) \subset K(\alpha,1) = K(\alpha)$ if $\beta_1 \leq \beta_2 \leq 1$. $K(\alpha_1,\beta) \subset K(\alpha_2,\beta) \subset K(0,\beta)$ if $\alpha_1 \geq \alpha_2 \geq 0$.

THEOREM 2. Let $p(z) \in K = K(0,1)$, and $f(z) \in K(\alpha,\beta)$, then

$$p * f(z) \in K(\alpha,\beta),$$

where * denotes the Hadamard product.

PROOF. We know that $1+zf''(z)/f'(z) = (zf'(z))'/f'(z)$ takes all its values in the convex domain $Q(\alpha,\beta)$. A result of Ruscheweyh and Sheil-Small [4] implies that

$p(z)*\{z(zf'(z))'\}/p(z)*(zf'(z))$ also takes all its values in $Q(\alpha,\beta)$ since we have $f(z) \in K(\alpha,\beta) \subset K(0,1) = K$ from Corollary 1. It is easy to see that

$$p(z)*(zf'(z)) = z(p*f)'(z),$$

$$p(z)*\{z(zf'(z))'\} = z\{p(z)*(zf'(z))\}' = z\{z(p*f)'(z)\}'$$

$$= z(p*f)'(z) + z^2(p*f)''(z).$$

Thus for each $z \in D$

$$1+z(p*f)''(z)/(p*f)'(z) = p(z)*\{z(zf'(z))'\}/P(z)*(zf'(z)) \in Q(\alpha,\beta),$$

which yields $p*f(z) \in K(\alpha,\beta)$. The proof is completed.

For $\alpha = 0$ and $\beta = 1$, Theorem 2 is the well known Polya-Schoenberg conjecture proved in [4].

COROLLARY 2. $K(\alpha,\beta) \subset S^*(\alpha,\beta)$.

PROOF. If $f(z) \in K(\alpha,\beta)$. Let $p(z) = \log(1-z)^{-1}$ in Theorem 2, we get

$$g(z) = \int_0^z f(t)/t\, dt \in K(\alpha,\beta),$$

which gives that $f(z) = zg'(z) \in S^*(\alpha,\beta)$.

LEMMA 1. $G(z) = k(\alpha,\beta,z)/zk'(\alpha,\beta,z)$ is an analytic and convex univalent function in D. Moreover, $G(z)$ is analytic and univalent on $\bar{D}$ except for $z=1$ when $\beta=1$ for which $\lim_{\substack{z\to 1\\ z\in D}} G(z) = \infty$.

PROOF. We may assume that $\beta \neq \frac{1}{2}$ and $\alpha \neq \frac{1}{2}/\beta$ since the convexity for $\beta = \frac{1}{2}$ or $\alpha = \frac{1}{2}/\beta$ can be deduced from the convexity for $\beta \neq \frac{1}{2}$ and $\alpha \neq \frac{1}{2}/\beta$.

From (3.1), we find that

$$G(z) = (2\beta-1+G_1(z))/(2\beta\alpha-1),$$

where

$$G_1(z) = z^{-1}\{(1+(1-2\beta)z)^{2\beta(1-\alpha)/(2\beta-1)}-1\}.$$

So we have

$$G_1(z) + 2\beta(1-\alpha) = 2\beta(1-\alpha)/z \int_0^z G_2(t)dt,$$

where

$$G_2(z) = 1-(1+(1-2\beta)z)^{(1-2\beta\alpha)/(2\beta-1)}.$$

Hence

$$1+zG_2''(z)/G_2'(z) = (1-(1-2\beta\alpha)z)/(1+(1-2\beta)z),$$

which yields that $G_2(z)$ is a convex univalent function. $G_1(z) + 2\beta(1-\alpha)$ is also convex follows from a result due to Libera [5]. Thus $G_1(z)$ is convex, and so is $G(z)$. This results in the conclusions as desired. When $\beta = 1$, we can get the other result easily. We come to the end of our proof.

THEOREM 3. Let $f(z) \in K(\alpha,\beta)$, we have sharp subordination

$$zf'(z)/f(z) \prec zk'(\alpha,\beta,z)/k(\alpha,\beta,z). \tag{3.4}$$

To prove Theorem 3, we need the following lemma due to Miller and Mocanu [6].

LEMMA A. Let $q(z) = a+q_1z+\dots$ be regular and univalent on $\bar{D}$ except for those points $\zeta \in \partial D$ for which $\lim_{z\to\zeta, z\in D} q(z) = \infty$, and let $p(z) = a+p_1z+\dots$ be analytic in D with $p(z) \not\equiv a$. If there exists a point $z_o \in D$ such that $p(z_o) \in q(\partial D)$ and $p(|z| < |z_o|) \subset q(D)$. Then

$$z_op'(z_o) = m\zeta q'(\zeta),$$

where $q^{-1}(p(z_o)) = \zeta = e^{i\theta}$ and $m \geq 1$.

PROOF OF THEOREM 3. Let $g(z) = f(z)/zf'(z)$, $G(z) = k(\alpha,\beta,z)/zk'(\alpha,\beta,z)$ and $H(z) = 1/G(z)$. The required result is equivalent to that

$$g(z) \prec G(z). \tag{3.5}$$

It is clear that (3.5) is to be the case if $g(z) \equiv 1$. So we assume that $g(z) \not\equiv 1$ next. We can easily check that

$$1+zf''(z)/f'(z) = 1/g(z)-zg'(z)/g(z),$$

$$1/G(z)-zG'(z)/G(z) = (1+(1-2\beta\alpha)z)/(1+(1-2\beta)z).$$

If (3.5) is not true, then by using Lemma 1 and Lemma A, there exists $z_o \in D$ such that

$$z_og'(z_o) = m\zeta G'(\zeta),\ g(z_o) = G(\zeta),$$

where $|\zeta| = 1$ and $m \geq 1$. Thus we have

$$1+z_of''(z_o)/f'(z_o) = 1/G(\zeta)-m\zeta G'(\zeta)/G(\zeta)$$

$$= m(1/G(\zeta)-\zeta G'(\zeta)/G(\zeta))-(m-1)/G(\zeta)$$

$$= m(1+(1-2\beta\alpha)\zeta)/(1+(1-2\beta)\zeta)-(m-1)H(\zeta). \tag{3.6}$$

From Corollary 2, we know that $k(\alpha,\beta,z) \in S^*(\alpha,\beta)$, which gives that

$$H(\zeta) \in \overline{Q(\alpha,\beta)}. \tag{3.7}$$

For $\beta = 1$, (3.7) is equivalent to that $\mathrm{Re}H(\zeta) \geq \alpha$. Thus (3.6) implies

$$\mathrm{Re}(1+z_of''(z_o)/f'(z_o)) = m\mathrm{Re}\{(1+(1-2\alpha)\zeta)/(1-\zeta)\} - (m-1)\mathrm{Re}H(\zeta)$$

$$\leq m\alpha - (m-1)\alpha = \alpha,$$

which contradicts that $f(z) \in K(\alpha,\beta)$.

For $\beta = \frac{1}{2}$, (3.7) becomes $|H(\zeta)-1| \leq 1-\alpha$. And it follows from (3.6)

$$|z_of''(z_o)/f'(z_o)| = |m(1+(1-\alpha)\zeta)-1-(m-1)H(\zeta)|$$

$$= |m(1-\alpha)\zeta-(m-1)(H(\zeta)-1)| \geq m(1-\alpha)-(m-1)(1-\alpha) = 1-\alpha,$$

which is impossible since $f(z) \in K(\alpha,\frac{1}{2})$.

For $\beta \neq \frac{1}{2},1$, (3.7) is the same as

$$|H(\zeta)-\alpha-(1-\alpha)/2(1-\beta)| \leq (1-\alpha)/2(1-\beta).$$

We get from (3.6) that

$$|1+z_o f''(z_o)/f'(z_o)-\alpha-\tfrac{1}{2}(1-\alpha)/(1-\beta)|$$

$$= |m(1+(1-2\beta\alpha)\zeta)/(1+(1-2\beta)\zeta)-\alpha-\tfrac{1}{2}(1-\alpha)/(1-\beta)-(m-1)H(\zeta)|$$

$$\geq m|(1+(1-2\beta\alpha)\zeta)/(1+(1-2\beta)\zeta)-\alpha-\tfrac{1}{2}(1-\alpha)/(1-\beta)|$$

$$-(m-1)|H(\zeta)-\alpha-\tfrac{1}{2}(1-\alpha)/(1-\beta)|$$

$$\geq m\tfrac{1}{2}(1-\alpha)/(1-\beta)-(m-1)\tfrac{1}{2}(1-\alpha)/(1-\beta)=\tfrac{1}{2}(1-\alpha)/(1-\beta),$$

which is also impossible. This completes the proof of Theorem 3.

For $\beta = 1$, (3.4) was first verified by MacGregor [7]. Our proof is much simpler than that in [7].

THEOREM 4. Let $f(z) = z+a_2z^2+\ldots$ be analytic in D. Then $f(z) \in K(\alpha,\beta)$ if and only if

$$\frac{1}{z}\ \{f * \frac{z+z^2(\beta+\beta\alpha-1+x)/\beta(1-\alpha)}{(1-z)^3}\} \neq 0 \qquad (|z|<1,\ |x| = 1).$$

PROOF. We only prove the result for $\beta < 1$. The result for $\beta = 1$ can be deduced from that for $\beta < 1$ by letting β tend to 1.

We know $f(z) \in K(\alpha,\beta)$ if and only if $1+zf''(z)/f'(z) \in Q(\alpha,\beta)$ $(z \in D)$. Since $1+zf''(z)/f'(z) = 1$ at $z = 0$, $1+zf''(z)/f'(z) \in Q(\alpha,\beta)$ is equivalent to

$$1+zf''(z)/f'(z) \neq \alpha+\tfrac{1}{2}(1-\alpha)(1+y)/(1-\beta) \qquad (|y| = 1),$$

which simplifies to

$$zf''(z)+f'(z)\tfrac{1}{2}(1-\alpha)(1-2\beta-y)/(1-\beta) \neq 0 \qquad (|y| = 1). \tag{3.8}$$

As

$$f'(z) = \frac{f(z)}{z} * \frac{1}{(1-z)^2},$$

$$zf''(z) = \frac{f(z)}{z} * \frac{2z}{(1-z)^3}.$$

We have

$$zf''(z)+f'(z)\tfrac{1}{2}(1-\alpha)(1-2\beta-y)/(1-\beta)$$

$$= \frac{f(z)}{z} * \{2z/(1-z)^3+(1-z)^{-2}\tfrac{1}{2}(1-\alpha)(1-2\beta-y)/(1-\beta)\}$$

$$= \tfrac{1}{2}(1-\alpha)(1-2\beta-y)(1-\beta)^{-1}\ \frac{f(z)}{z} * \{\frac{1}{(1-z)^3}\ (1+(4(1-\beta)/(1-\alpha)(1-2\beta-y)-1)z)\}$$

It is not difficult to verify that $\frac{1}{2}(1-\alpha)(1-2\beta-y)/(1-\beta) \neq 0$ and $2(1-\beta)/(1-2\beta-y) = 1-\frac{1}{2}(1-x)/\beta$ is a homotopic mapping from $|y| = 1$ to $|x| = 1$. Thus (3.8) is equivalent to

$$\frac{f(z)}{z} * \frac{1+z(\beta+\beta\alpha-1+x)/(1-\alpha)\beta}{(1-z)^3} \neq 0 \qquad (|x| = 1),$$

which is the same as the result desired. This completes the proof of Theorem 4.

For β=1, Theorem 4 was first given in [8].

4. APPLICATIONS.

With the help of principle of subordination, we can get the following results from Theorem 1 and Theorem 3. Here we omit their proof.

THEOREM 5. Let $f(z) \in K(\alpha,\beta)$ and $|z| = r < 1$, then we obtain sharp estimates.

$$2\beta(1-\alpha)r/(1+|1-2\beta|r) \leqq |zf''(z)/f'(z)| \leqq 2\beta(1-\alpha)r/(1-|1-2\beta|r),$$

$$|\arg(1+zf''(z)/f'(z))| \leqq \arcsin\{2\beta(1-\alpha)r/(1-(1-2\beta)(1-2\beta\alpha)r^2)\},$$

$$(1+(2\beta-1)r)^{2\beta(1-\alpha)/(1-2\beta)} \leqq |f'(z)| \leqq (1-(2\beta-1)r)^{2\beta(1-\alpha)/(1-2\beta)} \qquad (\beta \neq \tfrac{1}{2}),$$

$$e^{-(1-\alpha)r} \leqq |f'(z)| \leqq e^{(1-\alpha)r} \qquad (\beta = \tfrac{1}{2}),$$

$$|\arg f'(z)| \leqq (1-2\beta)^{-1}2\beta(1-\alpha)\arcsin(1-2\beta)r \qquad (\beta \neq \tfrac{1}{2}),$$

$$|\arg f'(z)| \leqq (1-\alpha)r \qquad (\beta = \tfrac{1}{2}),$$

$$\min_{|z|=r} |zk'(\alpha,\beta,z)/k(\alpha,\beta,z)| \leqq |zf'(z)/f(z)| \leqq rk'(\alpha,\beta,r)/k(\alpha,\beta,r),$$

$$|\arg\{zf'(z)/f(z)\} \leqq \max_{|z|=r} \arg\{zk'(\alpha,\beta,z)/k(\alpha,\beta,z)\}.$$

Using a traditional method, we get from Theorem 5 the following

COROLLARY 3. Let $f(z) \in K(\alpha,\beta)$ and $|z| = r < 1$, then we have sharp inequality

$$-k(\alpha,\beta,-r) \leqq |f(z)| \leqq k(\alpha,\beta,r).$$

Theorem 3 also has an application of getting the sharp order of starlikeness for the functions in $K(\alpha,\beta)$.

COROLLARY 4. If $f(z) \in K(\alpha,\beta)$, $|z| = r < 1$. Then

$$\mathrm{Re}\{zf'(z)/f(z)\} \geqq \min_{|z|=r} \mathrm{Re}\{zk'(\alpha,\beta,z)/k(\alpha,\beta,z)\}.$$

In particular, we have $f(z) \in S^*(s(\alpha,\beta))$, where

$$s(\alpha,\beta) = \inf_{|z|<1} \mathrm{Re}\{zk'(\alpha,\beta,z)/k(\alpha,\beta,z)\} > \alpha .$$

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