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THE SMIRNOV COMPACTIFICATION AS A QUOTIENT SPACE OF THE STONE-CECH COMPACTIFICATION

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ABSTRACT. For a separated proximity space, a decomposition of the Stone-Čech compactification is presented which produces the Smirnov compactification and its basic properties by elementary arguments without recourse to clusters or totally bounded uniformities.

KEY WORDS AND PHRASES. Proximity space, compactification, quotient space.
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1. INTRODUCTION.

It has long been recognised that

(i) every T_2 compactification of a $T_{3\frac{1}{2}}$ topological space can be obtained as a quotient space of its Stone-Čech extension, and

(ii) every (separated) proximity space can be densely embedded in a compact proximity space, its Smirnov compactification;

see, for example, [1] and [2]. The purpose of this note is to present an explicit construction whereby the Smirnov compactification can, as is implicit in the above results, be derived from the Stone-Čech. Since it is markedly simpler than the constructions usually employed, the procedure has pedagogical utility in addition to its intrinsic interest; the author has found it to be of considerable convenience in giving a brief introduction to proximity space theory to final year undergraduates who had completed a course in general topology.

2. CONSTRUCTION.

Given a separated proximity space (X,δ), with associated $T_{3\frac{1}{2}}$ topological space $(X,\tau(\delta))$ regarded as a (topological) subspace of its Stone-Čech compactification βX, let $\overline{S}$ and $\mathrm{int}(S)$ denote the closure and interior <u>in the space βX</u> of a subset S (of X or of βX). Recall the notation $A \ll B$ to mean $A \not\delta X \setminus B$ (for subsets A, B of X). The construction proceeds by identifying points of βX whenever they are indistinguishable (in a natural sense) from within (X,δ). We begin by observing the following result, generally obtained as a consequence of the Smirnov compactification (see, for example, [2, Theorem 7.12]), but which to avoid circularity can be obtained by an argument like that which establishes Urysohn's lemma.

LEMMA 1. If $A \,\bar{\delta}\, B$ then there is a continuous mapping $f\colon X \to [0,1]$ taking the values 0 and 1 throughout A and B, respectively.

PROPOSITION 1. The binary relation $\sim$ defined on βX thus:

$p \sim q$ if and only if there do not exist subsets A,B of X such that $p \in \bar{A}$, $q \in \bar{B}$, $A \,\bar{\delta}\, B$ is an equivalence relation.

PROOF. Reflexivity follows from Lemma 1 since the continuous extension over βX of such an f will map $\bar{A}$ and $\bar{B}$ to 0 and 1, implying $\bar{A} \cap \bar{B} = \phi$. Symmetry is immediate. For transitivity, suppose if possible that $p \sim q$, $q \sim r$ and $p \not\sim r$, and choose subsets A,C and B of X so that $p \in \bar{A}$, $r \in \bar{C}$, $A \,\bar{\delta}\, C$, $A \,\bar{\delta}\, B$, $X \setminus B \,\bar{\delta}\, C$. Since $q \in \bar{B} \cup \overline{X \setminus B}$ this contradicts either $p \sim q$ or $q \sim r$.

Now for each $p \in \beta X$ denote by $\theta(p)$ the equivalence class containing p, and by σX the set of all these equivalence classes, so that θ becomes a mapping from βX onto σX. Give σX the quotient topology induced by θ, and we have immediately that

$$\theta \text{ is continuous, } \sigma X \text{ is compact, } \theta(X) \text{ is dense in } \sigma X. \tag{2.1}$$

In the investigation of this quotient space it will be helpful to know that θ is closed mapping and thus the decomposition is upper semi-continuous, which is the point of Lemma 4 below. We first establish an alternative characterization (Lemma 3) of the relation $\sim$.

LEMMA 2. If $A \ll B$ in (X,δ) then $\bar{A} \subseteq \operatorname{int}(\bar{B})$ in βX.

PROOF. This is almost immediate from Lemma 1.

LEMMA 3. For p, $q \in \beta X$,

$p \not\sim q$ if and only if there are neighbourhoods N_p of p, N_q of q (in βX) such that $N_p \cap X \,\bar{\delta}\, N_q \cap X$.

PROOF. If such neighbourhoods exist then $p \in \overline{N_p \cap X}$ and $q \in \overline{N_q \cap X}$, hence $p \not\sim q$. Conversely if $p \not\sim q$ choose A, $B \subset X$ so that $p \in \bar{A}$, $q \in \bar{B}$ and $A \,\bar{\delta}\, B$. Using [2, Cor. 3.5 and Lemma 2.8] we may find closed subsets C,D of X such that $A \ll C$, $B \ll D$ and $C \,\bar{\delta}\, D$: then Lemma 2 shows that $\bar{C}$ and $\bar{D}$ are neighbourhoods of p and q whose traces on X are not δ-related.

LEMMA 4. Let A be a closed subset of βX; then so is $\theta^{-1}(\theta(A))$.

PROOF. If not, then there is a point u in the closure of $\theta^{-1}(\theta(A))$ with the property that for each $a \in A$, $\theta(u) \neq \theta(a)$: so that by Lemma 3 we can find open neighbourhoods U_a of u and N_a of a with $U_a \cap X \,\bar{\delta}\, N_a \cap X$. Now the open cover $\{N_a : a \in A\}$ of compact A has a finite subcover, say $\{N_{a(1)}, N_{a(2)}, \ldots, N_{a(n)}\}$; and the neighbourhood $U_{a(1)} \cap U_{a(2)} \cap .. \cap U_{a(n)}$ of u must intersect $\theta^{-1}(\theta(A))$ in at least one point v, where $v \sim a'$ for some $a' \in A$. Then (for some j between 1 and n) $a' \in N_{a(j)}$, so that $U_{a(j)}$ and $_{A(j)}$ are neighbourhoods of v and a', repectively, whose traces on X are not δ-related, giving the contradiction $v \not\sim a'$.

Standard quotient-space results obtain from Lemma 4 the following, where cl denotes closure <u>in the space σX:</u>

θ is closed, σX is T_2, and for each subset A of βX we have $\theta(\bar{A}) = cl(\theta(A))$. (2.2)

Being a compact T_2 space by (2.1) and (2.2), σX possesses a unique compatible proximity, the relation Δ between its subsets given by

$C \,\Delta\, D$ if and only if $cl(C) \cap cl(D) \neq \phi$.

It remains to examine the way in which θ embeds (X,δ) into $(\sigma X,\Delta)$, beginning with the following observation which establishes that θ acts injectively on X:

LEMMA 5. For each $x \in X$, $\theta(x) = \{x\}$.

PROOF. Consider any z in βX distinct from x. If we choose a closed neighbourhood Z of z not including x, then $X \cap (\beta X \setminus Z)$ is an open neighbourhood in X of x, so $\{x\} \not\delta\; X \setminus (X \cap (\beta X \setminus Z)) = X \cap Z$. Since $x \in \overline{\{x\}}$ and $z \in \overline{X \cap Z}$ this gives $x \not\sim z$.

The final verificational step in the construction is to check that θ is a proximity-isomorphism between (X,δ) and the proximity subspace $\theta(X)$ of $(\sigma X,\Delta)$:

PROPOSITION 2. For subsets A, B of X,

$A \,\delta\, B$ if and only if $cl(\theta(A)) \cap cl(\theta(B)) \neq \phi$.

PROOF. If there exists y in $cl(\theta(A)) \cap cl(\theta(B))$ then (2.2) shows that we can find $p \in \bar{A}$, $q \in \bar{B}$ such that $y = \theta(p) = \theta(q)$; and since $p \sim q$ we get $A \,\delta\, B$.

Conversely, suppose that $A \,\delta\, B$. We observe that the family of sets $\{A \cap C : C \gg B\}$ possesses the finite intersection property, whence the compactness of βX guarantees that it contains a point p which is common to their closures. For each neighbourhood N of p in βX, $N \cap X \,\delta\, B$ (since otherwise $B \ll X \setminus N$, and the choice of p yields a contradiction). It follows that the family

$\{B \cap M : M \gg N \cap X,$ N a variable neighbourhood of p$\}$

also possesses the finite intersection property. A second appeal to compactness produces $q \in \beta X$ common to their closures. Thus

each neighbourhood of q meets every such set $B \cap M$. (2.3)

Now if p,q were not $\sim$-related we would be able to find neighbourhoods P,Q (respectively) of them such that $P \cap X \not\delta\; Q \cap X$; however, this gives us $X \setminus Q \gg P \cap X$ from which (2.3) produces the contradiction that Q intersects $B \cap (X \setminus Q)$. Hence $p \sim q$ i.e. $\theta(p) = \theta(q)$. Since $p \in \bar{A}$ and (via (2.3)) $q \in \bar{B}$ we now see using (2.2) that

$\theta(p) \in \theta(\bar{A}) \cap \theta(\bar{B}) = cl(\theta(A)) \cap cl(\theta(B))$.

Summarizing, we have seen that σX is a compact (separated) proximity space possessing a dense subspace which is isomorphic to X; that is,

THEOREM. $(\sigma X,\Delta)$ is the Smirnov compactification of (X,δ).

NOTE. The above procedure, in addition to constructing the Smirnov compactification, provides a convenient base from which to establish its fundamental properties. For example, let there be given a proximity mapping f from (X,δ) into a compact separated proximity space (Y,δ'); and denote by f^* the continuous extension of f over βX. It is routine to confirm that the formula

$$f^{\sigma}(\theta(x)) = f^*(x)$$

gives a well-defined and continuous mapping f^{σ} from σX to Y, so f has a proximity mapping extension over σX. The essential uniqueness of the Smirnov compactification can be proved merely by checking that if (Σ,δ'') is any compact separable proximity space containing X as a dense subspace then the extension over σX of the inclusion of X in Σ is injective; and virtually the same argument shows that, given a T_2 compactification γX of a topological space X, the Smirnov compactification of X under the proximity induced by γX is indistinguishable from γX itself: whence the one-to-one correspondence between compatible proximities and T_2 compactifications follows.

REFERENCES

1. KELLEY, J.L. General Topology, Van Nostrand, 1955.
2. NAIMPALLY, S.A. and WARRACK, B.D. Proximity Spaces, Cambridge University Press, 1970.