Internat. J. Math. & Math. Sci.
VOL. 11 NO. 2 (1988) 343-350

SOME RESULTS CONCERNING EXPONENTIAL DIVISORS

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(Received November 18, 1986 and in revised form February 9, 1987)

ABSTRACT. If the natural number n has the canonical form $p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r}$ then $d = p_1^{b_1}p_2^{b_2}\cdots p_r^{b_r}$ is said to be an exponential divisor of n if $b_i|a_i$ for $i = 1,2,\ldots,r$. The sum of the exponential divisors of n is denoted by $\sigma^{(e)}(n)$. n is said to be an e-perfect number if $\sigma^{(e)}(n) = 2n$; $(m;n)$ is said to be an e-amicable pair if $\sigma^{(e)}(m) = m+n = \sigma^{(e)}(n)$; $n_0,n_1,n_2,\ldots$ is said to be an e-aliquot sequence if $n_{i+1} = \sigma^{(e)}(n_i)-n_i$. Among the results established in this paper are: the density of the e-perfect numbers is .0087; each of the first 10,000,000 e-aliquot sequences is bounded.

KEYS WORDS AND PHRASES. Exponential divisors, e-perfect numbers, e-amicable numbers, e-aliquot sequences.
1980 AMS SUBJECT CLASSIFICATION CODE. 10A20.

1. INTRODUCTION.

If n is a positive integer greater than one whose prime-power decomposition is given by

$$n = p_1^{a_1}p_2^{a_2}\cdots p_r^{a_r} \tag{1.1}$$

then d is said to be an "exponential divisor" of n if $d = p_1^{b_1}p_2^{b_2}\cdots p_r^{b_r}$ where $b_i|a_i$ for $i = 1,2,\ldots,r$. The sum of all of the exponential divisors of n is denoted by $\sigma^{(e)}(n)$. This function was first studied by Subbarao [1] who also initiated the study of exponentially perfect (or e-perfect) numbers.

The positive integer n is said to be an e-perfect number if $\sigma^{(e)}(n) = 2n$. If $\sigma^{(e)}(n) = kn$, where k is an integer which exceeds 2, n is said to be an e-multiperfect number. The properties of e-perfect and e-multiperfect numbers have been investigated by Straus and Subbarao [2] and Fabrykowski and Subbarao [3]. It has been proved, for example, that all e-perfect and e-multiperfect numbers are even. Also, if n is an e-perfect number and $3\nmid n$ then $2^{110}|\ n$ and $n > 10^{618}$.

While it is easy to show that there are an infinite number of e-perfect numbers, whether or not any e-multiperfect numbers exist is still an open question. Subbarao, Hardy and Aiello [4] have conjectured that there are no e-multiperfect numbers. They have proved that any which exist are very large.

In Section 2 of the present paper the density of the set of e-perfect numbers is investigated. Section 3 is devoted to a study of e-amicable pairs, integers m and n such that $\sigma^{(e)}(m) = m+n = \sigma^{(e)}(n)$. Finally, e-aliquot sequences $n_0, n_1, n_2, \ldots$ where $n_{i+1} = \sigma^{(e)}(n_i) - n_i$ for $i = 0,1,2,\ldots$ are studied in Section 4.

2. THE DENSITY OF THE e-PERFECT NUMBERS.

By definition, $\sigma^{(e)}(1) = 1$ and it is easy to see that $\sigma^{(e)}(n)$ is multiplicative. Therefore, since $\sigma^{(e)}(p) = p$ if p is a prime, we see that $\sigma^{(e)}(m) = m$ if m is squarefree.

Now suppose that n, as given by (1.1), is a powerful e-perfect number (so that $a_i \geq 2$ for $i = 1,2,\ldots,r$ and $\sigma^{(e)}(n) = 2n$). Then if $(m,n) = 1$ and m is squarefree then $\sigma^{(e)}(mn) = 2mn$ so that mn is an e-perfect number. Therefore, if x is a (fixed) positive number and $n_1 < n_2 < \ldots < n_s$ are the powerful e-perfect numbers which do not exceed x then E(x), the set of (all) e-perfect numbers less than or equal to x, is given by $E(x) = \bigcup_{i=1}^{s} A_i$ where

$$A_i = \{mn_i : (m,n_i) = 1,\ m \leq x/n_i \text{ and m is squarefree}\} \tag{2.1}$$

Let N be a positive integer and let X be a positive real number. If Q(N,X) is the number of positive, squarefree integers which do not exceed X and which are relatively prime to N, then E. Cohen (Lemma 5.2 in [5]) has shown that

$$Q(N,X) = \beta(N)\cdot X + O(\theta(N)\cdot X^{1/2}) \tag{2.2}$$

where $\beta(N) = (\zeta(2) \prod_{p|N} (1+1/p))^{-1}$ and $\theta(N)$ is the number of squarefree divisors of N. It is easy to see that $\theta(N) = \prod_{p|N} 2$. $\zeta(k)$ is the Riemann Zeta function, so that $\zeta(2) = \pi^2/6$, and the constant implied by the O-term is independent of N and X.

If Q(e,x) is the number of e-perfect numbers which do not exceed x (so that Q(e,x) is the cardinality of E(x)) it follows from (2.1) and (2.2) that

$$Q(e,x) = x \sum_{i=1}^{s} \beta(n_i)/n_i + O(x^{1/2} \sum_{i=1}^{s} \theta(n_i/n_i^{1/2}).$$

Therefore,

$$Q(e,x)/x = \sum_{i=1}^{s} \beta(n_i)/n_i + O(x^{-1/2} \sum_{i=1}^{s} \theta(n_i)/n_i^{1/2}). \tag{2.3}$$

The following results concerning powerful numbers will be needed in what follows. Proofs may be found in Golomb [6].

LEMMA 1. If $r_1 < r_2 < \ldots$ is the sequence of powerful numbers then $\sum_{i=1}^{\infty} 1/r_i$ is convergent.

LEMMA 2. If P(X) is the number of powerful numbers not exceeding x then $P(x) < 2.2x^{1/2}$ for large x.

Now let ε be a given positive number and let P_i denote the ith prime. There exists a positive integer k such that

$$2/P_k < \varepsilon \cdot (2.2K)^{-1}/3 \tag{2.4}$$

where K is the constant implied by the O-term in (2.3).

Since there are only a finite number of powerful e-perfect numbers which are divisible by fewer that k distinct primes (see Theorem 2.3 in [2]) there exists a positive integer J such that if $n_1 < n_2 < \dots$ is the sequence of powerful e-perfect numbers then for all $i > J$ n_i has at least k distinct prime factors and n_i has a prime factor, say Q_i, such that $Q_i \geq P_k$. Since n_i is powerful, $n_i^{1/2} \geq \Pi p$ where the product is taken over the distinct prime factors of n_i, and it follows from (2.4) that if $i > J$ then

$$\theta(n_i)/n_i^{\frac{1}{2}} \leq \prod_{p|n_i} 2/p < 2/Q_i \leq 2/P_k < \varepsilon\cdot(2.2K)^{-1}/3. \tag{2.5}$$

Splitting the sum in the O-term in (2.3) at $i = J$ (with J held fixed) we can take x large enough so that $x^{-1/2}\cdot K\cdot \sum_{i=1}^{J} \theta(n_i)/n_i^{1/2} < \varepsilon/3$. At the same time, since every n_i is powerful, we see from (2.5) and Lemma 2 that we can also take x large enough so that

$$x^{-1/2}\cdot K\cdot \sum_{i=J+1}^{s} \theta(n_i)/n_i^{1/2} < x^{-1/2}\cdot K\cdot \sum_{i=J+1}^{s} \varepsilon\cdot(2.2K)^{-1}/3$$

$$< x^{-1/2}\cdot P(x)\cdot\varepsilon\cdot(2.2)^{-1}/3 < \varepsilon/3.$$

Finally, since $\beta(n_i) < 1$ and every n_i is powerful we see from Lemma 1 that $\sum_{i=1}^{\infty} \beta(n_i)/n_i$ is convergent. (This series may be finite since whether or not the set of powerful e-perfect numbers is finite or infinite is an open question). It follows that we can take x (and consequently s) large enough so that the tail of this series is less that $\varepsilon/3$. Therefore, from (2.3) we have for all large values of x,

$$\left|Q(e,x)/x - \sum_{i=1}^{\infty} \beta(n_i)/n_i\right| < \varepsilon\ . \tag{2.6}$$

We have proved

THEOREM 1. Let Q(e,x) denote the number of e-perfect numbers which do not exceed x and let $n_1 < n_2 < n_3 < \dots$ be the sequence of powerful numbers. Then

$$\lim_{x\to\infty} Q(e,x)/x = \sum_{i=1}^{\infty} \beta(n_i)/n_i = C$$

where $\beta(n) = 6\pi^{-2}\prod_{p|n}(1+1/p)^{-1}$. Correct to ten decimal places, C = .0086941940.

(There are eight powerful e-perfect numbers less than 10^{10}: 36; 1800; 2700; 17,424; 1,306,800; 4,769,856; 238,492,800; 357,739,200. The approximate value of C given above was calculated using these eight numbers).

The "theoretical" density of the e-perfect numbers as given in Theorem 1 agrees very nicely with the following exact computational results: $Q(e,10^5)/10^5 = .00871$; $Q(e,10^6)/10^6 = .008690$; $Q(e,10^7)/10^7 = .0086940$; $Q(e,10^8)10^8 = .00869417$.

3. EXPONENTIALLY AMICABLE NUMBERS.

We shall say that m and n are exponentially amicable (or e-amicable) numbers if

$$\sigma^{(e)}(m) = m + n = \sigma^{(e)}(n). \tag{3.1}$$

LEMMA 3. If $(m;n)$ is an e-amicable pair and p is a prime, then $p|m$ if and only if $p|n$.

PROOF. Suppose that $p^a||m$ where $a \geq 1$. Then $p|\sigma^{(e)}(m)$ since $p|\sigma^{(e)}(p^a)$ and $\sigma^{(e)}$ is a multiplicative function. It is now obvious from (3.1) that $p|n$. By the same argument, if $p|n$ then $p|m$.

COROLLARY 3.1. If $(m;n)$ is an e-amicable pair then $m \equiv n \pmod 2$.

If $(m;n)$ is an e-amicable pair and there is no prime p such that $p||m$ and $p||n$ we shall say that m and n are primitive e-amicable numbers. It is easy to see that if $(m;n)$ is a primitive e-amicable pair and r is a squarefree positive integer such that $(m,r) = 1$, then $(rm;rn)$ is an amicable pair.

A search was made for all primitive e-amicable pairs $(m;n)$ such that $m < n$ and $m < 10^7$. The search required about 1.5 hours on the CDC CYBER 750 and three pairs were found. They are as follows: $(2^2 3^2 7 \cdot 19^2; 2^2 3^3 7^2 19)$; $(2^2 3^2 7 \cdot 61^2; 2^2 3^4 7^2 61)$; $(2^3 3^2 5^2 7 \cdot 19^2; 2^3 3^3 5^2 7^2 19)$.

This list suggests the following questions. Are there any odd e-amicable numbers? Are there any powerful e-amicable numbers? Is every e-amicable number divisible by at least four distinct primes? (It is easy to show that every e-amicable number has at least three different prime factors).

The following result can sometimes be used to generate new e-amicable pairs from known pairs.

THEOREM 2. Suppose that $(aM;aN)$ is an e-amicable pair such that $(a,M) = (a,N) = 1$. If $(b,M) = (b,N) = 1$ and $\sigma^{(e)}(a)/a = \sigma^{(e)}(b)/b$ then (bM,bN) is an e-amicable pair.

PROOF. $\sigma^{(e)}(bM) = \sigma^{(e)}(b) \cdot \sigma^{(e)}(M) = a^{-1}b\sigma^{(e)}(a) \cdot \sigma^{(e)}(M) = a^{-1}b\sigma^{(e)}(aM) = \sigma$ $a^{-1}b(aM + aN) = bM + bN$. Similarly, $\sigma^{(e)}(bN) = bM + bN$.

The results of a computer search for powerful numbers a and b such that $4 \leq a < b \leq 10000$ and $\sigma^{(e)}(a)/a = \sigma^{(e)}(b)/b$ are given in Table I.

TABLE I

$\sigma^{(e)}(a)/a$	a	b
3/2	2^2	$2^3 5^2$ or $2^4 11^2$
4/3	3^2	$3^3 5^2$
2	$2^2 3^2$	$2^3 3^5 5^2$ or $2^2 3^3 5^2$
39/32	2^6	$2^7 5^2$
5/3	$2^3 3^2$	$2^2 3^3$ or $2^3 3^3 5^2$
12/7	$2^2 7^2$	$2^3 5^2 7^2$
65/48	$2^7 3^2$	$2^6 3^3$
40/21	$2^3 3^2 7^2$	$2^2 3^3 7^2$

EXAMPLE. Since $(2^2 \cdot 3^2 \cdot 7 \cdot 19^2; 2^2 \cdot 3^3 \cdot 7^2 \cdot 19)$ is an e-amicable pair and since $\sigma^{(e)}(2^2)/2^2 = \sigma^{(e)}(2^4 \cdot 11^2)/2^4 \cdot 11^2$ it follows from Theorem 2 that $(2^4 \cdot 11^2 \cdot 3^2 \cdot 7 \cdot 19^2; 2^4 \cdot 11^2 \cdot 3^3 \cdot 7^2 \cdot 19)$ is an e-amicable pair.

4. EXPONENTIAL ALIQUOT SEQUENCES.

The function $s^{(e)}$ is defined by $s^{(e)}(n) = \sigma^{(e)}(n) - n$, the sum of the exponential aliquot divisors of n. $s^{(e)}(1) = s^{(e)}(r) = 0$ for every squarefree number r and we define $s^{(e)}(0) = 0$. A t-tuple of distinct natural numbers $(n_0;n_1;\ldots;n_{t-1})$ with $n_i = s^{(e)}(n_{i-1})$ for $i = 1,2,\ldots, t-1$ and $s^{(e)}(n_{t-1}) = n_0$ is called an exponential t-cycle. An exponential 1-cycle is an e-perfect number and an exponential 2-cycle is an e-amicable pair. A search was made for all exponential t-cycles with smallest member not exceeding 10^7. None with $t > 2$ was found.

The exponential aliquot sequence (or e-aliquot sequence) $\{n_i\}$ with leader n is defined by $n_0 = n, n_1 = s^{(e)}(n_0), n_i = s^{(e)}(n_{i-1}),\ldots$. Such a sequence is said to be terminating if n_k is squarefree for some index k (so that $n_i = 0$ for $i > k$). An exponential aliquot sequence is said to be periodic if there is an index k such that $(n_k;n_{k+1};\ldots;n_{k+t-1})$ is an exponential t-cycle. An e-aliquot sequence which is neither terminating nor periodic is unbounded.

An investigation was made of all aliquot sequences with leader $n \leq 10^7$. About 2.3 hours of computer time was required. 9,896,235 were found to be terminating and 103,765 were periodic (103,694 ended in 1-cycles and 71 ended in 2-cycles).

The fact that the first ten million exponential aliquot sequences are bounded might tempt one to conjecture that the set of unbounded e-aliquot sequences is empty. However, the following theorem shows that e-aliquot sequences exist which contain arbitrarily long strings of monotonically increasing terms. Therefore, whether or not unbounded e-aliquot sequences exist would seem to be a very open and difficult question.

THEOREM 3. Let N be a positive integer which exceeds 2. Then there exist infinitely many exponential aliquot sequences such that $n_0 < n_1 < n_2 < \ldots < n_{N-2}$.

PROOF. Let $q_1,q_2,\ldots, q_N$ be a sequence of N primes such that $q_1 = 2$, $q_2 = 3$ and $q_i^2 | (q_{i+1} + 1)$ for $i = 2,3,\ldots, N-1$. (Infinitely many such sequences exist since, by Dirichlet's theorem, the arithmetic progression $aq_i^2 - 1$ contains an infinite number of primes.) We shall write $q_{i+1} + 1 = K_i q_i^2$.

Now let $n_0,n_1,n_2,\ldots$ be the exponential aliquot sequence with leader n_0 given by $n_0 = q_1^2 q_2^2 \ldots q_N^2$. Then

$$\sigma^{(e)}(n_0) = \prod_{i=1}^{N} (q_i + q_i^2) = 3 \cdot q_1q_2 \ldots q_N \cdot \prod_{i=2}^{N} (1 + q_i)$$

$$= 3 \cdot q_1q_2 \ldots q_N \cdot \prod_{i=1}^{N-1} K_iq_i^2 ,$$

and

$$n_1 = \sigma^{(e)}(n_0) - n_0 = (3 \cdot q_1q_2 \ldots q_N \cdot K_1 \ldots K_{N-1} - q_N^2) \cdot \prod_{i=1}^{N-1} q_i^2 .$$

Therefore, $n_1 = M_1 \prod_{i=1}^{N-1} q_i^2$ where $(M_1,q_i) = 1$ for $i = 1,2,\ldots, N-1$.

Since $n_0/36$ is not squarefree, $n_1 = \sigma^{(e)}(n_0) - n_0 = \sigma^{(e)}(36) \cdot \sigma^{(e)}(n_0/36) - n_0 = 72 \cdot \sigma^{(e)}(n_0/36) - n_0 > 72 \cdot n_0/36 - n_0 = n_0$.

Similarly, we find that for $k = 2,3,\ldots, N-2$

$$n_k = M_k \prod_{i=1}^{N-k} q_i^2 \text{ where } (M_k,q_i) = 1 \text{ for } i = 1,2,\ldots, N-k$$

and

$$n_k = \sigma^{(e)}(n_{k-1}) - n_{k-1} = \sigma^{(e)}(36) \cdot \sigma^{(e)}(n_{k-1}/36) - n_{k-1}$$

$$> 72 \cdot n_{k-1}/36 - n_{k-1} = n_{k-1} .$$

Therefore, $n_o < n_1 < \ldots < n_{N-2}$.

REMARK 1. $n_{N-2} = 36M_{N-2}$ where $(6,M_{N-2}) = 1$. If M_{N-2} is not squarefree, then $n_{N-1} = 72 \cdot \sigma^{(e)}(M_{N-2}) - 36M_{N-2} > 72M_{N-2} - 36M_{N-2} = 36M_{N-2} = n_{N-2}$.

REMARK 2. The proof of Theorem 3 is modeled on that of Theorem 2.1 in [7].

Our next objective is to determine $M(\sigma^{(e)}(n)/n)$, the mean value of $\sigma^{(e)}(n)/n$. The mean value of an arithmetic function f is defined by $M(f) = \lim_{N\to\infty} N^{-1} \sum_{n=1}^{N} f(n)$.

We shall need the following lemma due to van der Corput (See Theorem A in [8].)

LEMMA 4. If f and h are arithmetic functions such that $f(n) = \sum_{d|n} h(d)$ and $\sum_{n=1}^{\infty} h(n)/n$ is absolutely convergent then $M(f) = \sum_{n=1}^{\infty} h(n)/n$.

We wish to apply this lemma to the function $f(n) = \sigma^{(e)}(n)/n$. By the Moebius inversion formula, $h(n) = \sum_{d|n} \mu(n/d)\sigma^{(e)}(d)/d$. h is multiplicative and $h(1) = 1$. If p is a prime and a is a positive integer then $h(p^a) = \sigma^{(e)}(p^a)/p^a - \sigma^{(e)}(p^{a-1})/p^{a-1}$. If $a < 6$ it is easy to verify that $|h(p^a)| < p^{-a/4}$. (For example, $|h(p^3)| = p^{-1} - p^{-2} < p^{-1} < p^{-3/4}$.) Suppose that $a \geq 6$. Then $|h(p^a)| = \sigma^{(e)}(p^a)/p^a - \sigma^{(e)}(p^{a-1})/p^{a-1}$ <u>or</u> $|h(p^a)| = \sigma^{(e)}(p^{a-1})/p^{a-1} - \sigma^{(e)}(p^a)/p^a$. Since $\sigma^{(e)}(p^m)/p^m < 1 + p/(p-1)p^{m/2}$ (see [2] or [4])and $\sigma^{(e)}(p^b)/p^b \geq 1$, $|h(p^a)| < p/(p-1)p^{(a-1)/2}$. Since $a \geq 6$ it follows easily that $|h(p^a)| < p^{-a/4}$. Since h is multiplicative, $|h(n)| \leq n^{-1/4}$ for every positive integer n. It follows that $\sum_{n=1}^{\infty} h(n)/n$ is absolutely convergent so that Lemma 4 applies if $f(n) = \sigma^{(e)}(n)/n$.

From Theorem 286 in [9] we have

$$\sum_{n=1}^{\infty} h(n)/n = \prod_{p} \{1 + h(p)/p + h(p^2)/p^2 + \ldots\}$$

$$= \prod_{p} \{1 + p^{-1}(\sigma^{(e)}(p)/p-1) + p^{-2}(\sigma^{(e)}(p^2)/p^2 - \sigma^{(e)}(p)/p) + \ldots\}$$

$$= \prod_{p} \{\sum_{j=0}^{\infty} \sigma^{(e)}(p^j)/p^{2j} - p^{-1} \sum_{j=0}^{\infty} \sigma^{(e)}(p^j)/p^{2j}\}$$

$$= \prod_{p} \{(1 - p^{-1}) \sum_{j=0}^{\infty} \sigma^{(e)}(p^j)/p^{2j}\}.$$

Now the last infinite series can be "split up" by first taking all the terms with numerator p^j to form the series $\sum_{j=0}^{\infty} p^j/p^{2j} = \sum_{j=0}^{\infty} 1/p^j$; then taking the remaining terms with numerators p to form the series $\sum_{j=2}^{\infty} p/p^{2j} = p^{-3}\sum_{j=0}^{\infty}(p^{-2})^j$; then taking the terms with numerators p^2 to form the series $\sum_{j=2}^{\infty} p^2/p^{4j} = p^{-6}\sum_{j=0}^{\infty}(p^{-4})^j$; then taking the terms with numerators p^3 to form the series $\sum_{j=2}^{\infty} p^3/p^{6j} = p^{-9}\sum_{j=0}^{\infty}(p^{-6})^j$; etc. It follows that

$$\sum_{n=1}^{\infty} h(n)/n = \prod_p \{(1 - p^{-1})((1 - p^{-1})^{-1} + p^{-3}(1 - p^{-2})^{-1} + p^{-6}(1 - p^{-4})^{-1} + p^{-9}(1 - p^{-6})^{-1} + \ldots)\}$$

$$= \prod_p \{(1 - p^{-1})((1 - p^{-1})^{-1} + (p^3 - p)^{-1} + (p^6 - p^2)^{-1} + (p^9 - p^3)^{-1} + \ldots)\}$$

$$= \prod_p \{1 + (1 - p^{-1}) \sum_{j=1}^{\infty} (p^{3j} - p^j)^{-1}\}.$$

From Lemma 4 we have

THEOREM 4. $M(\sigma^{(e)}(n)/n) = \prod_p \{1 + (1 - p^{-1}) \cdot \sum_{j=1}^{\infty} (p^{3j} - p^j)^{-1}\} = C.$

Correct to 6 decimal places, C = 1.136571.

(This approximate value of C was calculated using all primes less than 10^6 in the infinite product.)

Since $s^{(e)}(n) = \sigma^{(e)}(n) - n$ we have

COROLLARY 4.1. $M(s^{(e)}(n)/n) = .136571.$

Finally, since $n_{i+1}/n_i = s^{(e)}(n_i)/n_i$ we see that, in some sense, the average value of the ratio of two consecutive non-zero terms of an e-aliquot sequence is about .136571.

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