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EXPOSITORY AND SURVEY ARTICLE

RECENT RESULTS IN HYPERRING AND HYPERFIELD THEORY

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Dedicated to the memory of Joachim Ahrensdorf and of Marc Krasner

ABSTRACT. This survey article presents some recent results in the theory of hyperfields and hyperrings, algebraic structures for which the "sum" of two elements is a subset of the structure. The results in this paper show that these structures cannot always be embedded in the decomposition of an ordinary structure (ring or field) in equivalence classes and that the structural results for hyperfields and hyperrings cannot be derived from the corresponding results in field and ring theory.

KEYWORDS AND PHRASES. Hyperoperation, hyperring, hyperfield, quotient hyperring, partition hyperring.
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1. INTRODUCTION.

The purpose of this survey paper is to present some recent results in hyperring theory. At the same time, it is hoped that it will draw the attention of English speaking mathematicians to the work of Marc Krasner and of his students, who have published, mainly, in French.

The notion of a hyperoperation is a straightforward generalization of the notion of an operation. Given a non empty set S, a hyperoperation maps SxS into the set of all non empty subsets of S. If the hyperoperation is commutative and associative, then it is called a hyperaddition. And if we generalize the usual axioms for addition, we obtain structures that are generalizations of the usual algebraic structures and we call them Abelian hypergroups, hyperrings, hypermodules, and hyperfields (these terms are defined in section 2). The notion of a hypergroup appears at least as early as 1935 in the work of F. Marty [1,2], while the notion of a hyperring was introduced by Krasner [3], who used it as a technical tool in a study of his on the approximation of valued fields. Later, two students of Krasner, Mittas and Stratigopoulos, earned their theses by studying the structure of the hyperrings.

The thesis of Stratigopoulos and the articles based on his thesis that were subsequently published, fueled a discussion on the merits of studying these structures rather than channeling the efforts of mathematicians into more traditional subjects. During this discussion it was noted that all known hyperrings were obtained via a construction that Krasner had introduced. Krasner observed that if R is a ring and G is a subset of R such that (G,•) is a group, then G can define an equivalence relation

P in R. For each r in R, the P equivalence class of r, P(r), is the set of all s in R for which sG=rG. Let R/P stand for the set of all P-equivalence classes in R. Assume that rG=Gr for every r in R and that the unit of G is a unit for R [this last assumption is not strictly necessary but it greatly simplifies the exposition of what follows]. Then P(r)=rG for all r in R and it suffices to define

$$P(r)\#P(s)=\{\ P(t) \mid P(t) \text{ intersects } P(r)+P(s)\ \}$$

in order to introduce a hyperaddition over R/P [note that, under our assumptions, if P(t) intersects P(r)+P(s) then P(t) is a subset of P(r)+P(s)]. As it is implied above, # is the symbol for the hyperoperation and P(r)+P(s) stands by usual convention for the set of all possible sums one can obtain in R by adding a member of P(r) to a member of P(s) [Note that in the definition of P(r)#P(s) above, P(r), P(s), and P(t) are seen alternatively as elements of R/P and as subsets of R]. If we define the product of two equivalence classes P(r) and P(s) as being P(rs), we can easily prove that (R/P,#,•) is a hyperring. But, if all hyperrings could be embedded isomorphically to the type of hyperrings that Krasner exhibited, then some of the elaborate proofs of Stratigopoulos could have been obtained by very straightforward means.

This particular argument admits a mathematical answer, but it was not settled at the time (sixties). In 1980, Massouros proposed a construction that seemed to prove the existence of hyperrings that are not embeddable in quotient hyperrings (Krasner's construction). It was later shown that while Massouros' construction was not a quotient hyperring, it could be embedded in one. Massouros' construction turned out to be a watershed as far as attempts of attacking the above problem were concerned. Up to that point the problem was attacked by trying to induce ring partitions based on a group G such that at least one r in R fails to satisfy rG=Gr. But, a much more natural method of attack is to focus on the hyperaddition, the truly unusual feature in a hyperring. Moreover, Krasner's construction of a quotient hyperring is a special case of a decomposition of a ring into equivalence classes and the problem should also be addressed in the general case.

The author proposed a counting lemma [prop. 2, Section 3] that links, under some preconditions, the cardinalities of P(r), P(s), and P(r)#P(s). And, on the basis of this lemma, constructed hyperrings that are not embeddable in quotient hyperrings [Section 4]. One of the constructed hyperrings is generated by a set of orthogonal multiplicative idempotents and validates the importance of another of Stratigopoulos' results, i.e., a generalization of Jacobson's density theorem to hyperrings [4,5]. In the meantime, Massouros proposed a class of hyperfields which, he claimed, were likely not to be embeddable in quotient hyperfields (a quotient hyperfield is the structure R/P we obtain if R is actually a field). While it was shown that this claim was not true for all members of this class and a counterexample was constructed along the same lines that produced counterexamples for the hyperrings, Massouros [6] succeeded in showing that an infinite subclass of the class he proposed had the desired property. Furthermore, by showing that a Cartesian product of hyperfields is embeddable in a quotient hyperring, if and only if each hyperfield is embeddable in a quotient hyperfield, he provided a different method for creating counterexamples.

2. BASIC DEFINITIONS

The purpose of this section is to introduce the basic definitions and terminology that we will use in the subsequent sections (these definitions were introduced by F. Marty and Marc Krasner, but for ease of reference it is best to consult [7, 8, and

9; see also 10, 11, and 12]). We have:

DEFINITION 1. A **hyperoperation** over a non empty set R is a mapping of RxR into the set of all non empty subsets of R. A **hyperaddition,** to be denoted by "#" in what follows, is a hyperoperation which is commutative and associative. Given that # maps each pair of RxR onto a subset of R, we cannot define associativity unless we state what we mean by A#b when A is a non empty subset of R. Quite clearly, given #, we can extend it over all non-empty subsets of R as follows:

A#B={x¦ there is an a in A and a b in B such that x is in a#b}.

Therefore, associativity is well defined provided that we let A#b and a#B stand for A#{b} and {a}#B, respectively.

DEFINITION 2. A non empty subset M is called **Abelian hypergroup** provided that there is a hyperaddition # over M such that:

(1) There exists an element, called 0 in what follows, such that for every a in M, there is a unique element of M, called -a in what follows, such that 0 is in a#(-a), and

(2) for all a, b, and c in M, if a is in b#c, then b is in a#(-c).

As usual, one can prove that -(-a)=a, that a#0={a} and that 0 is unique.

DEFINITION 3. A subset N of an Abelian hypergroup (M,#) is said to be a **subhypergroup** of M if 0 is in N and if N is an Abelian hypergroup under #.

DEFINITION 4. A **hyperring** is a non empty set equipped with a hyperaddition, "#", and a multiplication, "•", such that (H,#) is an Abelian hypergroup, (H,•) is a semigroup having 0 as an absorbing element (both from the left and from the right), and the multiplication is distributive across the hyperaddition (both from the left and from the right).

Remark: As usual, we denote multiplication by symbol concatenation (ab stands for a "times" b) and we assume that

Ac={ ac ¦ a is in A} and that cA={ ca ¦ a is in A}

(otherwise it makes no sense to speak of distributivity).

DEFINITION 5. A subset h of a hyperring H is called a **subhyperring** of H iff h contains 0 and is a hyperring under the hyperaddition and the multiplication of H.

DEFINITION 6. Let H be a hyperring and M an Abelian hypergroup. M is called a **hypermodule over H,** or an **H hypermodule,** provided that for each a in M and each x in H there is a unique element of M to be denoted ax, such that the following relations hold for all a and b in M and all x and y in H:

(1) (a # b)x = ax # bx

(2) a(x # y) = ax # ay

(3) (ax)y = a(xy), and

(4) 0(M)x=a0(H)=0(M).

Remark that 0(M) and 0(H) represent M's and H's zeroes respectively.

DEFINITION 7. Let M be an H hypermodule. A subset N of M is called an **H hypermodule** of M if N is a subhypergroup of M and NH={ax ¦ a is in N and x is in H} is a subset of N.

DEFINITION 8. An H hypermodule M is called **irreducible** iff its only subhypermodules are {0(M)} and M. It is called **faithful,** iff Mx={0(M)} implies that x=0(H).

DEFINITION 9. A hyperring H is called **primitive,** if there is an H hypermodule M that is irreducible and faithful.

DEFINITION 10. A hyperring H is called a **hyperfield** iff H-{0} is a multiplicative group.

DEFINITION 11. Let (R,+,•) be a ring and G a subset of R. G shall be called a **multiplicative subgroup** of R iff (G,•) is a group. If, in addition, G is such that R=RG and rG=Gr for all r in R, then G shall be called a **normal subgroup** of R. We remark that only rings with an identity element admit normal subgroups.

As we already mentioned, a normal subgroup G of R induces an equivalence relation P in R and a partition of R in equivalence classes which inherits from R a hyperring structure. Hyperrings obtained via this construction are called quotient hyperrings and are denoted R/G. The results that follow answer the following questions:

(1) Are all hyperrings embeddable in quotient hyperrings?

(2) Are all hyperrings generated by a set of orthogonal idempotents embeddable into quotient hyperrings?

(3) Are all primitive hyperrings embeddable into quotient hyperrings? and

(4) Are all hyperfields embeddable in quotient hyperrings?

Actually, as we already mentioned, one can generalize the notion of a quotient hyperring as follows:

Assume that P is a relation in R and that for each r in R, P(r) is the equivalence class to which r belongs. Assume that for all a and b in R P(a)P(b) is a subset of P(ab). Let R/P be the set of all equivalence classes in R and for each subset X of R let the P-closure of X, P(X), be the set of all equivalence classes that intersect X. Clearly , the multiplication in R induces an associative multiplication in R/P provided that the product of any two classes P(a) and P(b) is defined to be P(ab), i.e., the P-closure of their set product in R. Similarly, R's addition induces a commutative hyperoperation & in R/P provided that one defines P(a)&P(b) to be the P-closure of the set sum P(a)+P(b) in R. If P is such that (R/P,&,•) is a hyperring (as a rule it is not), then R/P is called a **partition hyperring**. One can readily verify that (R/P,&,•) is a hyperring iff P satisfies the following conditions:

(a) P(0) is a bilateral ideal of R such that for every a in R a+P(0) is a subset of P(a),

(b) for every a in R P(-a)=-P(a), and

(c) P is such that & is associative and the multiplication in R/P is left and right distributive across &.

Clearly, condition (c) is a restatement of the problem and it would be interesting to derive conditions on P that ensure that (R/P,&,•) is a hyperring.

Under this light Krasner's original construction can be seen as a proof that if P is induced by a normal subgroup G, then R/P inherits from R a hyperring structure.

In the sections that follow we shall show that there are hyperrings that are not embeddable in partition hyperrings and that there are partition hyperrings that are not embeddable in quotient hyperrings. It turns out though, that the class of partition and quotient hyperfields are one and the same.

3. RESULTS ON THE HYPERRING/HYPERFIELD STRUCTURES.

As was already mentioned, Massouros proposed a hyperring that was not isomorphic to a quotient hyperring because it contained more than one unit from the right (a quotient hyperring R/G has a single unit, G's image). Nevertheless, a quotient hyperring R/G can have subhyperrings that do not contain G's image. Thus, such a subhyperring h could very well contain more than one unit, either from the left or from the right (evidently, not both!). Indeed, Massouros' construct was to consider a ring A with an identity element, 1, and to define a ring R, R=AxA, in which the addition is defined componentwise and the multiplication via the following rule:

$(a,b)(c,d)=(a(c+d),b(c+d))$.

If we let now $G=\{(1,0),(-1,0)\}$, G is not a normal subgroup of R (it fails to satisfy rG=Gr for all r in R). Nevertheless, G induces an equivalence relation P in R such that R/P inherits from R a hyperring structure. We observe that R/P equals $\{ \{r, -r\} \mid r \text{ in } R\}$ and that it has more than one unit from the right (all rG with r=(a,b) and a+b=1).

As we already mentioned, the existence of multiple units from the right shows that the hyperring in question is not isomorphic either to quotient hyperrings or to quotient subhyperrings that contain the image of the normal group that induces the hyperring structure. But, there exist quotient subhyperrings that do not contain a unit element and the above construction is embeddable in a quotient hyperring.

Indeed, assume that for every semigroup S and every ring A, A[S] is the semigroup ring of S over A, i.e., the set of all mappings of S into A that have finite support. This set can be endowed with a ring structure as in [13] (pages 158-159) where the algebra of a semigroup over a field is defined. Indeed, for any two such functions f and g it suffices to define

$(f+g)(s) = f(s)+g(s)$ for every s in S, and

$(fg)(r) = \Sigma f(s)g(t)$ where r is in S and (s,t) ranges over all pairs such that st=r.

We observe that for every subsemigroup of S, T, the elements of A[T] can be identified with the elements of A[S] whose support is a subset of T, i.e., that A[T] can be isomorphically embedded in A[S].

Let X be a left zero semigroup (xy=x for all x and y in X) of at least two elements, let X^e be the smallest semigroup with an identity element, e,that contains X, and assume that A has an identity element, 1, such that 1+1 is not zero. We observe that A is isomorphic to A[{e}] and can be isomorphically mapped into $A[X^e]$ (identify each a in A with the function that maps X to {0} and e to a); therefore, F={-1,1} is a normal subgroup of $A[X^e]$. If Y is any two-element subset of X, then we have:

(a) A[Y] is isomorphic to Massouros' ring and isomorphic to a subring of $A[X^e]$ (Y is a subsemigroup of X^e),

(b) F induces a partition of A[Y] such that A[Y]/F is isomorphic to a subhyperring of $A[X^e]/F$, and

(c) A[Y]/F is isomorphic to Massouros' hyperring.

We remark that F is not embeddable in A[Y]. But, if g maps Y onto {0,1}, then F introduces in A[Y] the same partition as {-g,g} and this group is isomorphic to the group Massouros used in order to partition A[Y].

In what follows we propose to use the following symbols and terminology:

(1) H^* will represent a hyperring whose elements are 0^*, a^*, b^*, ...

(2) R/P will represent a partition hyperring (it is assumed that P is such that R/P inherits from R a hyperring structure).

(3) H' is a subhyperrring of R/P. It is assumed that the elements of H' are 0',a', b',... . If H^* and H' are assumed to be isomorphic, then it is also assumed that the images of 0^*, a^*, b^*, ... are 0', a', b', ... respectively. We note that a' can also be seen as a subset of R since it is nothing more than a P equivalence

class.

(4) An equivalence P is said to be induced by a group G, iff the classes of P are of the form rG and G is a multiplicative subgroup of R.

The two next propositions link the cardinality of $a^*\#b^*$ to the cardinality of b' when seen as a subset of R (clearly, it is assumed that H^* is embeddable in a partition hyperring, R/P). As such, they are a blueprint for constructing counter-examples and for proving non-embeddability in partition hyperrings. The second proposition is, after all, one more "counting lemma" and therefore it provides a natural method for constructing counterexamples.

PROPOSITION 1. If P is an equivalence relation that induces a hyperring structure in R/P, then I, I=P(0), is an ideal of R. Furthermore, a+I is a subset of P(a) for every a in R and P induces a partition $P^*=P/I$ over $R^*=R/I$. Finally, R/P and R^*/P^* are isomorphic hyperrings.

COROLLARY. If a hyperring H is embeddable in a partition hyperring R/P, then one can assume without loss of generality that P(0)={0}.

PROPOSITION 2. Assume that a hyperring H is embeddable in a partition hyperring R/P for which P(0)={0} and assume that there are two elements a^* and b^* in H such that for every c^* in $a^*\#b^*$, $c^*\#(-c^*)$ and $b^*\#(-b^*)$ have only 0^* in common. Then, the cardinality of b' cannot exceed the cardinality of $a^*\#b^*$. (Clearly, b' is the image of b^* and when we speak of its cardinality, we view b' as a subset of R).

Indeed, if a is an element in a' and if b_1 and b_2 are distinct elements of b', then $a+b_1$ and $a+b_2$ belong to different equivalence classes in R/P. Therefore, there is an injection from b' into a'#b' in R/P, and as a result there is an injection from b' into $a^*\#b^*$. This injection is not always a surjection because a, the element in a', is arbitrary but fixed. But, if P is induced by a group G, then we observe that aG+bG=(a+bG)G. Therefore, the mapping we just described is a surjection and we have:

COROLLARY. Under the assumptions in proposition 2 and if P is induced by a group, then b' and $a^*\#b^*$ have the same cardinality.

In the next section, we will see how the above propositions can be used in the construction of counterexamples.

4. COUNTEREXAMPLES IN HYPERRING THEORY.

As we stated above, proposition 2 and its corollary can be used in the construction of hyperrings that are not embeddable in quotient hyperrings. In what follows we will show how this can be actually done by exhibiting the type of constructions that have been proposed in the last five years.

PROPOSITION 3. There are partition hyperrings that are not embeddable in quotient hyperrings.

Clearly, Abelian hypergroups can be seen as hyperrings if one defines that each and every product is zero. Let then Q be the set of all rational numbers, and let L be the set of all irreducible fractions of the form k/m with m=1 or m=2. Clearly, L is an additive subgroup of Q and, if equipped with the type of multiplication mentioned above (all products zero), then L is a ring. Let P be defined a follows:

i/k and j/m are equivalent iff either k=m=2 or i+j=0.

Then L/P is a partition hyperring H^* whose members are $0^*=\{0\}$, $x^*=\{(2i+1)/2 \mid i=0,-1,1,-2,2,-3,3,\ldots\}$, and $d^*(i)=\{-i,i\}$ for i=1,2,3,...

One can prove that L/P is not embeddable in a quotient hyperring R/G by contradiction. Indeed, corollary 2 can be used then to prove that for each i, d'(i) has two elements and that their sum is zero. Furthermore, we can prove by induction that these two elements can be chosen in such a way that d(i)={-id,id} for i=1,2,3,... If L/P were embeddable into R/G, then x' would be of the form x'=xG for every x in x'. If it were also true that x+x=0, then for every y in x', y+y=0. Furthermore, since for every i, $d^*(i)$ is in $x^*\#x^*$, we can prove that d above is of the form x1+y1 with x1 and y1 from x'. But, d+d is not zero while, under our assumptions, (x1+y1)+(x1+y1) is. Therefore, if x is in x', x+x cannot be zero. Furthermore, we observe that since x'+x'={0',d'(1),d'(2),...}, x+x must belong to some d'(i). This i cannot be even, because if i were equal to 2m, then both x+md and x-md would belong to x' ($x^*\#d^*(m)=x^*$). But then, either x+md or x-md must satisfy t+t=0, in contradiction to what we just proved. Thus, there is an m such that x+x is in d'(2m+1). If x+x=(2m+1)d, then it suffices to take z=x-md in order to obtain an element z in x'such that z+z=d. If on the other hand x+x= -(2m+1)d, one can achieve the same result by taking z=(m+1)d-x. Thus we can always assume that the element x we chose satisfies x+x=d. It ensues then that x'={ -x,x,-3x,3x,-5x,5x,....}. Indeed, if y is in x', then either y+y=(2m+1)d or y+y=-(2m+1)d for a well chosen m. In the first instance, it suffices to consider t, t=y-(2m+1)x. In the second, t=y+(2m+1)x. In either case t+t=0, and we can prove that t cannot belong either to x' or to any d'(i), i=1,2,.... Hence t=0 and therefore, x'={ -x,x,-3x,3x,-5x,5x,...}.

Finally, we observe that if x'=xG=(3x)G, then x=(3x)g for some g in G. But, since x'=xG, xg must be of the form (2k+1)x. Thus, x=3(2k+1)x which implies that (3k+1)d=(6k+2)x=0. But, 3k+1 is not zero, and if its absolute value is n, then (3k+1)d belongs to d'(n). Therefore, the above line of reasoning produces a contradiction and this implies that H^* is not embeddable in a quotient hyperring.

While the above construction quite conclusively shows that there are hyperrings that are not embeddable in quotient hyperrings, it clearly leaves open the possibility that all hyperrings generated by a set of multiplicative idempotents may be embeddable in quotient hyperrings and, as we already mentioned, a good part of the controversy centered on Stratigopoulos' generalization of Jacobson's theorem and the proof that he proposed. This question was settled by the following propositions:

PROPOSITION 4. Let T^* be a multiplicative group and let H^* be the disjoint union of $\{0^*,u^*,v^*\}$ and of T^*. Then H^* can be endowed with a hyperring structure if one defines an hyperaddition, #, and a multiplication as follows:

(1) For every a^* in H^*, $a^*\#0^*=0^*\#a^*=\{a^*\}$,

(2) For every a^* other than zero, $a^*\#a^*=\{0^*,a^*\}$.

(3) For all distinct a^* and b^*, $a^*\#b^*=H^*-\{a^*,b^*,0^*\}$, provided that neither a^* nor b^* is 0^*.

(4) for every a^* in H^*, $a^*0^*=0^*a^*=0^*$.

(5) $u^*u^*=u^*$, $v^*v^*=v^*$, and $u^*v^*=v^*u^*=0^*$

(6) for every t^* in T^*, $u^*t^*=t^*u^*=u^*$ and $v^*t^*=t^*v^*=v^*$.

(7) Over T^*, the multiplication of H^* and of T^* are identical.

The proof of this proposition is quite straightforward, albeit long. The important part is that we can prove the following proposition:

PROPOSITION 5. If H^* is embeddable in a partition hyperring R/P, then the following hold:

(a) u'U0' and v'U0' are, viewed as subsets of R, finite fields,

(b) u' and v' are isomorphic to subgroups of T^*,

(c) $f(u')$ and $f(v')$, the isomorphic images of u' and v' under these isomorphisms, are normal subgroups of T^*, and

(d) There are homomorphisms from u' onto $T^*/f(v')$ and from v' onto $T^*/f(u')$.

Before we delineate a proof of the above proposition, let us observe that one, among many, way of constructing hyperrings not embeddable in quotient hyperrings is to let T^* have a prime number elements q such that q+1 is not a power of 2, e.g., q=5. Indeed, if proposition 5 holds then either u' or v'would be isomorphic to T^* and we would have a finite field of q+1 elements. Since the latter is impossible, we will have produced a class of hyperrings that are not embeddable in partition hyperrings. A fortiori, they cannot be embedded in quotient hyperrings.

The proof of proposition 5 is relatively simple. If H^* were embeddable in a partition hyperring R/P, then we could use proposition 2 to prove that the images of u^* and v^*, u' and v' respectively, are finite subsets of R having at most q elements. Moreover, since u^* and v^* are multiplicative idempotents, it follows that u' and v' are multiplicative semigroups of R. We can then construct a semigroup homeomorhism f that maps u'xv'onto T^* by defining f as follows:

$$f(u,v)= t^* \text{ if and only if and } u+v \text{ is a member of } t'$$

(as we already indicated, t' is the isomorphic image of t* when H* is mapped into a subhyperring of R/P).

Since $u^*v^*=v^*u^*=0^*$ and 0' has a single element, R's zero, for any two pairs (u1,v1) and (u2,v2) from u'xv', (u1+v1)(u2+v2) equals u1u2+v1v2. Therefore, f(u1u2,v1v2)=f(u1,v1)f(u2,v2) and f is a semigroup homeomorphism from u'xv' onto T^*. Let now (u0,v0) be a multiplicative idempotent in u'xv' (such an idempotent exists because u'xv' is finite). Then, by fixing u to be u0, we obtain a homomorphism f(u0,v) that maps v' into a subset of T^*, f(v'). Similarly, by fixing v to be v0, we obtain a homomorphism, f(u,v0), that maps u' into a subset of T^*, f(u'). It is elementary then to check that f(u0,v) and that f(u,v0) are injections and that all finite semigroups of a group are groups. It ensues that u', v', and u'xv' are multilicative groups and that f is a group homomorphism from u'xv' onto T^*. It also follows that (u0,v0) is the unity of u'xv' and that {u0}xv' and u'x{v0} are normal subgroups of u'xv'. Given that u' is isomorphic to u'xv'/{u0}xv', u' is homomorphic to $T^*/f(v')$. Similarly, v' is homomorphic to $T^*/f(u')$.

It is easy now to check that u'U{0} and v'U{0} are finite fields (u' and v'are finite groups, u'+u'=u'U{0}, and v'+v'=v'U{v0}). We also observe that if the cardinality of T^* is a prime number, q, then either u' or v' is isomorphic to T^*. Indeed, if f(u') is not T^*, it must equal $\{e^*\}$, e^* being the identity of T^*. But then, the finite group v' is isomorphic to f(v'), a subgroup of T^*, and homomorphic to $T^*/\{e^*\}$ and the only way this can happen is if v' has q elements.

A similar construction to the one we just used can be employed in order to show that there exist hyperfields that are not embeddable into partition hyperrings. Given that each hyperfield is also an irreducible and faithful module over itself, it follows that there are primitive rings that are not embeddable into partition hyperfields. Indeed, let T^* be any finite group of m, m>3, elements and define a hyperfield structure over H^*, $H^*=T^*U\{0^*\}$, as follows:

(1) $a^*0^*=0^*a^*=0^*$ for every a^* in H^*,

(2) $a^* \# 0^* = 0^* \# a^* = \{a^*\}$ for every a^* in H^*,

(3) $a^* \# a^* = \{a^*,0^*\}$ for every a^* in T^*, and

(4) $a^* \# b^* = b^* \# a^* = T^*-\{a^*,b^*\}$ for all a^* and b^* in T^*, provided that a^* and b^* are distinct.

Structures that exhibit such properties are not that uncommon. Indeed, it suffices to consider C the field of complex numbers and R^*, the multiplicative group of all nonzero reals, in order to obtain a quotient hyperfield C/R^* that displays properties (3) and (4). But, when the underlying multiplicative group T^* is finite, then we can prove the following:

PROPOSITION 6. If H^*, as constructed above is embeddable in a partition hyper=-ring R/P, and if H' is the isomorphic image of H^*, then the following hold:

(1) The isomorphism maps e^*, the unit of T^*, onto a finite multiplicative subgroup of R that will be called e' in what follows,

(2) If H1 is the subset of R that corresponds to H', then e' and P induce the same partition on H1, and

(3) e'U{0} is a field of m-1 elements while H1 is a field of m(m-2)+1 elements (m-1 squared).

Clearly, if (1) through (3) hold, we can choose T* in such a way that H* cannot be embeddable in a partition hyperring. Indeed, all finite fields are commutative, their cardinality is a power of a prime, and the multiplicative group of their nonzero elements is cyclic. One can choose then either m or the structure of T* in such a way that H* is not embeddable into a partition hyperring.

The proof itself is rather simple. Proposition 2 applies and as a result, all non zero elements of H' correspond to finite subsets of R having m-2 elements or less. Let H1 be the union of all these subsets. We start by observing that e' is a finite set closed for multiplication, without divisors of zero (e'e'=e'). Furthermore, since e'#e'={e',0'}, if x and y are distinct elements of e', then x-y is in e'. It follows that for every a in e', the mappings x-->ax and x-->xa are injections from e'into e'. Therefore, e' is a group.

But, the same argument can be used for H1-{0}. Since H'-{0} is isomorphic to T^*, H1-{0} has no divisors of zero and is closed for multiplication. Given that if x and y are distinct elements of H1 x-y also belongs to H1, we deduce that for every a in H1-{0} the mappings x-->ax and x-->xa are injections. Since H1 is finite it follows that H1-{0} is a group, that e' is a subgroup of H1-{0}, and that these two groups share the same identity element, e.

Let x' be any nonzero element of H'. Since H' is a hyperfield, there is a y' such that y'x'=x'y'=e' in H'. It ensues that if y is any element of y', then yx' and x'y are subsets of e' in H1. The inverse of y in H1-{0}, x, is clearly an element of x'. By multiplying by x we obtain that x' is a subset of xe' and e'x in H1. On the other hand, since x'e'=e'x'=x' in H', xe' and e'x must be subsets of x' in H1. Hence x'=xe=ex for some x in x'. But, if this relation is true for one x in x', it is true for every x in x' (it suffices to remark that e' is a group). By now, we have all the results that we need. Since P is induced over H1 by e', each class in H1 has exactly m-2 elements (proposition 2) except for 0' that contains only 0. Therefore, H1 is a field of m(m-2)+1 elements while e'U{0} is a field (e*#e*={e*,0*}) of (m-2)+1 elements.

Propositions 1 through 6 are all original results taken from the manuscript the author submitted in 1982. Another interesting construction of hyperfields that are not embeddable in quotient hyperfields can be found in [6; see also 14-18]. The idea there is to take a group G and to introduce a hyperfield structure over H=GU{0} as follows:

(a) For every h in H, h#0=0#h={h},

(b) g1#g2={g1,g2} for every two distinct elements of G, and

(c) g#g=H-{g} for every g in G.

One can then prove that G can be chosen in such a way that H is not embeddable in a quotient hyperfield. Indeed, we have:

PROPOSITION 7. If G is not trivial and gg=e for every g in G, then H is not isomorphic to a quotient hyperfield.

This can be proven by contradiction. If H were isomorphic to a quotient hyperfield F/Q, then we have:

(a) Since g*g=e in G, Q contains all squares in F other than zero,

(b) Since for each element g in G, g#g=H-{g}, it follows that Q=-Q and that Q+Q=F-Q.

But, if all squares of F and their opposites are in Q, then we have a contradiction. If the characteristic of F is not two, then each element of F is the difference of two squares. If the characteristic of F is two, then the sum of two squares is a square. In either case, Q+Q cannot equal F-Q.

Finally, one can prove [6] that if a Cartesian product of hyperrings is embeddable in a quotient hyperring, then every term of the product that is a hyperfield must be isomorphic to a quotient hyperfield. Thus, one can produce hyperrings that are not embeddable in quotient hyperrings.

It must be noted that propositions six and seven are organically and timewise independent results. Organically, because the construction in proposition six is an outgrowth of the construction used in proposition five while the type of construction used in proposition seven is the result of studies of the following problem: "Given a field F and a non trivial multiplicative subgroup G of F^*, F^*=F-{0}, such that F^*/G is finite, is it possible to conclude that F=G-G?".

Timewise, because a construction somewhat more general than the one used in proposition seven was presented by the author of [6], without proof, as a hyperfield likely not to be embeddable in a quotient hyperfield; this counterexample candidate was proposed after the results described in propositions one through five were found, but before proposition six was conceived. On the other hand, proposition six was proven before the specific example and the proof that appear in proposition seven were found.

5. STRUCTURE THEOREMS AND OTHER RESULTS IN THE THEORY OF ALGEBRAIC STRUCTURES WITH A HYPEROPERATION.

As we pointed out in the Introduction, the structures and counterexamples that appear in section 4 prove that the theory of hyperrings is not a straightforward extension of ring theory. It would be appropriate then to mention, however briefly, several papers on the structure of hyperrings and on other related subjects.

A study of Abelian hypergroups (canonical in the author's terminology) appears in [7]. Some of the most notable results are obtained by considering Q, the smallest subhypergroup of H that contains all differences of the form x-x, x in H (remark: x-x is as a rule a subset of H, unless H is a group). The author proves, among other results, that if h is a subhypergroup of H, then the quotient hypergroup H/h is a group if and only if h contains Q. Similar structure results hold for for hyperrings [8]. Indeed, if q is a bilateral hyperideal of a hyperring H, then H/q is a ring if and only if q contains Q. We note that a hyperideal is a subhyperring q such that qH and Hq are subsets of q and that under this definition Q (the smallest subhypergroup of H that contains all x-x, x in H) is a bilateral ideal.

Another set of papers [4, 5, and 9] is concerned with the algebraic structure of hyperrings and with extending the theory of radicals from rings to hyperrings. These results are quite interesting, but they cannot be properly described without the introduction of a cascade of new definitions and the interested reader is invited to read the original publications.

In [19] the authors study Boolean hyperalgebras, hyperrings H having a multiplicative identity element, 1, and such that for every x in H xx=x. The authors show that under these assumptions x= -x and xy=yx for all x and y in H, and that for each x in H there is a unique x' such that xx'=0 and 1 is in x#x'. Furthermore, they introduce an ordering of H, x precedes y iff x=xy, under which x' can be seen as the complement of x and H can be seen as a Boolean algebra in which inf(x,y)=xy and sup(x,y)=(x'y')'. Finally it is shown that sup(x,z)=sup(y,z)=sup(x,y) for all x and y in H and for every z in x#y, and this paves the way for a new structure. A **strong Boolean hyperring** is a hyperring such that z is in x#y iff sup(x,y)=sup(x,z)=sup(y,z). It ensues that when one is dealing with a strong Boolean hyperring, the resultant lattice fully defines the structure of the Boolean hyperring.

Finally, we should direct the attention of the reader to the work of Marc Krasner [3 and 20-24]. Although his work falls outside the scope of this article, it must be noted that he was the first to use a hyperring structure as a technical tool. Indeed, if K is a field with a valuation (K admits an absolute value |.| such that for all x and y in K |x+y| does not exceed sup(|x|,|y|), then the elements of K with absolute value one or less form a ring R while the elements of K that have absolute value one form a multiplicative group G that is normal in R. Thus G introduces a partition of R, R/G, and a hyperring structure over R/G. This structure is then used to define a way in which a complete valuated field of characteristic p, p>0, can be approximated by complete valuated fields of characteristic zero.

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