Internat. J. Math. & Math. Sci.
VOL. 11 NO. 1 (1988) 129-142

ON THE QUASIUNIQUENESS OF SOLUTIONS OF DEGENERATE EQUATIONS IN HILBERT SPACE

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(Received December 22, 1981, and in revised form April 15, 1982)

ABSTRACT. In this paper, we study the quasiuniqueness (i.e., $f_1 \doteq f_2$ if $f_1 - f_2$ is flat, the function $f(t)$ being called flat if, for any $K > 0$, $t^{-K} f(t) \to 0$ as $t \to 0$) for ordinary differential equations in Hilbert space. The case of inequalities is studied, too.

The most important result of this paper is this:

THEOREM 3. Let $B(t)$ be a linear operator with domain D_B and $B(t) = B_1(t) + B_2(t)$ where $(B_1(t)x,x)$ is real and $\mathrm{Re}(B_2(t)x,x) = 0$ for any $x \in D_B$. Let for any $x \in D_B$ the following estimate hold:

$$\|B_1x - \frac{(B_1x,x)}{(x,x)}x\|^2 + \mathrm{Re}(B_1x,B_2x) + t(\dot{B}_1(t)x,x) \geq -Ct[|(\dot{B}_1(t)x,x)| + (x,x)]$$

with $C \geq 0$.

If $u(t)$ is a smooth flat solution of the following inequality in the interval $t \in I = (0,1]$.

$$\|t\frac{du}{dt} - B(t)u\| \leq t\phi(t)\|u(t)\|$$

with non-negative continuous function $\phi(t)$, then $u(t) \equiv 0$ in I. One example with self-adjoint $B(t)$ is given, too.

KEY WORDS AND PHRASES. Degenerate equations, differential equations in Hilbert space, quasiuniqueness, flat solutions.
1980 MATHEMATICS SUBJECT CLASSIFICATION CODE. 34G10, 35J70, 35K22, 35K65, 35R45.

0. INTRODUCTION.

In this paper we study the quasiuniqueness of solutions of the abstract equation of the form

$$t\frac{du}{dt} = B(t)u \ , \quad t \quad I = [0,T], \quad 0 < T < +\infty \tag{0.1}$$

Here $B(t)$ is an unbounded non-symmetric operator in Hilbert space. The quasiuniqueness of solutions of a somewhat more general problem of the form

$$\|t\frac{du}{dt} - B(t)u(t)\| \leq t\phi(t)\|u(t)\| \tag{0.2}$$

is studied too. Here $B(t)$ is of the same type as in (0.1), and $\phi(t)$ is a continuous non-negative function in the interval I.

Recall that by quasiuniqueness we mean uniqueness in the class of functions that differ by flat functions. We say that the function $u(t)$ is a flat function if

$$\forall k > 0 \ , \ t^{-k}u(t) \xrightarrow[t\to 0]{} 0 \ .$$

In Section 1, we study the simplest model, which is further developed in Section 2. In Section 3 the main theorems are obtained: Theorem 3 for the problem (0.2) and Theorem 4 for the problem (0.1). Our conditions of quasiuniqueness generalize the corresponding conditions of [1] (we do not present the analog of Theorem 1-1 of [1] since it is trivial). Theorem 2 of section 2 corresponds to Theorem 1-3 of [1] and generalizes Theorem 1-2 of the same paper. Our Theorems 3, 4 of Section 3 are a further generalization of Theorem 2, section 2 as well as of Theorems 1-2, 1-3 of [1]. Section 4 is devoted to remarks about previous sections. We point out that in the paper we used methods different from those of Alinhac-Baouendi in [1].

Problems (0.1), (0.2) and those which can be reduced to them were recently studied by a number of authors (see [1-6]). Thus in [4] an example of a particular equation which could be reduced to the form (0.1), where $B(t) = B(0)$ is self-adjoint, was considered, and the quasiuniqueness was proved for it. Further in [2] and [3], equation (0.1) was studied for $B(t) = B(0) + tB_1(t)$ with $B(0)$ bounded (Fuchs-type equation). In the paper [5], the quasiuniqueness was proved for a certain class of elliptic operators with a degeneration in a single point. Conditions which are difficult to verify were imposed, but a simple class of elliptic operators satisfying them was indicated. In our paper [6] elliptic equations with a possible degeneration on a hyperplane or in a single point are studied. In [6], the quasiuniqueness was proved for (0.1)-(0.2) with self-adjoint operator $B(t)$.

Methods employed here were first used by Agmon and Nirenberg ([7], [8]) for studying the Cauchy problem in the non-degenerate case.

1. MODEL CASE.

Let H be a Hilbert space with scalar product $(\cdot,\cdot)$ and norm $\|\cdot\|$, I is the interval $[0,T]$ with $0 < T < +\infty$, $\phi(t)$ a continuous non-negative function on I, $u(t) \in C^1(I,H)$, A, a linear operator in H, with domain D_A and $A = A_1 + A_2$, $A_1^* = A_1$ the self-adjoint part of A, and $A_2^* = -A_2$ the anti-self-adjoint part of A. We shall assume that $u(t) \in D_A$ and that $Au(t) \in C(I,H)$. Set $D = t\frac{\partial}{\partial t}$.

THEOREM 1. Let $u(t)$ be a solution of the inequality

$$\|Du(t) - Au(t)\| \le t\phi(t)\|u(t)\| \ . \tag{1.1}$$

We suppose that all the conditions introduced above hold and that commutator $[A_1,A_2] = 0$. Let $u(t)$ be a flat function (i.e., $\forall\, k > 0,\ t^{-k}u(t) \xrightarrow[t\to 0]{} 0$). Then $u(t) \equiv 0$ in I.

PROOF. Let

$$q(t) = (u(t),u(t)) \tag{1.2}$$

$$f(t) = Du(t) - Au(t) \ , \tag{1.3}$$

and let $(t_1,t_0]$ be a subinterval of I such that $q(t) > 0$ for $t_1 < t \le t_0$,

$$\psi(t) = 2\mathrm{Re}(f(t),u(t))/q(t) \tag{1.4}$$

$$s(t) = \exp - \int_{t_0}^{t} \frac{\psi(\tau)}{\tau}\, d\tau \tag{1.5}$$

$$\ell(t) = \log[q(t) \cdot s(t)] \;. \tag{1.6}$$

LEMMA 1.1. Suppose that all the conditions of Theorem 1 and (1.2)-(1.6) hold. Then $\ell(t)$ is twice differentiable and satisfies the following second-order differential inequality in the interval $(t_1, t_0]$:

$$D^2\ell(t) + 2t^2\phi^2(t) \geq 0 \;. \tag{1.7}$$

PROOF. From (1.6) we have

$$t\dot{s}(t) = -s(t)\psi(t)$$

and

$$Dq(t) = t\dot{q}(t) = 2\mathrm{Re}(t\dot{u},u) = 2\mathrm{Re}(Au,u) + \psi q = 2(A_1u,u) + \psi q$$

$$D\ell(t) = \frac{t\dot{s}q + st\dot{q}}{sq} = \frac{t\dot{q} - q}{q} = \frac{2}{q(t)}(A_1u,u) \;. \tag{1.8}$$

Next it follows from (1.8) that $\ell(t)$ is twice differentiable, and

$$D\,\ell(t) = \frac{2}{q}D(A_1u,u) - \frac{2}{q^2}(A_1u,u)Dq$$

$$= \frac{4}{q}\mathrm{Re}(A_1u,Du) - \frac{2}{q^2}(A_1u,u)[2(A_1u,u) + \psi q]$$

$$= \frac{4}{q}\mathrm{Re}(A_1u,Du) - \frac{2\psi}{q}(A_1u,u) - \frac{4}{q^2}(A_1u,u)^2$$

$$= \frac{4}{q}(A_1u,A_1u) - \frac{2}{q^2}(A_1u,u)^2 + \frac{4}{q}\mathrm{Re}(A_1u,f) - \frac{2\psi}{q}(A_1u,u) + \frac{4}{q}\mathrm{Re}(A_1u,A_2u) \;.$$

Now

$$\frac{4}{q}[\|A_1u\|^2 - q^{-1}(A_1u,u)^2] = \frac{4}{q}\|A_1u - \frac{(A_1u,u)}{q}u\|^2 \;,$$

and hence we find

$$D^2\ell(t) = \frac{4}{q}\|A_1u - \frac{(A_1u,u)}{q}u\|^2 + \frac{4}{q}\mathrm{Re}(A_1u,f) - \frac{2\psi}{q}(A_1u,u) + \frac{4}{q}\mathrm{Re}(A_1u,A_2u) \;.$$

From (1.4),

$$\frac{4}{q}\mathrm{Re}(A_1u,f) - \frac{2\psi}{q}(A_1u,u) = \frac{4}{q}\mathrm{Re}\left[(A_1u,f) - \frac{(A_1u,u)(u,f)}{q}\right]$$

$$= \frac{4}{q}\mathrm{Re}(A_1u - \frac{(A_1u,u)}{q}u, f)$$

$$\geq - \frac{2}{q}\|A_1u - \frac{(A_1u,u)}{q}u\|^2 - \frac{4}{q}\|f\|^2 \;,$$

and from (1.1)-(1.3),

$$\|f(t)\| \leq t\phi(t)q^{\frac{1}{2}}(t) \;,$$

so we have

$$D^2\ell(t) \geq \frac{2}{q}\|A_1u - \frac{(A_1u,u)}{q}u\|^2 - 2t^2\phi^2(t) + \frac{4}{q}\mathrm{Re}(A_1u,A_2u) \;.$$

$$\mathrm{Re}(A_1u,A_2u) = \frac{1}{2}([A_1,A_2]u,u) = 0$$

and hence we find

$$D^2\ell(t) + 2t^2\phi^2(t) \geq 0 \;.$$

Lemma 1.1 is proved.

LEMMA 1.2. Let $\ell(\tau)$ be a solution of (1.7). Then

$$\ell(t) \geq \ell(t_0) + [t_0\dot{\ell}(t_0) + 4c]\ln\frac{t}{t_0}\ , \tag{1.9}$$

where

$$c = \int_0^{t_0} t\phi^2(t)dt\ . \tag{1.10}$$

PROOF. Let

$$\ell_1(t) = D\ell(t)\ . \tag{1.11}$$

Then we have

$$D\ell_1(t) + 4t^2\phi^2(t) = f(t) \geq 0$$

and

$$\ell_1(t) = \ell_1(t_0) - 4\int_{t_0}^{t} \tau\phi^2(\tau)d\tau + \int_{t_0}^{t}\frac{f(\tau)}{\tau}d\tau\ ,$$

$$\ell(t) = \ell(t_0) + \ell_1(t_0)\ln\frac{t}{t_0}\ 4\int_{t_0}^{t}\frac{d\tau}{\tau}\int_{t_0}^{t} s\phi^2(s)ds + \int_{t_0}^{t}\frac{d\tau}{\tau}\int_{t_0}^{t}\frac{f(s)}{s}ds\ .$$

On condition that $t, \tau \leq t_0$, we have

$$\ell(t) = \ell(t_0) - \ell_1(t_0)\ln\frac{t_0}{t} - 4\int_t^{t_0}\int_\tau^{t_0}\frac{s\phi^2(s)}{\tau}dsd\tau + \int_t^{t_0}\int_\tau^{t_0}\frac{f(s)}{s\cdot\tau}dsd\tau\ .$$

We assume that $f(s) \geq 0$, $s, \tau \in I$, i.e., $s \cdot \tau \geq 0$.

We have $\frac{f(s)}{s\cdot\tau} \geq 0$, and for $t \leq t_0$, $\tau \leq t_0$,

$$\int_t^{t_0}\frac{d\tau}{\tau}\int_t^{t_0}\frac{f(s)}{s}\,ds \geq 0\ .$$

Next it follows from (1.10) and (1.11) that

$$\ell_1(t_0) = t\dot{\ell}(t)\Big|_{t=t_0} = D\ell(t)\Big|_{t=t_0} = t_0\dot{\ell}(t_0)\ ,$$

$$\int_\tau^{t_0} s\phi^2(s)ds \leq \int_0^{t_0} s\phi^2(s)ds = c$$

and

$$4\int_t^{t_0}\frac{d\tau}{\tau}\int_\tau^{t_0} s\phi^2(s)ds \leq 4c\int_t^{t_0}\frac{d\tau}{\tau} = 4c\ \ln\frac{t_0}{t}\ .$$

Now we have

$$\ell(t) \geq \ell(t_0) + [t_0\dot{\ell}(t_0) + 4c]\ln\frac{t}{t_0}$$

and

$$\exp(\ell(t)) \geq \exp\ell(t_0)\cdot\exp\{[t_0\dot{\ell}(t_0) + 4c]\ln\frac{t}{t_0}\}$$

$$= \exp\ell(t_0)\cdot(t/t_0)^{t\dot{\ell}(t)\big|_{t=t_0}\ 4c}\ .$$

From the last formula,

$$\exp\ell(t) \geq \exp\ell(t_0)\cdot(t/t_0)^{2\nu+2\mu}\ , \tag{1.12}$$

where

$$\nu = 2c \quad \text{is dependent on} \quad \phi(t) \quad \text{only} \tag{1.13}$$

$$\mu = \frac{1}{2}t_0\dot{\ell}(t_0) \quad \text{is dependent on} \quad \ell(t) \quad \text{only.} \tag{1.14}$$

Lemma 1.2 is proved.

We now turn to

PROOF OF THEOREM 1. From (1.2)-(1.6),

$$q(t) = \exp \ell(t)/s(t) \ , \ s(t_0) = 1 \ , \ q(t_0) = \exp \ell(t_0) \ ,$$

$$|\psi(\tau)| \le 2\tau\phi(\tau) \ , \ 1/s(t) = \exp\left(\int_{t_0}^{t} \frac{\psi(\tau)}{\tau}d\tau\right)$$

and

$$\left|\int_{t_0}^{t}\frac{\psi(\tau)}{\tau}d\tau\right| \le \int_{t}^{t_0}\left|\frac{\psi(\tau)}{\tau}\right|d\tau \le 2\int_0^{t_0}\phi(\tau)d\tau = 2d \ ,$$

where

$$d = \int_0^{t_0} \phi(\tau)d\tau \ , \tag{1.15}$$

and we have

$$s^{-1}(t) \ge \exp(-2d)$$

and

$$q(t) \ge e^{-2d}\exp\ell(t) \ge \exp\ell(t_0)e^{-2d}(t/t_0)^{2\nu+2\mu} \ . \tag{1.16}$$

From (1.2) and (1.6) it follows that

$$\|u(t)\| \ge M\|u(t_0)(t/t_0)^{\nu+\mu} \ , \tag{1.17}$$

where ν, μ are defined in (1.13) and (1.14), and $M = e^{-d}$

Assume that the flat-function $u(t)$ satisfying (1.1) is not identically zero. From (1.17) we have that $u(t)$ is not a flat function. This is a contradiction. Therefore Theorem 1 is proved.

2. QUASIUNIQUENESS FOR PROBLEM (0.2).

THEOREM 2. Let $B(t)$ be a linear operator with domain $D_{B(t)}$. We shall assume that $u(t) \in D_{B(t)}$ and $B(t) = B_1(t) + B_2(t)$, $B_1^*(t) = B_1(t)$, $B_2^*(t) = -B_2(t)$ and $\mathrm{Re}(B(t)u,u) = (B_1(t)u,u)$, $\mathrm{Re}(B_2(t)u,u) = 0$, that $u(t) \in C^1(I,H)$, $B(t)u(t) \in C^1(I,H)$, and $\phi(t)$ denotes a non-negative continuous function in the interval I.

We assume that the function $B(t)x$ is differentiable for $0 \le t \le T$ for all $x \in D_{B(t)}$, and set

$$\frac{d}{dt}B(t)x = \mathcal{B}(t)x \ . \tag{2.1}$$

Let $u(t)$ be a solution of

$$\|Du(t) - B(t)u(t)\| \le t\phi(t)\|u(t)\| \ . \tag{2.2}$$

such that

$$\|t^{-1}[B_1,B_2]u + \mathcal{B}_1u\| \le \gamma(t)\|B_1(t)u(t)\| + \beta(t)\|u(t)\| \ , \ t \in I \ , \tag{2.3}$$

or

$$\frac{1}{t}([B_1,B_2]u,u) + (\mathcal{B}_1(t)u,u) \ge -\gamma(t)|(B_1u,u) - \beta(t)\|u(t)\|^2 \tag{2.3a}$$

where $\gamma(t)$, $\beta(t)$ are non-negative continuous functions in the interval I. If $u(t)$ is a flat-function, then $u(t) \equiv 0$ in I.

PROOF OF THEOREM 2. Let

$$q(t) = (u(t),\ u(t)) \tag{2.4}$$

$$f(t) = t\dot{u}(t) - B(t)u(t) \tag{2.5}$$

and let $(t_1,t_0]$ be a subinterval of I such that $q(t) > 0$ for $t_1 < t \le t_0$,

$$\psi(t) = 2\mathrm{Re}(f(t),u(t))/q(t) \tag{2.6}$$

$$s(t) = \exp\{-\int_{t_0}^{t} \frac{\psi(\tau)}{\tau}d\tau\} \tag{2.7}$$

$$p(t) = s(t)q(t) \tag{2.8}$$

$$\ell(t) = \log p(t)\ . \tag{2.9}$$

LEMMA 2.1. Suppose that all the conditions of Theorem 2 hold. Then $\ell(t)$ is twice differentiable and satisfies the following second-order differential inequality in the interval $(t_1,t_0]$:

$$D^2\ell(t) + 2t\gamma(t)|D\ell(t)| + 2t\beta(t) + 4t^2\phi^2(t) + 2t^2\gamma^2(t) \ge 0\ . \tag{2.10}$$

PROOF OF LEMMA 2.1.

$$t\dot{q}(t) = 2\mathrm{Re}(t\dot{u}(t),u(t)) = 2\mathrm{Re}(f,u) + 2\mathrm{Re}(B(t)u,u)$$
$$= \psi q + 2(B_1(t)u,u)$$

$$t\dot{\ell}(t) = \frac{t\dot{p}(t)}{p(t)} = \frac{t\dot{q}s + qt\dot{s}}{sq} = \frac{2(B_1(t)u,u)s + \psi qs - \psi sq}{sq}$$
$$= \frac{2}{q}(B_1(t)u,u) = \frac{2}{q}(B_1u,u)\ . \tag{2.11}$$

Next it follows from (2.11) that $\ell(t)$ is twice differentiable, and

$$D^2\ell(t) = \frac{2}{q}D(B_1u,u) - \frac{2}{q^2}(B\ u,u)Dq \tag{2.12}$$

$$= \frac{2}{q}[(DB_1u,u) + 2\mathrm{Re}(B_1u,Du)] - \frac{2}{q^2}(B_1u,u)[2(B_1u,u) + \psi q]$$

$$= \frac{4}{q}\mathrm{Re}(B_1u,f) + \frac{4}{q}(B_1u,B_2u) + \frac{2}{q}(DB_1u,u) - \frac{4}{q^2}(B_1u,u)^2 - \frac{2\psi}{q}(B_1u,u) + \frac{4}{q}\mathrm{Re}(B_1u,B_2u)$$

Now

$$\frac{4}{q}[\|B_1u\|^2 - q^{-1}(B_1u,u)^2] = \frac{4}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2$$

and

$$D^2\ell(t) = \frac{4}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2 + \frac{4}{q}\mathrm{Re}(B_1u,f) + \frac{2}{q}(DB_1u,u) - \frac{2\psi}{q}(B_1u,u) + \frac{4}{q}\mathrm{Re}(B_1u,B_2u)\ .$$

From (2.6),

$$\frac{4}{q}\mathrm{Re}(B_1u,f) - \frac{2\psi}{q}(B_1u,u) = \frac{4}{q}\mathrm{Re}[(B_1u,f) - (\frac{(B_1u,u)}{q}u,f)]$$
$$= \frac{4}{q}\mathrm{Re}(B_1u - \frac{(B_1u,u)}{q}u,f)$$
$$\ge -\frac{2}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2 - \frac{4}{q}\|f\|^2\ .$$

(a) Case (2.3). From (2.3) we have

$$([B_1,B_2]u,u) + (DB_1u,u) \ge -\gamma(t)t\|B_1(t)u(t)\|\cdot\|u(t)\|\cdot\|-\beta(t)\cdot t\|u(t)\|^2$$

and

$$D^2\ell(t) \ge \frac{2}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2 - \frac{4}{q}\|f\|^2 - \frac{2}{q}[\gamma(t)\cdot t\|B_1u\|\cdot\|u\| - \beta(t)\cdot\|u\|^2]$$

or

$$D^2\ell(t) + 4\phi^2t^2 + 2t\beta(t) \ge \frac{2}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2 - \frac{2t\gamma(t)}{q(t)}\|B_1u\| \cdot \|u\| .$$

For fixed t, we have

(i) $|(B_1u,u)| \ge \frac{1}{2}\|B_1u\| \cdot \|u\|$, or

(ii) $|(B_1u,u)| < \frac{1}{2}\|B_1u\| \cdot \|u\|$.

In case (i),

$$\frac{2}{q}\|B_1u\| \cdot \|u\| \le \frac{4|(B_1u,u)|}{q} = 2|D\ell(t)|$$

and

$$D^2\ell(t) + 2t\gamma(t)|D\ell(t)| + 4t^2\phi^2(t) + 2t\beta(t) \ge 0 . \tag{2.13}$$

In case (ii),

$$(B_1u,u)^2 < \frac{1}{4}\|B_1u\|^2\|u\|^2 = \frac{1}{4}\|B_1u\|^2q ,$$

and from the inequality

$$\|a + b\|^2 \ge \frac{1}{2}\|a\|^2 - \|b\|^2$$

it follows that

$$\|B_1u - \frac{(B_1u,u)}{q}u\|^2 \ge \frac{1}{2}\|B_1u\|^2 - \frac{(B_1u,u)^2}{q^2}q \ge \frac{1}{4}\|B_1u\|^2$$

and

$$D^2\ell(t) + 4\phi^2t^2 + 2t\beta(t) \ge \frac{1}{2q}[\|B_1u\|^2 - 4t\gamma(t)\|B_1u\| \cdot \|u\|]$$

$$= \frac{1}{2q}[\|B_1u\| - 2t\gamma\|u\|]^2 - \frac{1}{2q}4t^2\gamma^2(t)\|u\|^2 \ge -2t^2\gamma^2(t) ,$$

and

$$D^2\ell(t) + 4t^2\phi^2 + 2t^2\gamma^2 + 2t\beta \ge 0 . \tag{2.14}$$

(b) Case (2.3a). From (2.3a) we have

$$([B_1,B_2]u,u) + (DB_1u,u) \ge -t\gamma(t)|(B_1u,u)| - t\beta(t)q(t)$$

and

$$D^2\ell(t) \ge \frac{2}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2 - \frac{4}{q}\|f\|^2 - \frac{2}{q}t\gamma(t)|(B_1u,u)| - 2t\beta(t) .$$

From (2.5) we find

$$\|f(t)\| \le t\phi(t)q^{\frac{1}{2}}(t)$$

and from this and (2.11),

$$D^2\ell(t) + 4t^2\phi^2(t) + 2t\gamma(t)|D\ell(t)| \ge \frac{2}{q}\|B_1u - \frac{(B_1u,u)}{q}u\| \ge 0 . \tag{2.15}$$

From (2.13)-(2.15) it follows that

$$D^2\ell(t) + a(t)|D\ell(t)| + b(t) \ge 0$$

where

$$a(t) = 2t\gamma(t)$$

$$b(t) = 4t^2\phi^2(t) + 2t^2\gamma^2(t) + 2t\beta(t) .$$

Lemma 2.1 is proved.

LEMMA 2.2. Let $\ell(t)$ be a twice differentiable function in the interval I, satisfying the following second-order differential inequality

$$\begin{gathered} D^2\ell(t) + ta(t)|D\ell(t)| + tb(t) \ge 0 \ , \ t \in I \\ a(t) \le M \ , \ b(t) \le M \ , \ \forall\, t \in I \end{gathered} \tag{2.16}$$

where $a(t)$, $b(t)$ are non-negative continuous functions in I. Then

$$\ell(t) \ge \ell(t_0) + c_1 \ln\frac{t}{t_0} + c_2 \ln\frac{t}{t_0} \ , \tag{2.17}$$

where c_1 is a constant depending on M, t_0, t, and c_2 is a constant depending only on M and t_0. Hence,

$$\exp \ell(t) \ge \exp \ell(t_0) \cdot t^{\nu} \cdot t^{\mu} \ , \tag{2.18}$$

with ν, μ non-negative, ν a constant depending on a and b only, and μ a constant depending on a, b and $t_0\dot{\ell}(t_0)$.

PROOF. From (2.16), it follows that

$$D^2\ell(t) + Mt|D\ell(t)| + Mt \ge 0 \tag{2.19}$$

is true. We change the variable using the formula

$$t = e^{-\tau} \tag{2.20}$$

and for $\ell(\tau)$ we have

$$\ell(\tau) + Me^{-\tau}|\dot{\ell}(\tau)| + Me^{-\tau} \ge 0 \ .$$

From Lemma 1.2 of [2] we get

$$\ell(\tau) \ge \ell(\tau_0) + \min\{0, \dot{\ell}(\tau_0)\} e^{Me^{\tau_0}} (\tau - \tau_0) - Me^{\tau_0} e^{\tau_0} (\tau - \tau_0) \ . \tag{2.21}$$

From (2.20) and (2.21) we have

$$\begin{aligned} \ell(\tau) &\ge \ell(t_0) + \min\{0, t_0\dot{\ell}(t_0)\}\exp(\tfrac{M}{t_0})\ln\frac{t_0}{t} - M\exp(\tfrac{M}{t_0})t_0^{-1}\ln\frac{t_0}{t} \\ &= \ell(t_0) + \max\{0, -t_0\dot{\ell}(t_0)\}\exp(\tfrac{M}{t_0})\ln\frac{t}{t_0} + M\exp(\tfrac{M}{t_0})t_0^{-1}\ln\frac{t}{t_0} \\ &= \ell(t_0) + \mu(t_0)\ln\frac{t}{t_0} + \nu(t_0)\ln\frac{t}{t_0} \ , \end{aligned} \tag{2.22}$$

where

$$u(t_0) = \max\{0, -t\,\dot{\ell}(t_0)\}\exp(M/t_0) \tag{2.23}$$

depends only on M, t_0, $\dot{\ell}(t_0)$, and

$$v(t_0) = M\exp(M/t_0)t_0^{-1} \ , \tag{2.24}$$

which depends on M and t_0 only. From (2.22) we have

$$e^{\ell(t)} = e^{\ell(t_0)} \cdot (t/t_0)^{\nu} \cdot (t/t_0)^{\mu} \ . \tag{2.25}$$

Lemma 2.2 is proved.

REMARK. The theorem proved above corresponds to Theorem 1 of paper [1]. Our condition (2.3) exactly coincides with the condition (1.6) of [1]. Simultaneously the condition (2.3) is weaker than the corresponding condition (1.4) in [1]

Indeed, the condition (1.4) of [1] is of the form

$$t^{-1}([B_1,B_2]u,u) + (\dot{B}_1u,u) \ge -\lambda(B_1u,u) - C(u,u)$$

with $\lambda \ge 0$, $C \ge 0$. At the same time our condition (2.3) reads

$$t^{-1}([B_1,B_2]u,u) + (\dot{B}_1u,u) \ge -\gamma(t)|(B_1u,u)| - \beta(t)(u,u) \ ,$$

with $\gamma(t)$, $\beta(t)$ non-negative continuous functions in the interval I.

3. MAIN THEOREMS

It can be easily seen from the proof of Theorem 2 for the case (2.3a) that the following is true:

THEOREM 3. Let $B(t)$ be a linear operator with domain $D_B(t)$,

$$B(t) = B_1(t) + B_2(t) ,$$

where

$$B_1(t) = B_1^*(t) \text{ is the self-adjoint part of } B(t),$$

$$B_2(t) = -B_2^*(t) \text{ is the anti-self-adjoint part of } B(t).$$

We shall assume that $u(t) \in D_B(t)$, that $u(t) \in C^1(I,H)$, $B(t)u(t) \in C^1(I,H)$. Let $\phi(t)$ denote a non-negative continuous function in the interval I. Let a flat-function $u(t)$ be a solution of

$$\|Du(t) - B(t)u(t)\| \le t\phi(t)\|\mathcal{L}u(t)\| \tag{3.1}$$

such that

$$\begin{aligned} &\|B_1u - \frac{(B_1u,u)}{(u,u)}u\|^2 + ([B_1,B_2]u,u) + ((DB_1)u,u) \\ &\ge -\gamma(t)t|(B_1u,u)| - \beta(t)t\|u(t)\|^2 , \end{aligned} \tag{3.2}$$

where $\gamma(t)$, $\beta(t)$ are non-negative continuous functions in the interval I. Then $u(t) \equiv 0$ in I.

Now consider, instead of inequality (3.1), the equation

$$t\frac{du}{dt} = B(t)u(t) , \tag{3.3}$$

with the same assumptions regarding $B(t)$ as in Theorem 3. The following is true:

THEOREM 4. Let, with the assumptions of Theorem 3, $u(t)$ be a flat-function and solution of the equation (3.3). Then, if

$$\begin{aligned} &2\|B_1u - \frac{(B_1u,u)}{(u,u)}u\|^2 + ([B_1,B_2]u,u) + (t\dot{B}_1u,u) \\ &\ge -\gamma(t)t|(B_1u,u)| - \beta(t)t\|u(t)\|^2 , \end{aligned} \tag{3.4}$$

then $u(t) \equiv 0$ in I.

PROOF. Let

$$q(t) = (u(t),u(t)) , \tag{3.5}$$

$$\ell(t) = \log q(t) \tag{3.6}$$

and

$$t\dot{q}(t) = 2\mathrm{Re}(t\dot{u},u) = 2\mathrm{Re}(B(t)u(t),u(t)) = 2(B_1u,u)$$

$$t\dot{\ell}(t) = \frac{t\dot{q}(t)}{q(t)} = \frac{2}{q}(B_1u,u) . \tag{3.7}$$

Next it follows from (3.7) that $\ell(t)$ is twice differentiable, and

$$\begin{aligned} D^2\ell(t) &= \frac{2}{q}D(B_1u,u) - \frac{2}{q^2}(B_1u,u)Dq \\ &= \frac{2}{q}(t\dot{B}_1u,u) - \frac{4}{q}\mathrm{Re}(B_1u,Du) - \frac{2}{q^2}(B_1u,u) \cdot 2(B_1u,u) \\ &= \frac{2}{q}(t\dot{B}_1u,u) - \frac{4}{q}\|(B_1u\|^2 + \frac{4}{q}\mathrm{Re}(B_1u,B_2u) - \frac{4}{q^2}(B_1u,u)^2 \end{aligned} \tag{3.8}$$

Now

$$\frac{4}{q}\|B_1u\|^2 - \frac{4}{q^2}(B_1u,u)^2 = \frac{4}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2$$

and

$$\frac{4}{q}\mathrm{Re}(B_1u,B_2u) = \frac{2}{q}[(B_1u,B_2u) + (B_2u,B_1u)]$$
$$= \frac{2}{q}((-B_2B_1 + B_1B_2)u,u)$$
$$= \frac{2}{q}([B_1B_2]u,u)$$

From this we find

$$D^2\ell(t) = \frac{4}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2 + \frac{2}{q}([B_1,B_2]u,u) + \frac{2}{q}(t\dot{B}_1u,u) \ , \tag{3.9}$$

and from (3.4) we have

$$D^2(t) = \frac{2}{q}\{2\|B_1u - \frac{(B_1u,u)}{q}u\|^2 + \frac{2}{q}([B_1,B_2]u,u) + \frac{2}{q}(t\dot{B}_1u,u)\}, \tag{3.10}$$
$$\geq -\frac{2t}{q}[\gamma(t)|(B_1u,u)| + \beta(t)\|u\|^2 \ ,$$

and from (3.7) it follows that

$$D^2\ell(t) \geq -\gamma(t)t|D\ell(t)| - 2t\beta(t) \tag{3.11}$$

or

$$D^2\ell(t) + \gamma(t)t|D\ell(t)| + 2t\beta(t) \geq 0 \ . \tag{3.12}$$

From (3.12) and Lemma 2.2 it follows immediately that $u(t) \equiv 0$ in the interval I.

4. REMARKS.

REMARK 1. Our key step in proving all the theorems was to obtain an inequality for $\ell(t)$ of the form

$$D^2\ell(t) + t\alpha(t)|D\ell(t)| + t\beta(t) \geq 0 \ . \tag{4.1}$$

Therefore, it follows from (2.12) in the case of equation (3.3) (problem (0.1), i.e., when $f = \psi = 0$), that the following equality holds:

$$D^2\ell(t) = \frac{4}{q}\|B_1u\|^2 + \frac{2}{q}(t\dot{B}_{1_t}u,u) - \frac{4}{q^2}(B_1u,u)^2 + \frac{2}{q}([B_1,B_2]u,u)$$
$$= \frac{4}{q}\|B_1u - \frac{(B_1u,u)}{q}u\|^2 + \frac{2}{q}(t\dot{B}_1u,u) + \frac{2}{q}([B_1,B_2]u,u) \ . \tag{4.2}$$

In the course of deducing (4.1), one obtains the condition (3.4),

$$2\|B_1u - \frac{(B_1u,u)}{q}u\|^2 + (t\dot{B}_1u,u) + ([B_1,B_2]u,u) \geq -t\gamma(t)|B_1u,u| - t\beta(t)\|u(t)\|^2 \ .$$

Let us point out that it seems to us that this condition, obtained from (4.2), must be close enough to being necessary (for quasiuniqueness).

REMARK 2. For $t = 0$, (3.4) reduces to

$$2\|B_1u - \frac{(B_1u,u)}{q}u\|^2 + ([B_1,B_2]u,u)\Big|_{t=0} \geq 0 \ , \tag{4.3}$$

and since

$$B_1 = \frac{B + B^*}{2} \ , \ B_2 = \frac{B - B^*}{2} \ ,$$

it follows that

$$(B_1u,u)^2 = \frac{1}{4}\{(Bu,u)^2 + |(Bu,u)|^2 + (B^*u,u)^2 + |(B^*u,u)|^2\} \ ,$$

$$2\mathrm{Re}(B_1u,B_2u) = ([B_1,B_2]u,u) = \frac{1}{2}\{\|Bu\|^2 - \|B^*u\|^2\} \ ,$$

and $$\|B_1u\|^2 = \frac{1}{4}\{\|B\|^2 + \|B^*u\|^2\} + \frac{1}{2}Re(Bu,B^*u) .$$

In this case our condition reduces to the following one:

$$\frac{1}{2}\|Bu\|^2 + \frac{1}{2}\|B^*u\|^2 + Re(Bu,B^*u) - \frac{1}{2q}\{(Bu,u)^2 + |(Bu,u)|^2\}$$
$$- \frac{1}{2q}\{(B^*u,u)^2 + |B^*u,u)|^2\} + \frac{1}{2}\|Bu\|^2 - \frac{1}{2}\|B^*u\|^2\Big|_{t=0} \geq 0 .$$

For this condition to hold it is sufficient to have

$$\|Bu\|^2 - \|B^*u\|^2\Big|_{t=0} \geq 0$$

or

$$[B_1,B_2]\Big|_{t=0} \geq 0 , \tag{4.4}$$

i.e., if u is a flat solution of (0.2) and (4.4) holds, then $u \equiv 0$ in I (see also Remark 6).

REMARK 3. The method of the proof of the theorems concerning the quasiuniqueness of the solution of (0.1)-(0.2) presented in sections 2 and 3 allows one to assert, even in cases when there is no quasiuniqueness, that a given solution is trivial if the appropriate conditions are true for this solution. We have in mind the conditions (2.3), (2.3a), (3.2), (3.4).

It may quite happen that these conditions do not hold for all the solutions of (0.1)-(0.2). On the other hand, if for some specific solution $u(t)$ of (0.1) or (0.2) the appropriate condition does hold, then its triviality follows from the flatness of this specific $u(t)$. Quasiuniqueness of solution of (0.1)-(0.2) follows in the case when these conditions are satisfied by the whole class of possible solutions.

REMARK 4. It follows from Theorems 3 and 4 of section 3 that the quasiuniqueness takes place:

(i) If B does not depend on t, and $B_2(t) = 0$, i.e., for any constant symmetric operator B.

(ii) If $B_2(t) = 0$ and $B_1(t)$ satisfies the condition

$$C\|B_1u - \frac{(B_1u,u)}{(u,u)}u\|^2 + (t\dot{B}_{1_t}u,u) \geq -\gamma(t)t|(B_1u,u)| - t\beta(t)\|u(t)\|^2 . \tag{4.5}$$

Here $B_1(t)$ can be replaced by $B(t)$ and $C = 1$ for the problem (0.2) and $C = 2$ for the problem (0.1) correspondingly.

(iii) If $B_1(t) = 0$.

On the other hand, in case (iii) there even exists a classical uniqueness in the case of (0.1). This stems from the following:

$$t\frac{du}{dt} = B(t) ,$$

$$(t\frac{du}{dt},u(t)) = Re(B(t)u,u) = 0 ,$$

$$\frac{1}{2}t\frac{d}{dt}(u,u) = \frac{1}{2}t\frac{dq}{dt} = 0 ,$$

and if $q(0) = 0$, then $q(t) = 0$ for all t.

REMARK 5. The conditions in Theorems 2-4 do not seem natural, at any rate not at first sight. The following conditions seem more natural:

(i) $(\dot{B}_1u,u) \geq -\gamma(t)\{|(B_1u,u)| + \|u\|^2\}$ with $\gamma(t)$ a continuous function in I.

(ii) $Re(B_1u,B_2u) = \frac{1}{2}([B_1,B_2]u,u) \geq -\gamma(t)\{|(B_1u,u)| + \|u\|^2\}$ with the same $\gamma(t)$.

Then we have, from (2.12),

$$D^2\ell(t) + \frac{4}{q}\|f\|^2 \geq \frac{2}{q}\{\|B_1u - \frac{(B_1u,u)}{q}u\|^2 + (t\dot{B}_{1_t}u,u) + ([B_1,B_2]u,u)\}$$

$$\geq -\frac{2}{q}\gamma(t)t\{|(B_1u,u)| + \|u\|^2\} - \frac{2}{q}2\gamma(t)\{|B_1u,u)| + \|u\|^2\}$$

or

$$D^2\ell(t) + 4t^2\phi^2(t) + [2\gamma(t) + t\gamma(t)]|D\ell(t)| + 2[2\gamma(t) + t\gamma(t)]\|u\|^2 \geq 0 \tag{4.6}$$

or, introducing a new function

$$\beta(t) = 2\gamma(t) + t\gamma(t) ,$$

one obtains for $\ell(t)$,

$$D^2\ell(t) + \beta(t)|D\ell(t)| + 2\beta(t) + 4t^2\phi^2(t) \geq 0$$

or

$$2\ell(t) + \beta(t)|D\ell(t)| + \alpha(t) \geq 0 . \tag{4.7}$$

One can show, using the above reasoning, that

(a) in the case $\gamma(t) = t^{\varepsilon}\gamma_1(t)$, $\varepsilon > 0$, with $\gamma_1(t)$ bounded in I, there is quasiuniqueness,

(b) in the case $\gamma(t) = M + t^{\varepsilon}\gamma_1(t)$, $M > 0$, $\varepsilon > 0$, with $\gamma_1(t)$ bounded in I, the following estimate can be obtained:

$$\|u(t)\|^2 \geq Ct\|u(t_0)\|^2\exp[-(\nu + \mu)(t_0/t)^{2M}] \tag{4.8}$$

where

$$\nu - \min\frac{\alpha(\tau)}{\beta^2(t)} = \frac{1}{2M}, \quad \mu = \min\{0,t_0\dot{\ell}(t_0)\}\frac{1}{2M} .$$

Let us point out that in (4.8) we have a flat function.

(c) in the case of $\forall\ \varepsilon > 0$, $\exists\ \delta > 0 : \forall\ t\ [0,\delta]\ \gamma(t) < \varepsilon$.

Then the following estimate can be obtained:

$$\|u(t)\|^2 \geq Ct\|u(t_0)\|^2\exp[-(\nu + \mu)(t_0/t)^{2\varepsilon}] ,\ \forall\ \varepsilon > 0 \tag{4.9}$$

where

$$\nu = \frac{1}{2\varepsilon} , \quad \mu = \min\{0,t_0\dot{\ell}(t_0)\}\frac{1}{2\varepsilon} , \quad t_0 \in [0,\delta] .$$

We point out that in this case we have a flat function in (4.9).

REMARK 6. Consider the two following terms with our conditions:

$$\|B_1u - \frac{(B_1u,u)}{(u,u)}u\|^2 + ([B_1,B_2]u,u) . \tag{4.10}$$

We may have that the first non-negative term may improve the possible negativity of the second one. Unfortunately this is not the case. Let $\{e_k\}$ be an orthonormal basis of the eigenvectors of the operator B_1, assumed independent of t. Assume that the expression (4.10) is non-negative. Since the first term is identically zero on $\{e_k\}$, we then have

$$([B_1,B_2]e_k,e_k) \geq 0 . \tag{4.11}$$

Taking into account the orthogonality of $\{e_k\}$ and (4.11), we obtain the following:

$$([B_1,B_2]u,u) \geq 0 . \tag{4.12}$$

i.e., the appropriate term is non-negative.

5. EXAMPLE.

Let us consider the following equation

$$t\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}[-K_1(x)\frac{\partial u}{\partial x}] + t\frac{\partial}{\partial x}[-K_2(x)\frac{\partial u}{\partial x}] = A(t)u \tag{5.1}$$

where $x \in \Omega = [-1,1]$, $t \in (0,1]$. Here H is Hilbert space $L_2(\Omega)$ with condition

$$u|_{\partial\Omega} = 0 \tag{5.2}$$

and standard scalar product.

$K_1(x)$, $K_2(x)$ are smooth enough (from C^2) real-valued functions. In this case $A(t)$ will be the self-adjoint operator on $D_{A(t)} \subset H$ and

$$(A(t)u,u) = (K_1(x)\frac{\partial u}{\partial u},\frac{\partial u}{\partial u}) + t(K_2(x)\frac{\partial u}{\partial u},\frac{\partial u}{\partial u}) \tag{5.3}$$

$$(A(t)u,u) = (K_2(x)\frac{\partial u}{\partial u},\frac{\partial u}{\partial u}) . \tag{5.4}$$

Let us consider the following $K_1(x)$ and $K_2(x)$

$$K_1(x) = \begin{cases} x^3 & x \le 0 \\ 0 & x > 0 \end{cases} \tag{5.5}$$

$$K_2(x) = \begin{cases} x^3 & x \le 0 \\ x^4 & x > 0 \end{cases} \tag{5.6}$$

Then from (5.3)-(5.4) we have

$$(A(t)u,u) = (1 + t)(x^3 \frac{\partial u}{\partial x},\frac{\partial u}{\partial x})|_{\text{on } (-1,0)} + t(x^4 \frac{\partial u}{\partial x},\frac{\partial u}{\partial x})|_{\text{on } (0,+1)} \tag{5.7}$$

and

$$(A(t)u,u) = (x^3 \frac{\partial u}{\partial x},\frac{\partial u}{\partial x})|_{\text{on } (-1,0)} + t(x^4 \frac{\partial u}{\partial x},\frac{\partial u}{\partial x})|_{\text{on } (0,+1)} \tag{5.8}$$

Theorem 1 of [1] does not work in this case:

i) $A(t)$ is not negative;

ii) $([\lambda A(t) + A(t)]u,u)$ is not positive for all $\lambda > 0$;

iii) there is not an estimate of type $\|A(t)u\| \le C(\|Au\| + \|u\|)$ for any $C > 0$.

But from (5.7)-(5.8) we have that

$$(A(t)u,u) \ge -2(A(t)u,u) \tag{5.9}$$

and from Theorem 2, we obtain that the quasiuniqueness takes place for equation (5.1) under our assumptions (5.5)-(5.6).

REMARK. Of course, it is possible to construct an example of this type with C^∞ coefficients. It is possible also to construct an example of this type for non-self-adjoint operator $A(t)$.

ACKNOWLEDGEMENT. I wish to thank Professor Y. Kannai for the very fruitful discussions and guidance.

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