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ON SOLVABILITY OF BOUNDARY VALUE PROBLEM FOR ELLIPTIC EQUATIONS WITH BITSADZE-SAMARSKIĬ CONDITION

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ABSTRACT. In this paper we investigate the solvability of a non-local problem for a linear elliptic equation, which is also known as the boundary value problem with the Bitsadze-Samarskiĭ condition. We prove the existence and uniqueness of a classical solution to this problem. In the final part of this paper we propose an L^2-approach which gives a rise to weak solutions in a weighted Sobolev space. The crucial point in proving the existence of weak solutions is a suitable modification of the Bitsadze-Samarskiĭ condition.

KEY WORDS AND PHRASES. *Elliptic equations, non-local problems, Bitsadze-Samarskiĭ conditions.*
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1. INTRODUCTION. In recent years several authors have studied the solvability of non-local problems for elliptic and parabolic equations [1-9]. The importance of non-local problems appears to have been first noted in the literature by Bitsadze-Samarskiĭ. The problem studied in these papers constitutes a direct generalization of the classical boundary value problems. The most significant feature of nonlocal problems is that the boundary condition relates values of a solution on the boundary to its values on some part of the interior of the region. This type of the boundary value problem is often referred to as the boundary value problem with the Bitsadze-Samarskii condition [7],[8]. The problem (2.1),(2.2) discussed in this article arises from the mathematical description of some processes in a plasma (see paper [6] for full account of physical aspects of non-local problems).

The paper is organized as follows. In Section 2 we give the uniqueness and existence theorem of the classical solutions of the problem (2.1),(2.2). Our method is based on the maximum principle developed in papers [4] and [5]. Section 3 contains a discussion of the solvability of the non local problem for harmonic functions in a disc in R_2. The results of this section slightly improve the explicit formulae derived by Bitsadze (see [1] and [2]) for harmonic functions associated with some non-local problem. In a general case of a linear elliptic equation we reduce the problem (2.1),(2.2) to the solvability of the integral equation of the second kind. The final sections 4 and 5 are devoted to the study of the non-local problem for a linear elliptic equation with a parameter, whose principal part is in a divergence form. This allows us to remove some restrictions on the coefficient β appearing in the boundary condtion (2.2). On the other hand this also suggests further extensions of the solvability of the problem (2.1),(2.2) in a weighted Sobolev space $\tilde{W}^{1,2}(Q)$. We adopt here the L^2-approach to the Dirichlet problem with L^2-boundary data from [13] and [10].

2. UNIQUENESS AND A PRIORI ESTIMATE.

We consider a linear equation of the elliptic type

$$Lu = \sum_{i,j=1}^{n} a_{ij}(x)\, D_{ij}u + \sum_{i=1}^{n} b_i(x)D_iu + c(x)u = f(x) \tag{2.1}$$

in Q, where Q is a bounded domain in R_n. The purpose of this paper is to investigate the following non-local problem: given continuous functions h and β defined on the boundary ∂Q of Q find a solution $u \in C^2(Q) \cap C(\bar{Q})$ satisfying the boundary condition

$$u(x) - \beta(x)\ u(\phi(x)) = h(x) \text{ on } \partial Q, \tag{2.2}$$

where ϕ is a given continuous mapping of ∂Q into Q.

Throughout this section we make the following assumption

(A) The coefficients of the operator L are bounded in Q and there exists a constant $\gamma > 0$ such that

$$\gamma|\xi|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\ \xi_i\xi_j$$

for all $x \in Q$ and $\xi \in R_n$.

Moreover we assume that Q satisfies an interior sphere condition at each point of ∂Q (see [11], p. 33-35).

The uniqueness of the problem (2.1),(2.2) is a consequence of the strong maximum principle.

PROPOSITION 1. Let $|\beta(x)| \le 1$ on ∂Q and $c(x) \le 0$ in Q and suppose that either

(a) $-1 \le \beta(x_0) < 1$ at some point $x_0 \in \partial Q$

or

(b) $c(x_1) < 0$ at some point $x_1 \in Q$.

Then the problem (2.1),(2.2) has at most one solution in $C^2(Q) \cap C(\bar{Q})$.

PROOF. It is sufficient to show that if $f(x) \equiv 0$ on Q and $h(x) \equiv 0$ on ∂Q then $u \equiv 0$ is the only solution of the problem (2.1),(2.2). It is clear that under each of the assumptions (a) or (b) any constant solution must be identically equal to 0. If $u \not\equiv 0$ then u must be a non-constant solution and by the strong maximum principle ([11], Theorem 3.5) we may assume that

$$u(x_2) = \max_{\bar{Q}} u(x) > 0 \text{ with } x_2 \in \partial Q.$$

If $\beta(x_2) = 0$ we get a contradiction. Therefore it remains to consider two cases

$$\text{(i)} \quad 0 < \beta(x_2) \le 1 \quad \text{and} \quad \text{(ii)} \quad -1 \le \beta(x_2) < 0.$$

In the first case $u(\phi(x_2)) = \dfrac{u(x_2)}{\beta(x_2)} \ge u(x_2)$, which is impossible since $\phi(x_2) \in Q$. In the second case (ii) we have

$$u(\phi(x_2)) = \frac{u(x_2)}{\beta(x_2)} < 0$$

and by the strong maximum principle u takes on a negative minimum at $x_3 \in \partial Q$, that is

$$u(x_3) = \min_{\bar{Q}} u(x) < 0$$

and we may assume that $\beta(x_3) < 0$ since otherwise we get a contradiction. Hence

$$u(\phi(x_3)) = \frac{u(x_3)}{\beta(x_3)} > 0.$$

Now we distinguish two cases either

$$u(x_2) \le |u(x_3)| \quad \text{or} \quad u(x_2) > |u(x_3)|.$$

We show that both cases lead to a contradiction. Indeed, in the first case we have

$$u(x_2) \le |u(x_3)| = |\beta(x_3)||u(\phi(x_3))| \le |u(\phi(x_3))| = u(\phi(x_3)),$$

which is impossible. In the second case we have

$$|u(x_3)| < u(x_2) = \beta(x_2)\ u(\phi(x_2)) \leq |u(\phi(x_2))|.$$

Since both values $u(x_3)$ and $u(\phi(x_2))$ are negative u attains its negative minimum at $\phi(x_2) \in Q$ and we arrive at a contradiction.

Inspection of the proof of Proposition 1 shows that the following version of the maximum principle holds true.

PROPOSITION 2. Suppose that $c(x) \leq 0$ in Q and $0 \leq \beta(x) \leq 1$ on ∂Q. Let $Lu \leq 0$ (≥ 0) in Q, and $u(x) - \beta(x)\ u(\phi(x)) \geq 0$ (≤ 0) on ∂Q. Then $u(x) \geq 0$ (≤ 0) on $\bar{Q}$.

As an immediate consequence we deduce an a priori estimate

THEOREM 1. Suppose that $c(x) \leq -d$ in Q and $0 \leq \beta(x) \leq \alpha$ where $d > 0$ and $0 < \alpha < 1$ are constants. If u is a solution of the problem (2.1),(2.2), then

$$|u(x)| < \frac{1}{d}\sup_Q|f(x)| + \frac{1}{1-\alpha}\sup_{\partial Q}|h(x)| \text{ for all } x \in \bar{Q}.$$

PROOF. Let us define

$$v(x) = u(x) - \frac{1}{d}\sup_Q|f(x)| - \frac{1}{1-\alpha}\sup_{\partial Q}|h(x)|,$$

then we have

$$Lv = f - \frac{c}{d}\sup_Q|f(x)| - \frac{c}{1-\alpha}\sup_{\partial Q}|h(x)| \geq f + \sup_Q|f(x)| \geq 0$$

in Q and

$$v(x) - \beta(x)\ v(\phi(x)) = h(x) - \frac{1}{d}\sup_Q|f(x)| - \frac{1}{1-\alpha}\sup_{\partial Q}|h(x)| +$$

$$+ \frac{\beta(x)}{1-\alpha}\sup_{\partial Q}|h(x)| + \frac{\beta(x)}{\alpha}\sup_Q|f(x)| \leq$$

$$\leq (1 - \frac{1}{1-\alpha} + \frac{\alpha}{1-\alpha})\sup_{\partial Q}|h(x)| = 0$$

on ∂Q. Hence by Proposition 2

$$u(x) \leq \frac{1}{d}\sup_Q|f(x)| + \frac{1}{1-\alpha}\sup_{\partial Q}|h(x)|$$

on Q. Similarly we can establish the inequality

$$u(x) \geq -\frac{1}{d}\sup_Q|f(x)| - \frac{1}{1-\alpha}\sup_{\partial Q}|h(x)|$$

on Q, considering the auxiliary function

$$w(x) = u(x) + \frac{1}{d} \sup_{Q} |f(x)| + \frac{1}{1-\alpha} \sup_{\partial Q} |h(x)|.$$

REMARK 1. If $c \equiv 0$ on Q and $\beta(x) = 1$ on ∂Q, then any two solutions of the problem (2.1),(2.2) differ by a constant.

3. EXISTENCE OF CLASSICAL SOLUTIONS.

We commence by considering a particular case of the non-local problem (2.1),(2.2) which consists of finding a harmonic function u on $B(0,1)$ and satisfying the boundary condition

$$u(z) - \beta u(\phi(z)) = h(z) \quad \text{on } \partial B(0,1), \tag{3.1}$$

where $B(0,1)$ is an open disc in R_2 of radius 1 centred at 0, β is a constant in the interval $[-1,1]$ and h is a continuous function on $\partial B(0,1)$. The mapping ϕ is given by $\phi(z) = \phi_*(\delta z)$ with $0 < \delta < 1$, where ϕ_* is a univalent analytic function on $B(0,1)$ such that $|\phi_*(z)| \le 1$ in $B(0,1)$ and $\phi_*(0) = 0$. By virtue of Schwarz's lemma we have

$$|\phi_*(z)| \le |z| \text{ for all } z \in B(0,1). \tag{3.2}$$

The function ϕ maps disc $B(0,1)$ conformally and univalently onto certain set contained in $B(0,1)$. Letting $\phi_0(z) = z$ and $\phi_k(z) = \phi(\phi_{k-1}(z))$ for $k = 1,2,\ldots$ we have

$$|\phi_k(z)| < \delta^k |z| \text{ in } B(0,1), \tag{3.3}$$

for $k = 1,2,\ldots$. Since u and $u(\phi(z))$ are harmonic functions we have the following representation formula

$$u(z) - \beta u(\phi(z)) = \operatorname{Re} \frac{1}{\pi i} \int_{\partial B(0,1)} \left[\frac{1}{t-z} - \frac{1}{2t} \right] h(t)\,dt \equiv F(z) \tag{3.4}$$

Suppose first that $-1 < \beta < 1$. Iterating (3.4) we get

$$u(z) = \beta^n u(\phi_n(z)) + \sum_{k=1}^{n} \beta^{k-1} F(\phi_{k-1}(z)). \tag{3.5}$$

It follows from (3.3) that

$$|F(\phi_n(z))| = \left|\operatorname{Re} \frac{1}{2\pi i} \int_{\partial B(0,1)} \frac{t+\phi_n(z)}{t-\phi_n(z)t} h(t)\,dt\right| \le$$

$$< \frac{1+\delta}{2\pi(1-\delta)} \int_0^{\pi} |h(e^{is})|\,ds$$

for $n = 1,2,\ldots$ and consequently letting $n \to \infty$ in (3.5) we obtain

$$u(z) = \sum_{n=1}^{\infty} \beta^{n-1} F(\phi_{n-1}(z))$$

uniformly on $\overline{B(0,1)}$.

Let us now consider the case $\beta = -1$. It follows from (3.4) that

$$2u(0) = \mathrm{Re}\, \frac{1}{\pi i} \int_{\partial B(0,1)} \frac{h(t)}{2t}\, dt \tag{3.6}$$

and the functional equation (3.4) can be written in the form

$$u(z) + u(\phi(z)) - 2u(0) = \mathrm{Re}\, \frac{1}{\pi i} \int_{\partial B(0,1)} \left[\frac{1}{t-z} - \frac{1}{2t} - \frac{1}{2t}\right] h(t)\, dt$$

$$= \mathrm{Re}\, \frac{1}{\pi i} \int_{\partial B(0,1)} \frac{h(t)}{t(t-z)}\, dt \cdot z \equiv \mathrm{Re}[\Phi(z)\cdot z].$$

Iterating the last equation we obtain

$$u(z) = u(\phi_{2n}(z)) + \sum_{j=0}^{2n-1} (-1)^j \,\mathrm{Re}[\Phi(\phi_j(z))\ \phi_j(z)] \tag{3.7}$$

and

$$u(z) = 2u(0) - u(\phi_{2n-1}(z)) + \sum_{j=0}^{2n-2} (-1)^j \,\mathrm{Re}[\Phi(\phi_j(z))\ \phi_j(z)]. \tag{3.8}$$

It is easy to see that

$$|\Phi(\phi_n(z))| \le \frac{\delta}{\pi(1-\delta)} \int_0^{2\pi} |h(e^{is})|\, ds$$

for all $n = 1,2,\ldots$ and $z \in \overline{B(0,1)}$. Since $|\phi_j(z)| < \delta^j$ in $\overline{B(0,1)}$ the series $\sum_{j=0}^{\infty} (-1)^j \,\mathrm{Re}[\phi(\phi_j(z))\phi_j(z)]$ converges uniformly on $\overline{B(0,1)}$. Letting $n \to \infty$ in (3.7) and (3.8) we obtain the same limit in both cases

$$u(z) = u(0) + \sum_{j=0}^{\infty} (-1)^j \,\mathrm{Re}[\Phi(\phi_j(z))\ \phi_j(z)]$$

and invoking (3.6) we get that

$$u(z) = \mathrm{Re}\, \frac{1}{4\pi i} \int_{\partial B(0,1)} \frac{h(t)}{t}\, dt + \sum_{j=0}^{\infty} (-1)^j \,\mathrm{Re}[\Phi(\phi_j(z))\phi_j(z)].$$

Finally let $\beta = 1$, then

$$u(z) - u(\phi(z)) = \mathrm{Re}\, \frac{1}{\pi i} \int_{\partial B(0,1)} \left[\frac{1}{t-z} - \frac{1}{2t}\right] h(t)\, dt.$$

By Remark 1 any two solutions differ by a constant. If $z = 0$, then

$$0 = u(0) - u(\phi(0)) = \mathrm{Re}\,\frac{1}{\pi i} \int_{\partial B(0,1)} \frac{f(t)}{t}\,dt,$$

which gives a necessary and sufficient condition for the solvability of the problem (3.1). Hence

$$u(z) - u(\phi(z)) = \mathrm{Re}[\Phi(z)z].$$

Iterating this functional equation we obtain

$$u(z) = u(\phi_n(z)) + \sum_{j=1}^{n-1} \mathrm{Re}[\Phi(\phi_j(z))\ \phi_j(z)].$$

Letting $n \to \infty$ we obtain

$$u(z) = u(0) + \sum_{j=1}^{\infty} \mathrm{Re}[\Phi(\phi_j(z))\ \phi_j(z)].$$

To determine a solution in a unique way we may impose an additional condition $u(0) = C$, where $C > 0$ is a given constant. The case $\beta = 1$ was considered by Bitsadze in [1] and [2] under the assumption that h is Hölder continuous on ∂Q.

In a general case we reduce the problem (2.1),(2.2) to the Fredholm integral equation of the second kind.

THEOREM 2. Suppose that the assumptions of Proposition 1 hold and that $D^2_{ij} a_{ij}$ $(i,j = 1,\dots,n)$, $D_i b_i$ $(i = 1,\dots,n)$ and c are Hölder continuous on $\bar{Q}$. Then the problem (2.1),(2.2) admits a unique solution u in $C^2(Q) \cap C(\bar{Q})$.

PROOF. We try to find a solution in the form

$$u(x) = \int_{\partial Q} \frac{dG(x,y)}{dn_y} v(y)\,dS_y - \int_Q G(x,y)\,f(y)\,dy, \tag{3.9}$$

where $v \in C(\partial Q)$ is to be determined, G is the Green function for the operator L and $\frac{dG}{dn_y}$ denotes the conormal derivative. The boundary condition (2.2) leads to the Fredholm integral equation of the second kind.

$$v(x) - \int_{\partial Q} \beta(x) \frac{dG(\phi(x),y)}{dn_y} v(y)\,dS_y = h(x) +$$

$$+ \int_Q \beta(x)\,G(\phi(x),y)\,f(y)\,dy. \tag{3.10}$$

Since $\phi(\partial Q) \subset Q$ the kernel $\beta(x)\,\frac{dG(\phi(x),y)}{dn_y}$ is continuous function on $\partial Q \times \partial Q$. By Proposition 1 the homogeneous equation corresponding to (3.10) has only trivial solution. Hence by the Fredholm alternative there exists a unique solution $v \in L^2(\partial Q)$ which by the continuity of the kernel belongs to $C(\partial Q)$. Consequently the formula (3.9) gives a solution to the problem (2.1),(2.2).

4. ENERGY ESTIMATE.

In this section we consider the elliptic equation in the form

$$Mu + \lambda u = - \sum_{i,j=1}^{n} D_i(a_{ij}(x)\, D_j u) + \sum_{i=1}^{n} b_i(x)\, D_i u + c(x)u +$$

$$+ \lambda u = f(x) \text{ in } Q, \tag{4.1}$$

with the boundary condition (2.2).

Throughout this section we assume that β is a continuous function on ∂Q and $\phi:\partial Q \to Q$ is a C^1-mapping with the positive Jacobian. Further we assume that a_{ij}, $D_i a_{ij}$, b_i $(i,j = 1,\dots,n)$ c and f are Hölder continuous on $\bar{Q}$ and that Q is a bounded domain with the boundary of class C^2.

The objective of this section is to show that the problem (4.1),(2.2) has a unique solution for large values of the parameter λ.

For small $\delta > 0$ we define $Q_\delta = Q \cap \{x;\ \min_{y \in \partial Q} |x-y| > \delta\}$.

According to Lemma 14.16 in [11] (p. 355), the distance $r(x) = \text{dist}(x,\partial Q)$ belongs to $C^2(\bar{Q}-Q_{\delta_o})$ if δ_o is sufficiently small. Denote by $\rho(x)$ the extension of the function $r(x)$ into $\bar{Q}$ satisfying the following properties

$\rho(x) = r(x)$ for $x \in \bar{Q} - Q_{\delta_o}$, $\rho \in C^2(\bar{Q})$, $\rho(x) > \frac{3\delta_o}{4}$ in Q_{δ_o},

$\gamma_1^{-1}\, r(x) \le \rho(x) \le \gamma_1\, r(x)$ in Q for some constant $\gamma_1 > 0$, $\partial Q_\delta = \{x;\ \rho(x) = \delta\}$ for $\delta \in (0,\delta_o)$ and finally $\partial Q = \{x; \rho(x) = 0\}$.

THEOREM 3. There exist positive constants λ_o, C and d such that if u is a solution in $C^2(Q) \cap C(\bar{Q})$ of the problem (4.1),(2.2) for $\lambda \ge \lambda_o$ then

$$\int_Q |Du(x)|^2\, r(x)dx + \int_Q u(x)^2 r(x)dx + \sup_{0<\delta\le d} \int_{\partial Q_\delta} u(x)^2\, dS_x \le$$

$$\le C\Big(\int_{\partial Q} h(x)^2\, dS_x + \int_Q f(x)^2\, dx\Big).$$

Proof. We follow the proof of Theorem 5 in [13]. Multiplying (4.1) by

$$v(x) = \begin{cases} u(x)\ (\rho(x)-\delta) & \text{on } Q_\delta \\ 0 & \text{on } Q - Q_\delta \end{cases}$$

and integrating by parts we obtain

$$\frac{1}{2}\int_{\partial Q_\delta} \sum_{i,j=1}^{n} a_{ij}(x)\, D_i\rho\, D_j\rho\, u^2\, dS_x = -\frac{1}{2}\int_{Q_\delta} \sum_{i,j=1}^{n} D_i(a_{ij}(x)\, D_j\rho) u^2 dx +$$

$$+ \int_Q \sum_{i=1}^{n} b_i(x)\, D_j u\cdot u(\rho-\delta)dx + \int_{Q_\delta} (c(x)+\lambda)\, u^2(\rho-\delta)dx - \int_{Q_\delta} fu(\rho-\delta)dx.$$

Applying Hölder's inequality we easily obtain

$$\sup_{0<\delta\le d} \int_{\partial Q_\delta} u^2\, dS_x \le C_1\left[\int_Q |Du|^2 \rho\, dx + \int_Q u^2\, dx + \right.$$

$$\left. + \lambda \int_Q u^2 \rho\, dx + \int_Q f^2\, dx\right], \tag{4.2}$$

where d and C_1 are positive constants.

Similarly

$$\int_Q |Du|^2 \rho\, dx + \lambda \int_Q u^2 \rho\, dx \le C_2\left[\int_Q u^2\, dx + \int_Q f^2\, dx + \int_{\partial Q} u^2\, dS_x\right] \tag{4.3}$$

for some $C_2 > 0$. It follows from (2.2) that

$$\int_Q |Du|^2 \rho\, dx + \lambda \int_Q u^2 \rho\, dx \le C_3\left[\int_Q u^2\, dx + \int_Q f^2\, dx + \right.$$

$$\left. + \int_{\partial Q} h^2\, dS_x + \int_{\partial Q} u(\phi(x))^2 dS_x\right], \tag{4.4}$$

where $C_3 > 0$. The estimates (4.2) and (4.4) yield that

$$\int_Q |Du|^2 \rho\, dx + \lambda \int_Q u^2 \rho\, dx + \sup_{0<\delta<d} \int_{\partial Q} u^2\, dS_x \le$$

$$\le C_4\left[\int_Q u^2\, dS_x + \int_Q f^2\, dx + \int_{\partial Q} h^2\, dS_x + \int_{\partial Q} u(\phi(x))^2\, dS_x\right], \tag{4.5}$$

for some $C_4 > 0$. Since ϕ is a C^1-mapping with the positive Jacobian it is obvious that

$$\int_{\partial Q} u(\phi(x))^2\, dS_x \le C_5 \int_{\phi(\partial Q)} u^2\, dS_x \le C_6\left[\int_{Q_\phi} |Du|^2\, dx + \int_{Q_\phi} u^2\, dx\right], \tag{4.6}$$

where $C_5 > 0$ and $C_6 > 0$ are constants and Q_ϕ is a domain containing $\phi(\partial Q)$ with $\mathrm{dist}(Q_\phi, \partial Q) > 0$. Consequently by virtue of the Caccioppoli inequality we have

$$\int_{\partial Q} u(\phi(x))^2\, dS_x \le C_7\left[\int_Q u^2\, dx + \int_Q f^2\, dx\right] \tag{4.7}$$

for some $C_7 > 0$. We now observe that

$$\int_Q u^2\, dx \le \frac{1}{d_1} \int_Q u^2 \rho\, dx + d \sup_{0<\delta\le d} \int_{\partial Q} u^2\, d S_x, \tag{4.8}$$

where $d_1 = \inf_{Q_d} \rho(x)$. Choosing d sufficiently small and λ sufficiently large we easily derive the desired estimate from (4.5),(4.6),(4.7) and (4.8).

Repeating the argument of Theorem 2 we deduce the following

THEOREM 4. There exists a positive constant λ_o such that for every $\lambda \ge \lambda_o$ the problem (4.1),(2.2) admits a unique solution in $C^2(Q) \cap C(\bar{Q})$.

5. WEAK SOLUTIONS.

The energy estimate from Section 4 shows that one can expect solutions of the problem (4.1),(2.2) in a weighted Sobolev space defined by

$$\tilde{W}^{1,2}(Q) = \{u \in W^{1,2}_{loc}(Q);\ \int_Q |Du(x)|^2\, r(x)\, dx + \int_Q u(x)^2\, dx < \infty\}$$

and equipped with the norm

$$||u||^2_{\tilde{W}^{1,2}} = \int_Q [\,|Du(x)|^2\, r(x) + u(x)^2]\, dx.$$

We recall briefly that a function u is said to be a weak (generalized) solution of (4.1) if $u \in W^{1,2}_{loc}(Q)$ and it satisifies

$$\int_Q [\sum_{i,j=1}^n a_{ij} D_i u D_j v + \sum_{i=1}^n b_i D_i u \cdot v + (c+\lambda)uv]dx = \int_a f\, v\, dx \tag{5.1}$$

for each $v \in W^{1,2}(Q)$ with compact support in Q.

To proceed further we need some terminology. It follows from the regularity of the boundary ∂Q that there exists a number δ_o such that for $\delta \in (0,\delta_o)$ the domain Q_δ (defined in Section 4) with the boundary ∂Q_δ possesses the following property: to each $x_o \in \partial Q$ there exists a unique point $x_\delta(x_o) \in \partial Q_\delta$ such that $x_\delta(x_o) = x_o - \delta\nu(x_o)$, where $\nu(x_o)$ is the outward normal to ∂Q at x_0. The above relation gives a one-to-one mapping of class C^1, of ∂Q onto ∂Q_δ.

It is known that elements of the space $\tilde{W}^{1,2}(Q)$, in general, do not have traces on the boundary ∂Q (see [12]). However, by Theorem 4 in [13], if $u \in \tilde{W}^{1,2}(Q)$ is a solution of (4.1) then there exists a function $\varsigma \in L^2(\partial Q)$ such that

$$\lim_{\delta \to 0} \int_{\partial Q} [u(x_\delta) - \varsigma(x)]^2 \, dS_x = 0.$$

Therefore, as in the paper [13], we adopt the following L^2-approach to the problem (4.1),(2.2).

Let $h \in L^2(\partial Q)$. A weak solution $u \in W^{1,2}_{loc}(Q)$ of (4.1) is a solution of the non-local problem with the boundary condition (2.2) if

$$\lim_{\delta \to 0} \int_{\partial Q} [u(x_\delta) - \beta(x)\, u(\phi(x)) - h(x)]^2 \, dS_x = 0. \qquad (5.2)$$

It follows from Theorem 1 in [13] that if $u \in W^{1,2}_{loc}(Q)$ is a solution of the problem (4.1),(2.2) (with the boundary condition (2.2) understood in the sense of (5.2) then $u \in \tilde{W}^{1,2}(Q)$. We mention also that $u(\phi(x))$ is understood in the sense of trace, which is well defined since $u \in W^{1,2}_{loc}(Q)$ (see [14],chap.6).

We now are in a position to establish the existence result in $\tilde{W}^{1,2}(Q)$ of the problem (4.1),(2.2).

THEOREM 5. Let $h \in L^2(\partial Q)$. Then there exists a positive constant λ_o such that for $\lambda \geq \lambda_o$ the problem (4.1), (2.2) in $W^{1,2}_{loc}(Q)$ admits a unique solution.

PROOF. Let h_m be a sequence in $C^1(\partial Q)$ such that $\lim_{m\to\infty} \int_{\partial Q} (h_m - h)^2 \, dS_x = 0$. Let λ_o be a constant from Theorem 4 and assume that $\lambda \geq \lambda_o$. For each $m \geq 1$ Theorem 4 guarantees the existence of the unique solution $u_m \in C^2(Q) \cap C(\bar{Q})$ of the problem (4.1),(2.2) with $h = h_m$. Moreover we have for each u_m

$$\int_Q |Du_m|^2 \, r dx + \int_Q u_m^2 \, dx \leq C\left(\int_Q f^2 \, dx + \int_{\partial Q} h_m^2 \, dS_x\right),$$

where $C > 0$ is a constant independent of m. Since the sequence u_m is bounded in $\tilde{W}^{1,2}(Q)$, there exists a subsequence, which we relabel as u_m, converging weakly in $\tilde{W}^{1,2}(Q)$ to a function u. By Theorem 4.11 in [15], $\tilde{W}^{1,2}(Q)$ is compactly embedded in $L^2(Q)$ and therefore we may assume that u_m

converges to u in $L^2(Q)$. It is obvious that u is a weak solution of (4.1). By Theorem 4 in [13] u has a trace $\varsigma \in L^2(\partial Q)$ in the sense of L^2-convergence, that is

$$\lim_{\delta\to 0} \int_{\partial Q} [u(x_\delta) - \varsigma(x)]^2\, dS_x = 0.$$

To complete the proof we show that $\varsigma(x) = \beta(x)\, u(\phi(x)) + h(x)$ a.e. on ∂Q. Let $\Phi \in C^1(Q)$. It is easy to show $\Phi(x)\, \rho(x)$ is a legitimate test function in (5.1) and integrating by parts we obtain

$$\int_{\partial Q} \varsigma\, \Phi \sum_{i,j=1}^{n} a_{ij}\, D_i\rho\, D_j\rho\, dS_x = -\int_Q \sum_{i,j=1}^{n} D_i(a_{ij}\, D_j\rho\, \Phi)\, u\, dx +$$

$$+ \int_Q \sum_{i=1}^{n} b_i\, D_i u\, \Phi dx + \int_Q (c+\lambda) u\, \Phi\, \rho\, dx - \int_Q f\, \Phi\, u\, \rho\, dx \equiv$$

$$\equiv F(u). \qquad (5.3)$$

Similarly

$$\int_{\partial Q} [h_m(x) + \beta(x) u_m(\phi(x))] \sum_{i,j=1}^{n} a_{ij}\, D_i\rho\, D_j\rho\, dS_x \equiv F(u_m). \quad (5.4)$$

Applying the estimate (4.6) and the obvious analogue of the energy estimate to $u_p - u_q$ we obtain

$$\int_{\partial Q} [u_p(\phi(x)) - u_q(\phi(x))]^2\, dS_x \le C[\int_{Q_\phi} |Du_p - Du_q|^2\, dx +$$

$$+ \int_Q |u_p - u_q|^2\, dx] \le \frac{C}{k} \int_Q |Du_p - Du_q|^2 \rho\, dx + C \int_Q |u_p - u_q|^2\, dx \le$$

$$\le \tilde{C} \int_{\partial Q} |h_p - h_q|^2\, dS_x,$$

where $k = \inf_{\bar{Q}_\phi} \rho(x)$ and $\tilde{C} > 0$ is a constant independent of p and q. Hence by the continuity of weak solutions on Q we may assume that $u_m(\phi(x)) \to u(\phi(x))$ as $m \to \infty$ in $L^2(\partial Q)$. Combining this with the fact that $F(u_m) \to F(u)$ as $m \to \infty$ we deduce from (5.3) and (5.4) that $\varsigma(x) = \beta(x) u(\phi(x)) + h(x)$ a.e. on ∂Q and this completes the proof.

REMARK 2. It is worth noting that the non-local problem of the type (4.1),(2.2) has been studied in [9] for the higher order elliptic equations. The corresponding boundary datum h in [9] belongs to the space $H^{\frac{1}{2}}(\partial Q)$. Since $H^{\frac{1}{2}}(\partial Q)$ is a proper subspace $L^2(\partial Q)$, Theorem 5 cannot be deduced from the results of the paper [9].

REFERENCES.

1. BITSADZE, A.V., 'On the theory of nonlocal boundary value problems', Soviet Mat. Dokl. 30(1) (1984), 8-10.

2. BITSADZE, A.V., 'On a class of conditionally solvable non-local boundary value problems for harmonic functions', Soviet Mat. Dokl. 31(1) (1985), 91-94.

3. BITSADZE, A.V. and SAMARSKIĬ, A.A., 'On some simple generalizations of linear elliptic boundary problems', Soviet Math. Dokl. 10(2) (1969), 398-400.

4. CHABROWSKI, J., 'On non-local problems for parabolic equations', Nagoya Math. J., 93 (1984), 109-131.

5. CHABROWSKI, J., 'On the non-local problem with a functional for parabolic equation', Funkcialaj Ekvacioj, 27(1) (1984), 101-123.

6. SAMARSKII, A.A., 'Some problems in differential equation theory', Differencial'nye Uravneniya, 16(11), (1980), 1925-1935.

7. SKUBACZEVSKI, A.L., 'Solvability of elliptic problems with Bitsadze-Samarskii boundary conditions'. Differencial'nye Uravneniya, 21(4) (1985), 701-706.

8. SKUBACZEVSKI, A.L., 'Non-local elliptic problems with a parameter', Mat. USSR Sb. 49(1) (1984), 197-206.

9. SKUBACZEVSKI, A.L., 'Some non-local elliptic boundary-value problems', Differencial'nye Uravneniya 18(9) (1982), 1590-1599.

10. MIKHAILOV, V.P., 'Boundary values of elliptic equations in domains with a smooth boundary', Mat Sb. 101 (143) (1976), 163-188.

11. GILBARG, P. and TRUDINGER, N.S., 'Elliptic partial differential equations of second order', Grundlehrnen der Mathematischen Wissenschaften 224, Springer-Verlag, Berlin-Heidelberg-New York-Tokyo, 1983 (Second Edition).

12. POULSEN, Ebbe Thue, 'Boundary value properties connected with some improper Dirichlet integrals', Math. Scand. 8 (1960), 5-14.

13. CHABROWSKI, J. and THOMPSON, B., 'On the boundary values of the solutions of linear elliptic equations', Bull. Austral. Math. Soc. 27 (1983), 1-30.

14. KUFNER, A., JOHN, O. and FUČIK, S., 'Function spaces', Leyden, Noordhoff, Prague, Academia, 1977.

15. MEYER, R.D., 'Some embedding theorems for generalized Sobolev spaces and applications to degenerate elliptic differential operators', J. Math. Mech. 16 (1967), 739-760.