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ON SOME CLASSES OF ANALYTIC FUNCTIONS

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ABSTRACT: Let m_1, m_2 be any numbers and let V_{m_1,m_2} be the class of functions of analytic in the unit disc $E=\{z:|z|<1\}$ for which

$$f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}}$$

where S_1 and S_2 are analytic in E with $S_1'(0)=(S_2'(0)=1$. Moulis [1] gave a sufficient condition and a necessary condition on parameters m_1 and m_2 for the class V_{m_1,m_2} to consist of univalent functions if S_1 and S_2 are taken to be convex univalent functions in E. In fact he proved that if $f\varepsilon V_{m_1,m_2}$ where S_1 and S_2 are convex and $m_1 = \frac{k+2}{4} e^{-i\alpha}(1-\rho)\cos\alpha$, $m_2 = \frac{k-2}{4}(1-\rho)e^{-i\alpha}\cos\alpha$, $2|m_1+m_2| \leq 1$, then f is univalent in E.

In this paper we consider the class V_{m_1,m_2} in more general way and show that it contains the class of functions with bounded boundary rotation and many other classes related with it. Some coefficient results, arclength problem, radius of convexity and other problems are proved for certain cases. Our results generalize many previously known ones.

KEY WORDS AND PHRASES. Univalent functions, boundary rotation, radius of convexity.
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1. INTRODUCTION.

Let $V_k^\alpha(\rho)$ be the class of all functions f, analytic in $E=\{z:|z|<1\}$, $f'(0)=1$, $f(0)=0$, $f'(z)\neq$ such that for $z=re^{i\theta}$, $0\leq r<1$

$$\int_0^{2\pi} \left|\mathrm{Re}\ \frac{e^{i\alpha}(zf'(z))'-\rho\cos\alpha}{1-\rho}\right| d\theta \leq k\pi \cos\alpha,$$

where $k\geq 2$, $0\leq\rho<1$, α real and $|\alpha| < \frac{\pi}{2}$.

The class $V_k^\alpha(\rho)$ has been introduced and studied by Moulis in [1]. For $\rho=0$, we obtain the class V_k^α introduced and studied in [2]. $\rho=0$ and $\alpha=0$ give us the well known class V_k of functions with bounded boundary rotation first introduced and discussed by Paatero [3] and Lowner [4]. Functions in V_k^α and $V_k^\alpha(\rho)$ may not possess boundary rotation.

Also a class $T_k^\alpha(\rho)$ of analytic functions which is a generalization of $V_k^\alpha(\rho)$ has been discussed in [5]. A function f, analytic in E, $f(0)=0=f'(0)-1$ is in $T_k^\alpha(\rho)$ if for $z\varepsilon E$, there exists a function g in $V_k^\alpha(\rho)$ such that

$$\operatorname{Re}\frac{f'(z)}{g'(z)} > 0$$

The cases when $\rho=0$ and $\rho=0$, $\alpha=0$ have been discussed in [6] and [7] respectively.

Definition 1.1

Let m_1 and m_2 be any numbers and S_1 and S_2 be analytic functions in E with $S_1(0)=0=S_2(0)$ and $S_1'(0)=1=S_2'(0)$. Then $f\varepsilon V_{m_1,m_2}$ if and only if

$$f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}} \tag{1.1}$$

We have the following special cases.

Case A. Let $m_1=\frac{k+2}{4}$, $m_2=\frac{k-2}{4}$, $k\geq 2$ in (1.1). Then

(i) $V_{m_1,m_2}=V_k$, the class of functions with bounded boundary rotation if S_1 and S_2 are convex univalent functions. This was proved by Brannan in [8].

(ii) $V_{m_1,m_2}\equiv T_k^0(0) = T_k$ if S_1 and S_2 are close-to-convex univalent functions, see [7].

(iii) V_{m_1,m_2} coincides with V_k^α if zS_1' and zS_2' are α-spiral-like functions. This result is shown in [2].

(iv) $V_{m_1,m_2}\equiv T_k^\alpha$ if S_1 and $S_2\varepsilon T_2^\alpha(0)$, see [6] and $V_{m_1,m_2}\equiv T_k^\alpha(\rho)$ if S_1 and $S_2\varepsilon T_2^\alpha(\rho)$, see [5].

Case B. Let S_1 and S_2 be convex univalent functions in (1.1). Then we have the following subcases:

(i) If $m_1=\frac{k+2}{4}e^{-i\alpha}\cos\alpha$, $m_2=\frac{k-2}{4}e^{-i\alpha}\cos\alpha$, then $f\varepsilon V_k^\alpha$ in (1.1). See [2].

(ii) If $m_1 = \frac{k+2}{4}(1-\rho)e^{-i\alpha}\cos\alpha$, $m_2=\frac{k-2}{4}(1-\rho)e^{-i\alpha}\cos\alpha$, then $f\varepsilon V_k^\alpha(\rho)$ in relation (1.1). This is shown in [1].

2. MAIN RESULTS

We now proceed to prove the main results for the class V_{m_1,m_2}. Wherever needed, certatin restrictions on the parameters m_1 and m_2 and on analytic functions S_1 and S_2 will be imposed.

Theorem 2.1

Let $f\ \varepsilon\ V_{m_1,m_2}$ such that

$$f'(z) = \frac{(S_1'(z))^{m_1}}{(S'_2(z))^{m_2}},$$

where S_1 and S_2 are convex univalent in E. Let

$$I_\lambda(r) = \frac{1}{2\pi}\int_0^{2\pi} |f'(re^{i\theta})|^\lambda d\theta, \tag{2.1}$$

where $0\le r<1$ and $2m_1\lambda>1$; $m_1, m_2>0$

Then

$$\lim_{r\to 1}\text{Sup}\ (1-r)^{2m_1\lambda-1}\ I_\lambda(r) \le A(m_1,m_2,\lambda)$$

where

$$A(m_1,m_2,\lambda) = \frac{2^{2m_2\lambda}\Gamma(m_1\lambda+\frac{1}{2})}{\pi^{\frac{1}{2}}(2m_1\lambda-1)\Gamma(m_1\lambda)}$$

Proof

$$I_\lambda(r) = \frac{1}{2\pi}\int_0^{2\pi}\frac{|S_1'(z)|^{m_1\lambda}}{|S_2'(z)|^{m_2\lambda}}d\theta,\ S_1,\ S_2 \text{ are convex functions, } m_1, m_2>0.$$

Then $|S_2'(z)| \ge \frac{1}{(1+r)^2}$ by the distortion theorems for convex functions [9] and S_1' is subordinate to $(1-z)^{-2}$ in E. Consequently

$$I_\lambda(r) \le \frac{1}{2\pi}(1+r)^{2m_2\lambda}\int_0^{2\pi}\frac{1}{|1-re^{i\theta}|^{2m_1\lambda}}d\theta = (1+r)^{2m_2\lambda} J_{2m_1\lambda}(r), \text{ say} \tag{2.2}$$

Now it has been shown by Pommerenke in [10] that

$$J_p(r) \simeq \frac{\Gamma(p-1)}{2^{p-1}\Gamma^2(p)}\ \frac{1}{(1-r)^{p-1}},\ p>1,\quad r\to 1$$

$$= \frac{\Gamma(\frac{1}{2}p+\frac{1}{2})}{\pi^{\frac{1}{2}}(p-1)\Gamma(\frac{1}{2}p)}\cdot\frac{1}{(1-r)^{p-1}} \tag{2.3}$$

using the recurrence and duplication formulae for the Gamma function. Substitution of (2.3) in (2.2) completes the proof.

Corollary 2.1

Let $m_1 = \frac{k+2}{4}$, $m_2 = \frac{k-2}{4}$. Then $f\in V_k$

and

$$\lim_{r\to 1}\text{Sup}\ (1-r)^{(\frac{1}{2}k-1)\lambda-1}\ I_\lambda(r) \le A(k,\lambda), \text{ where}$$

$$A(k,\lambda) = \frac{2^{(\frac{1}{2}k-1)\lambda}\Gamma(\frac{1}{4}k\lambda+\frac{1}{2}\lambda+\frac{1}{2})}{\pi^{\frac{1}{2}}(\frac{1}{2}k\lambda+\lambda-1)\Gamma(\frac{1}{4}k\lambda+\frac{1}{2}\lambda)}$$

This result was proved in [8].

Theorem 2.2

Let $f\varepsilon V_{m_1,m_2}$, and S_1, S_2 be convex functions. Let $L(r)$ denote the length of the arc $f(|z|=r)$ given by the formula for $z=re^{i\theta}$.

$$L(r)=\int_0^{2\pi} |zf'(z)|d\theta$$

Then, for $m_1>\frac{1}{2}$, $m_2>0$, we have

$$L(r)=0(1).\frac{1}{(1-r)^{2m_1-1}},$$

where $0(1)$ is a constant depending only on m_1 and m_2.

The proof follows immediately from Theorem 2.1 by taking $\lambda=1$.

From Theorem 2.1 and the standard inequality [9,p.11].

$$|a_n|<\frac{e}{n} I_1(1-\frac{1}{2}),$$

we have the following.

Theorem 2.3

Let $f\varepsilon V_{m_1,m_2}$ and be given by (1.1) with $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ where S_1 and S_2 in (2.1) are convex, $m_1>\frac{1}{2}, m_2>0$. Then for $n\geq 2$

$$\lim_{n\to\infty} \text{Sup}\ n^{2-2m_1}|a_n|\leq\frac{e2^{2m_2}\Gamma(m_1+\frac{1}{2})}{\pi^{\frac{1}{2}}(2m_1-1)\Gamma(m_1)}$$

Corollary 2.2

If $m_1=\frac{k+2}{4}$, $m_2=\frac{k-2}{4}$ in Theorem 2.3 then $f\varepsilon V_k$ and

$$\lim_{n\to\infty} \text{Sup}\ [n^{1-\frac{1}{2}k}|a_n|]\leq\frac{e\ 2^{\frac{1}{2}k}\ \Gamma(\frac{k}{4}+1)}{\pi^{\frac{1}{2}}(k)\Gamma(\frac{k}{4}+\frac{1}{2})}$$

This result was proved in [8].

Theorem 2.4

Let $f\varepsilon V_{m_1,m_2}$ with S_1 and S_2 convex and $m_1>1$, $m_2>0$. Let f be given by $f(z)\quad z+\sum_{n=2}^{\infty} a_n z^n$. Then for $n\geq 1$

$$\left||a_{n+1}|-|a_n|\right|\leq C(m_1,m_2)n^{2m_1-3},$$

where $C(m_1,m_2)$ is a constant depending only on m_1 and m_2.

Proof

For $z_1\varepsilon E$ and $n\geq 1$, we have

$$|(n+1)z_1a_{n+1}-na_n|=\frac{1}{2\pi r^{n+1}}\int_0^{2\pi}|z-z_1||zf'(z)|d\theta,\quad z=re^{i\theta}$$

$$=\frac{1}{2\pi r^n}\int_0^{2\pi}|z-z_1|\frac{|S_1'(z)|^{m_1}}{|S'_2(z)|^{m_2}}d\theta \tag{2.4}$$

It is known [9] that for convex univalent functions S_2

$$|S_2'(z)| \geq \frac{1}{(1+r)^2} \tag{2.5}$$

Also, by a result of Golusin [11], there exists a $z_1 \varepsilon E$ with $|z_1| = r$ such that for all z, $|z| = r$

$$|z-z_1||S_1^*(z)| \leq \frac{2r^2}{1-r^2}, \tag{2.6}$$

where $S_1^*(z) = zS_1'(z)$ is univalent

Using (2.5) and (2.6), (2.4) becomes

$$|(n+1)z_1 a_{n+1} - na_n| \leq \frac{(1+r)^{2m_2}}{2\pi r^{n-1}} \left(\frac{2r^2}{1-r^2}\right) \int_0^{2\pi} |S_1'(z)|^{m_1-1} d\theta$$

$$\leq \frac{(1+r)^{2m_2-1}}{\pi r^{n-3}} \cdot \frac{1}{(1-r)^{2m_1-2}}$$

where we have used subordination for the function S_1'.

Putting $|z_1| = r$, $r = \frac{n}{n+1}$, we obtain the required result.

Corollary 2.3

Taking $m_1 = \frac{k+2}{4}$, $m_2 = \frac{k-2}{4}$, $k \geq 2$, we obtain $f \varepsilon V_k$ and $||a_{n+1}| - |a_n|| \leq C(k)\, n^{\frac{k}{2}-2}$, where $C(k)$ is a constant depending only on k.

Now we give the radius of convexity problem for the class V_{m_1,m_2} where the functions S_1 and S_2 are in V_k.

Theorem 2.5

Let $f \varepsilon V_{m_1,m_2}$ such that

$$f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}},$$

where $S_1, S_2 \varepsilon V_k$ and $m_1, m_2 \geq 0$ and real. Then f is convex for $|z| < r$ where r is the least positive root of

$$[1+m_2(1-\tfrac{k}{2})] - k(m_1+m_2)r + [2m_1 - m_2(1+\tfrac{k}{2}) - 1]r^2 = 0 \tag{2.7}$$

Proof

From definition it easily follows that

$$\frac{(zf'(z))'}{f'(z)} = m_1 \frac{(zS_1'(z))'}{S_1'(z)} - m_2 \frac{(zS_2'(z))'}{S_2'(z)} + (1-m_1+m_2)$$

Now, for $S_1 \varepsilon V_k$ it is known [12] that

$$\text{Re}\, \frac{(zS_1'(z))'}{S_1'(z)} \geq \frac{1-kr+r^2}{1-r^2} \tag{2.8}$$

Also, by the Paatero representation theorem [3] we have, for $S_2 \varepsilon V_k$,

$$\frac{(zS_2'(z))'}{S_2'(z)} = \frac{k+2}{4} h_1(z) - \frac{k-2}{4} h_2(z), \ \mathrm{Re}\, h_i(z) \geq 0, \ i=1,2, \text{ and } h_i(0) = 1$$

so that

$$\mathrm{Re}\, \frac{(zS_2'(z))'}{S_2'(z)} \leq \left| \frac{(zS_2'(z))'}{S_2'(z)} \right| \leq \frac{k}{2}\, \frac{1+r}{1-r}$$

Thus, using (2.8) and (2.9), we have

$$\mathrm{Re}\, \frac{(zf'(z))'}{f'(z)} \leq \frac{[1+m_2(1-\frac{k}{2})]-k(m_1+m_2)r+[2m_1-m_2(1+\frac{k}{2})-1]r^2}{1-r^2}$$

and this gives us the required result.

Corollary 2.4

If $k=2$, then $S_1, S_2 \varepsilon V_2 = C$, the class of convex functions and equation (2.7) reduces to

$$1-2(m_1+m_2)r+(2m_1-2m_2-1)r^2=0$$

and in this case if $m_1 = \frac{k+2}{4}$, $m_2 = \frac{k-2}{4}$ then V_{m_1,m_2} reduces to V_k and equation (2.7) reduces to the known result

$$1-kr+r^2 = 0$$

which was given in[12].

Corollary 2.5

If $m_1 = \alpha > 0$, $m_2=0$, then f is convex for $|z| < r$, where r is the least positive root of

$$1-k\alpha r + (2\alpha - 1)r^2 = 0$$

This result has been proved in [13].

Theorem 2.6

Let $f \varepsilon V_{m_1,m_2}$ such that

$$f'(z) = \frac{(S_1'(z))^{m_1}}{(S_2'(z))^{m_2}},$$

and $S_1, S_2 \varepsilon V_k$, $m_1, m_2 \geq 0$, $m_1 - m_2 \leq 1$.

Then $f \varepsilon V_{k'}$, where $k' = \{ m_1(k-2)+m_2(k+2)+2 \}$

From the above result, we deduce the following:

(i) If $S_1, S_2 \varepsilon V_2$, then $f \varepsilon V_{4m_2+2}$ and in this case if $m_1 = \frac{k+2}{4}$, $m_2 = \frac{k-2}{4}$, we have the well known result [8] that $f \varepsilon V_k$.

(ii) If $m_1 = \alpha$, $m_2=0$, $0 \leq \alpha \leq 1$, then $f \varepsilon V_{\alpha(k-2)+2}$.

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