SOME GENERAL CLASSES OF COMATCHING GRAPHS

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ABSTRACT. Some sufficient conditions are given for two graphs to have the same matching polynomial (comatching graphs). Several general classes of comatching graphs are given. Also, techniques are discussed for extending certain pairs of comatching graphs to larger pairs of comatching graphs.

KEY WORDS AND PHRASES. Matching polynomial, comatching graphs, weight, chains, cycles, attaching graphs to other graphs.
1980 AMS SUBJECT CLASSIFICATION CODE. 05A99, 05C99.

1. INTRODUCTION.

The graphs considered here will be finite and without loops or multiple edges. Let G be such a graph. We define a *matching* in G to be a spanning subgraph of G, whose components are nodes and edges only. Let us associate with each node in G, an indeterminate or *weight* w_1, and with each edge, a weight w_2. Also, let us associate with each matching in G with k edges, the weight $w_1^{p-2k} w_2^k$. Then, the *matching polynomial* of G is

$$M(G;\underline{w}) = \sum_k a_k \, w_1^{p-2k} \, w_2^k,$$

where a_k is the number of matchings in G with k edges, $\underline{w}$ is a weight vector (w_1, w_2), and the summation is taken over all values of k. The basic results on matching polynomials are given in Farrell [1].

We define two graphs to be *comatching* if and only if they have the same matching polynomial. In this paper, we investigate non-isomorphic comatching graphs. As far as we know, no investigation has been made into this interesting property of graphs. We will give some general results on comatching graphs. Then we will introduce various classes of comatching graphs. In many cases, we will give results concerning the construction of pairs of larger comatching graphs from certain "basic" pairs of comatching graphs (*comatching pairs*).

Let us denote the node set of G by $V(G)$ and the edge set by $E(G)$. Let $A \subseteq V(G)$. Then $G - A$ will denote the graph obtained from G by removing the nodes in A. When A is a singleton $\{a\}$, we will write $G - a$. Let H be a subgraph of

of G. Then G - H will denote the graph obtained from G by removing the nodes in
H. For brevity, we will write *m(G)* for M(G;$\underline{w}$).

2. SOME GENERAL RESULTS.

The following lemma is taken from [1]. It is called *the fundamental theorem for matching polynomials*.

LEMMA 1. Let G be a graph and xy an edge in G, where x,y $\in$ V(G). Let G' be
the graph obtained from G by deleting xy and G" the graph G - {x,y}. Then

$$M(G;\underline{w}) = M(G';\underline{w}) + w_2 M(G'';\underline{w}).$$

We will refer to the algorithm implied by this lemma, as the *reduction process*.
G' will be called the *reduced graph* and G", the *incorporated graph*. The following is
called the Component Theorem. It was also proved in [1].

LEMMA 2. Let G be a graph consisting of r components G_1, G_2,... , G_r. Then

$$m(G) = \prod_{i=1}^{r} m(G_i).$$

An immediate consequence of the definitions given in Section 1 is the following
theorem.

THEOREM 1. Two graphs are comatching if and only if they have the same numbers of the
same kinds of matchings.

Trivially, we have the following corollary in which $\overline{G}$ represents the complement
of G.

COROLLARY 1.1. (G, G) is a comatching pair, for all graphs G. Also, if G is self-
complementary, (G, $\overline{G}$) is a comatching pair.

The following are the smallest non-trivial comatching pair (H_1, H_2) and the
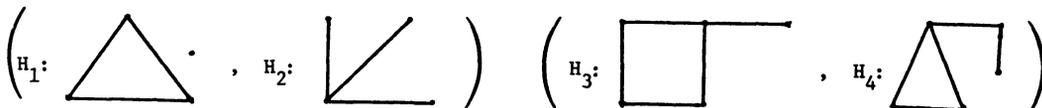smallest connected comatching pair (H_3, H_4).

Figure 1

Notice that $\overline{H}_3 = H_4$ and therefore H_3 is not self-complementary. H_3 is the small-
est (connected) non-selfcomplementary graph with the property that $(H_3, \overline{H}_3)$ is a
comatching pair. It can be easily confirmed that

$$m(H_1) = m(H_2) = w_1^4 + 3 w_1^2 w_2$$

and that

$$m(H_3) = m(H_4) = w_1^5 + 5 w_1^3 w_2 + 4w_1 w_2^2 .$$

Let G be a graph. Two nodes u and v in G are called *pseudosimilar* if
G - u and G - v are isomorphic, but no automorphism of G maps u onto v. We
will denote by $G_a + H_b$ the graph formed from two connected graphs ·G and H

by identifying node   a   of   G   with node   b   of   H.

THEOREM 2.   Let   G   be a graph with pseudosimilar nodes   u   and   v.   Let   H   be a graph containing a node   x.   Then

$$m(G_u + H_x) = m(G_v + H_x).$$

PROOF.   We will prove the result by induction on the valency $d(x)$ of node   x.   Suppose that $d(x) = 0$.   Then   H   is a node and in this case we get

$$G_u + H_x = G_v + H_x = G$$

and the result follows trivially.   Let $d(x) = 1$ and let   y   be the node adjacent to node   x.   Apply the reduction process to $G_u + H_x$ using edge xy.   Then   G'   (see Theorem 1) will consist of two components,   G   and   H − x.   G"   will consist of the components   G − u   and   H − {x,y}.   Hence by using the Component Theorem we get

$$m(G_u + H_x) = m(G)\, m(H - x) + w_2\, m(G - u)\, m(H - \{x,y\}). \qquad (2.1)$$

Apply the reduction process to $G_v + H_x$ by deleting edge xy.   Then   G'   will consist of the components   G   and   H − x.   G"   will consist of the components   G − v   and   H − {x,y}.   Therefore we get

$$m(G_v + H_x) = m(G)\, m(H - x) + w_2\, m(G - v)\, m(H - \{x,y\}). \qquad (2.2)$$

But from the definition of pseudosimilar nodes,

$$G - u \cong G - v$$

$$\therefore\ m(G - u) = m(G - v).$$

Hence from Equations (2.1) and (2.2), we get

$$m(G_u + H_x) = m(G_v + H_x).$$

Let us assume that the result holds for all graphs   H   with $d(x) < n$,   and let $d(x) = n$ in   H.   Apply the reduction process to $G_u + H_x$ by deleting edge xy, where   y   is one of the nodes in   H   which is adjacent to node   x.   Then   G'   will be the graph $G_u + H'_x$,   where   H'   is the graph obtained from   H   by deleting edge xy.   G"   will consist of two components,   G − u   and   H − {x,y}.   Hence

$$m(G_u + H_x) = m(G_u + H'_x) + w_2\, m(G - u)\, m(H - \{x,y\}). \qquad (2.3)$$

Similarly, by applying the reduction process to $G_v + H_x$,   we get

$$m(G_v + H_x) = m(G_v = H'_x) + w_2\, m(G - v)\, m(H - \{x,y\}). \qquad (2.4)$$

But from the induction hypothesis,

$$m(G_u + H'_x) = m(G_v + H'_x).$$

Therefore from Equations (2.3) and (2.4), we get

$$m(G_u + H_x) = m(G_v + H_x).$$

Hence the result holds for $d(x) = n$.   It follows by the Principle of Induction, that the result is true for all   n.   ▯

Theorem 2 is a useful result for constructing comatching pairs, since the graphs $G_u + H_x$ and $G_v + H_x$ will generally be non-isomorphic.

3. A GENERAL TECHNIQUE FOR CONSTRUCTING COMATCHING GRAPHS.

Let   G   and   H   be connected graphs.   We *attach*   H   to   G   by identifying a node   x   of   H   with a node   y   of   G.   In the resulting graph,   G   and   H   will be subgraphs

with exactly one node in common - the node z formed in the identification process. z
will be called the *node of attachment*. Let us denote $G(H)$, the graph obtained from a
given graph G , by attaching <u>in the same manner</u>, a copy of the graph H to each of
the nodes of G . It is clear that if G contains p_1 nodes and q_1 edges, and H
contains p_2 nodes and q_2 edges, then $G(H)$ will contain $p_1 p_2$ nodes and $(q_1 +$
$p_1 q_2)$ edges. If H is attached to only <u>some</u> of the nodes of G , the resulting graph
will be denoted by $G^*(H)$.

By a *chain*, we will mean a tree with nodes of valencies 1 and 2 only. The chain
with n nodes will be denoted by P_n. The following lemma gives a simple technique for
constructing a larger comatching pair, starting with a given comatching pair, which
satisfies certain conditions.

LEMMA 3. Let (A,B) be a comatching pair. Suppose that there exist nodes a and b
in A and B respectively, such that $(A - a, B - b)$ is also a comatching pair.
Then $(P_2(A), P_2(B))$ is a comatching pair, when a and b are used in the attachment
process.

PROOF. Apply the reduction process to $P_2(A)$ by deleting the edge belonging to P_2.
The reduced graph will consist of two components, each being the graph A . The incor-
porated graph will consist of two copies of A - a. We therefore get

$$m(P_2(A)) = m(A)^2 + w_2 \, m(A - a)^2. \qquad (3.1)$$

Similarly, we get

$$m(P_2(B)) = m(B)^2 + w_2 \, m(B - b)^2. \qquad (3.2)$$

Since (A,B) and $(A - a, B - b)$ are comatching pairs,

$$m(A) = m(B) \quad \text{and} \quad m(A - a) = m(B - b). \qquad (3.3)$$

Hence the result follows from Equations (3.1) amd (3.2).   ⬜

The following theorem gives a generalization of the lemma. It can be easily
proved by induction on the number of nodes in the chain.

THEOREM 3. Let (A,B) be a comatching pair satisfying the conditions of Lemma 3.
Then $(P_n(A), P_n(B))$ is also a comatching pair, when the attachments are made using
nodes a and b.

It is unnecessary to attach A and B to every node of P_n in order to obtain
a comatching pair. This is implied by the following theorem, which is a further gener-
alization of the lemma.

THEOREM 4. Let (A,B) be a comatching pair satisfying the conditions of Lemma 3.
Then $(P_n^*(A), P_n^*(B))$ is also a comatching pair, where the attachments are made to the
same nodes of P_n in forming the two graphs $P_n^*(A)$ and $P_n^*(B)$, and nodes a and b
respectively are used in all the attachments.

PROOF. We can apply the reduction process to $P_n^*(A)$ by always deleting edges incident
to nodes of attachment, until all the reduced and incorporated graphs are either graphs
of the form $P_r(A)$ or A - a. The reduction process can then be applied to $P_n^*(B)$ in
the identical manner. The result will then follow from Lemma 1, the Component Theorem

and Theorem 3. ☐

Theorem 4 can be generalized even further. Consider the graph $G^*(A)$. We can apply the reduction process to this graph until the reduced and incorporated graphs are of the form $P_r^*(B)$ or $B - b$. Then, by using Lemma 3, the Component Theorem and Theorem 4, we obtain the following general result.

THEOREM 5. Let (A,B) be a comatching pair such that $(A - a, B - b)$ is also a comatching pair, where a and b are nodes in A and B respectively. Then $(G^*(A), G^*(B))$ is also a comatching pair, when the attachments are made using nodes a and b of A and B respectively.

Theorem 5 provides us with a general construction technique for large families of comatching graphs. The following is a comatching pair (A,B) satisfying the conditions of Lemma 3.
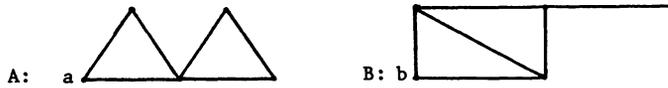

Figure 2

It can be eaily confirmed that

$$m(A) = m(B) = w_1^5 + 6w_1^3 w_2 + 5w_1 w_2^2$$

and that

$$m(A - a) = m(B - b) = w_1^4 + 4w_1^2 w_2 + w_2^2 .$$

The following theorem shows that instead of attaching the graphs A and B (of Lemma 3) to other graphs, we could instead attach other graphs to A and B, in order to construct new comatching pairs.

THEOREM 6. Let (A,B) be a comatching pair satisfying the conditions of Lemma 3. Let $A^*(G)$ and $B^*(G)$ be the graphs obtained by attaching a graph G to A and B respectively. Then $(A^*(G), B^*(G))$ is a comatching pair.

PROOF. Apply the reduction process to $A^*(G)$ by deleting one of the edges of G which is incident to the node of attachment – node z. Let G_1' be the reduced graph. The incorporated graph will consist of two components $G - y$ and $A - a$, where y is the node used to attach G to A. We can apply the reduction process to G_1' and to subsequent G_i' in the same manner until the reduced graph becomes disconnected with two components $G - y$ and A. Hence we will get

$$m(A^*(G)) = m(G - y) m(A) + d_y m(G - y) m(A - a), \qquad (3.3)$$

where d_y is the valency of node y in G.

By applying the reduction process to $B^*(G)$, in an identical manner, we get

$$m(B^*(G)) = m(G - y) m(B) = d_y m(G - y) m(B - b). \qquad (3.4)$$

Since $m(A) = m(B)$ and $m(A - a) = m(B - b)$, the result follows from Equations (3.3) and (3.4).   ☐

4. COMATCHING COMPLEXES OF CHAINS AND CYCLES.

We will denote by $G_{r,s}$, the graph formed by attaching the chain P_r (using an end-node) to the cycle C_s with s nodes.

THEOREM 7. $(G_{a,b}, G_{b-1,a+1})$ is a comatching pair, for all positive integers a and b.

PROOF. Apply the reduction process to each graph in turn, by deleting an edge which is incident to the node of attachment. In both cases, the reduced graph will be P_{a+b-1} and the incorporated graph will consist of two components P_{a-1} and P_{b-2}. Hence the result follows from Lemma 1.   □

Let x and y be the nodes of $G_{a,b}$ and $G_{b-1,a+1}$ respectively, which are adjacent to the node of attachment. Then it can be easily confirmed that $G_{a,b}$ - x will be P_{a+b-2} and that $G_{b-1,a+1}$ - y will also be P_{a+b-2}. Hence these graphs satisfy the conditions of Lemma 3. Theorem 7 therefore gives us a general technique, not only for constructing comatching pairs, but for constructing graphs which satisfy Lemma 3. Hence we can construct comatching complexes of chains and cycles, by using Theorem 5.

The following theorem shows that the graph consisting of the components P_r and C_{r+1} has the same matching polynomial as the chain P_{2r+1}. It can be proved by appropriate applications of the reduction process.

THEOREM 8. For all integers r,

$$m(P_r) \, m(C_{r+1}) = m(P_{2r+1})$$

5. A GENERAL COMATCHING QUADRUPLE.

We will give a technique for constructing four non-isomorphic graphs with the same matching polynomial i.e., *a comatching quadruple*. Let (A,B) be a comatching pair, satisfying the conditions of Lemma 3. Let G be a graph containing two nodes x and y. Construct four graphs as follows. H_1 is the graph obtained by attaching a graph A (using node a) to G, at the nodes x and y. Similarly H_2, H_3 and H_4 are the graphs formed by attaching to G, at nodes x and y respectively, the graphs A and B, B and A and B and B respectively. Then we have the following theorem.

THEOREM 9. The graphs H_1, H_2, H_3 and H_4 all have the same matching polynomial i.e., (H_1, H_2, H_3, H_4) is a comatching quadruple.

PROOF. The result can be established by applying the reduction process to the graphs until the attached subgraphs A and B become disconnected. This is accomplished by deleting the edges of G which are incident to the nodes of attachment. The reduced graphs will either be A or B. The incorporated graphs will be A - a, B - b and G - {x,y} . The result will then follow from Lemma 3.   □

6. DISCUSSION.

We have given several classes of comatching graphs. Also, we have given some general results which can be used to construct larger and larger classes from any given class. We have found other kinds of comatching graphs, but have not mentioned them here, since they do not appear to be very interesting.

REFERENCES

1. FARRELL, E. J. An Introduction to Matching Polynomials, *J. Comb. Theory B*, 27(1979) 75-86.