Internat. J. Math. and Math. Sci.
Vol. 10 No. 2 (1987) 395-403

FUNCTIONAL EQUATION OF A SPECIAL DIRICHLET SERIES

IBRAHIM A. ABOU-TAIR

Department of Mathematics
Islamic University - Gaza
Gaza - Strip

(Received December 10, 1985 and in revised form May 1, 1986)

ABSTRACT. In this paper we study the special Dirichlet series

$$L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin(\frac{2\pi n}{3})n^{-s} \quad , \; s \in C$$

This series converges uniformly in the half-plane Re(s) > 1 and thus represents a holomorphic function there. We show that the function L can be extended to a holomorphic function in the whole complex-plane. The values of the function L at the points 0,±1,-2,±3,-4,±5,... are obtained. The values at the positive integers 1,3,5,... are determined by means of a functional equation satisfied by L.

KEY WORDS AND PHRASES. Dirichlet Series, Analytic Continuation, Functional Equation, Γ-Function.
1980 AMS SUBJECT CLASSIFICATION CODE. 30B50, 30B40.

1. INTRODUCTION.

By a Dirichlet series we mean a series of the form

$$\sum_{n=1}^{\infty} a_n n^{-s}$$

where the coefficients a_n are any given numbers, and s is a complex variable [1], [2].

In this paper we study the special Dirichlet series

$$L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin(\frac{2\pi n}{3})n^{-s} \quad , \; s \in C$$

which converges uniformly in the half-plane Re(s) > 1 and thus represents an analytic function there. In section 1 we study the analytic behaviour of the function L beyond the half-plane Re(s) > 1, and prove that the function L can be extended to a holomorphic function in the whole complex-plane. Moreover values of L at the points -m (m=0,1,2,3,...) are obtained at the end of this section. The values of L at the positive integers 1,3,5,... are determined by means of the functional equation

$$L(s) = \frac{2}{\sqrt{3}}(\frac{2\pi}{3})^{s-1}\Gamma(1-s)\cos(\frac{1}{2}\pi s)L(1-s) \quad , s \in C$$

satisfied by the function L, which we prove in section 2.

2. ANALYTIC CONTINUATION OF L.

$$L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin(\tfrac{2\pi n}{3})n^{-s} \quad , (s \in \mathbb{C}) \tag{2.1}$$

is uniformly convergent in the half-plane Re(s) > 1 and so it represents an analytic function there. The aim of this section is to extend L to the whole complex plane and to prove that L is holomorphic in C.

LEMMA 2.1. For all values of s in the half-plane Re(s) > 1

$$L(s) = \frac{1}{\Gamma(s)}\int_0^\infty G(t)t^{s-1}dt \quad ,\text{where}$$

$$G(t) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin(\tfrac{2\pi n}{3})e^{-nt} \quad ,Re(t)>o$$

$$= \frac{1}{e^{t}+e^{-t}+1}$$

PROOF. Consider the Euler's integral

$$\Gamma(s) = \int_0^\infty e^{-t}t^{s-1}dt$$

Substitution of $nt, n \in N$, for t in the above integral yields

$$n^{-s}\,\Gamma(s) = \int_0^\infty e^{-nt}t^{s-1}dt \quad ,Re(s)>o$$

Thus for Re(s) > 1, we get

$$\Gamma(s)L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin(\tfrac{2\pi n}{3})\int_0^\infty e^{-nt}t^{s-1}dt$$

i.e.

$$\Gamma(s)L(s) = \frac{2}{\sqrt{3}}\int_0^\infty\sum_{n=1}^{\infty} \sin(\tfrac{2\pi n}{3})e^{-nt}t^{s-1}dt \quad ,$$

Thus

$$\Gamma(s)L(s) = \int_0^\infty G(t)t^{s-1}dt$$

Now

$$G(t) = \frac{1}{i\sqrt{3}}\sum_{n=1}^{\infty}((\varepsilon)^n-(\bar{\varepsilon})^n)e^{-nt} \quad ,\text{where } \varepsilon = e^{2\pi i/3}.$$

i.e.

$$G(t) = \frac{1}{i\sqrt{3}}\left(\sum_{n=1}^{\infty}(\varepsilon)^n e^{-nt} - \sum_{n=1}^{\infty}(\bar{\varepsilon})^n e^{-nt}\right) \quad ,Re(t)>0 .$$

Thus

$$G(t) = \frac{1}{i\sqrt{3}}\left(\frac{1}{(1-\varepsilon e^{-t})} - \frac{1}{(1-\bar{\varepsilon}e^{-t})}\right).$$

By using the identities $\varepsilon - \bar{\varepsilon} = i\sqrt{3}$, $\varepsilon + \bar{\varepsilon} + 1 = 0$ and $\varepsilon\bar{\varepsilon} = 1$, we get

$$G(t) = \frac{1}{e^{t}+e^{-t}+1}$$

The function $G(t) = (e^t + e^{-t} + 1)^{-1}$ is analytic near $t=0$; therefore it can be expanded as a power series in t. So we have

LEMMA 2.2. G(t) has the Taylor series expantion

$$G(t) = \sum_{n=0}^{\infty} a_n t^{2n} \qquad , \ |t| < 2\pi/3$$

where the coefficients a_n satisfy the recursion formula

$$a_0 = 1/3 \ , \ 3a_n + 2\sum_{k=1}^{n} \frac{1}{(2k)!} a_{n-k} = 0 \ , n \geq 1 \tag{2.2}$$

PROOF. Since G is an even function, the expantion of G can be expressed as

$$G(t) = \sum_{n=0}^{\infty} a_n t^{2n}$$

which is valid near zero (in fact valid in the disk $|t| < \frac{2}{3}\pi$ which extends to the nearest singularities $t = \pm\frac{2\pi}{3}$ of G(t)). The relation $G(t)(e^t + e^{-t} + 1) = 1$ gives

$$\left(\sum_{n=0}^{\infty} a_n t^{2n}\right)\left(1 + 2\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\right) = 1$$

i.e.

$$\sum_{n=0}^{\infty} a_n t^{2n} + 2\left(\sum_{n=0}^{\infty} a_n t^{2n}\right)\left(\sum_{n=0}^{\infty} \frac{t^{2n}}{(2n)!}\right) = 1$$

i.e.

$$\sum_{n=0}^{\infty} a_n t^{2n} + 2\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} \frac{1}{(2k)!} a_{n-k}\right) t^{2n} = 1$$

Thus for the coefficients a_n we have the recursion formula

$$a_0 = 1/3 \ , \ 3a_n + 2\sum_{k=1}^{n} \frac{1}{(2k)!} a_{n-k} = 0 \ , n \geq 1 \quad .$$

This completes the proof of the lemma.

The coefficient a_n can be determined successively by (2.2). The first few are easily determined to be

$$a_0 = \frac{1}{3} \ , \qquad a_1 = -\frac{1}{9}$$

$$a_2 = \frac{1}{36} \ , \qquad a_3 = -\frac{7}{1080}$$

THEOREM 2.1. The function L defined by

$$L(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} G(t) t^{s-1} dt \qquad , \mathrm{Re}(s) > 1$$

can be extended to a holomorphic function in the whole complex plane.

PROOF. Let us define P and Q for Re(s) > 1 by

$$P(s) = \int_0^1 G(t) t^{s-1} dt$$

$$Q(s) = \int_1^{\infty} G(t) t^{s-1} dt$$

The integral

$$\int_1^\infty G(t)t^{s-1}dt$$

exists and converges uniformly in any finite region of the s-plane, since the function

$$(e^{-t}\ t^{Re(s)+1})/(e^{-t}+\ e^{-2t}\ +1)$$

is bounded for all values of Re(s), and we can compare the integral with that of $1/t^2$. Thus Q is an entire function. Recall from Lemma 2.2 that

$$G(t) = \sum_{n=0}^{\infty} a_n t^{2n} \qquad , t \in [0,1]$$

the convergence being uniform on [0,1]. We deduce for Re(s) > 1 that

$$P(s) = \sum_{n=0}^{\infty} \int_0^1 a_n t^{2n+s-1} dt$$

$$= \sum_{n=0}^{\infty} \frac{1}{2n+s} a_n$$

Thus P is a meromorphic function on C with simple poles at 0,-2,-4,-6,... . Since $1/\Gamma$ is an entire function we may now extend L to the whole of C by

$$L(s) = \frac{P(s)}{\Gamma(s)} + \frac{Q(s)}{\Gamma(s)} \tag{2.3}$$

Since Q and $1/\Gamma$ are entire functions, the singularities of L can only be those of P/Γ . We have seen that P has simple poles at 0,-2,-4,-6,... . Since $1/\Gamma$ has simple zeros at 0,-2,-4,... it follows that L is regular for all values of s in the complex plane. This completes the proof of the theorem.

LEMMA 2.3. (i) L has zeros at -1,-3,-5,...

(ii) The values of L at 0,-2,-4,-6,... are given by

$$L(-2m) = (2m)!\,a_m \qquad , m = 0,1,2,3,4,\ldots .$$

PROOF. (i) This follows immediately from the fact that $1/\Gamma$ has zeros at 0,-1, -2,-3,..., and thus

$$L(1-2m) = \frac{P(1-2m)}{\Gamma(1-2m)} + \frac{Q(1-2m)}{\Gamma(1-2m)} = 0 \qquad , m \in N .$$

(ii) As in (i) we use the partial fraction (2.3) of L to get

$$L(-2m) = \lim_{s \to -2m} \frac{P(s)}{\Gamma(s)} + \frac{Q(s)}{\Gamma(s)}$$

$$= \lim_{s \to -2m} \frac{P(s)}{\Gamma(s)} = \lim_{s \to -2m} \frac{1}{\Gamma(s)} \sum_{n=0}^{\infty} \frac{1}{2n+s} a_n$$

i.e.

$$L(-2m) = \lim_{s \to -2m} \frac{1}{\Gamma(s)} \cdot \frac{1}{2m+s} a_m .$$

Since Γ has simple poles at the points $-m$ $(m=0,1,2,3,\ldots)$ with residues $(-1)^m/m!$, we get

$$\lim_{s\to -2m} (2m+s)\ \Gamma(s) = \mathrm{Res}(\Gamma, -2m) = \frac{1}{(2m)!}$$

Thus

$$L(-2m) = (2m)!\, a_m \qquad , m = 0,1,2,3,\ldots$$

where a_m can be determined successively by (2.2).

3. DERIVATION OF THE FUNCTIONAL EQUATION OF L.

In this section we derive the equation

$$L(s) = \frac{2}{\sqrt{3}}\left(\frac{2}{3}\pi\right)^{s-1} \Gamma(1-s)\cos\left(\frac{1}{2}\pi s\right)L(1-s) \quad , s \in \mathbb{C}.$$

where L is the Dirichlet series (2.1)

$$L(s) = \frac{2}{\sqrt{3}}\sum_{n=1}^{\infty} \sin\left(\frac{2\pi n}{3}\right)n^{-s} \quad , \quad s \in \mathbb{C}$$

Finally we determine the values of L at $1,3,5,\ldots$, by the use of the functional equation obtained above.

LEMMA 3.1. There exists an integral function I such that

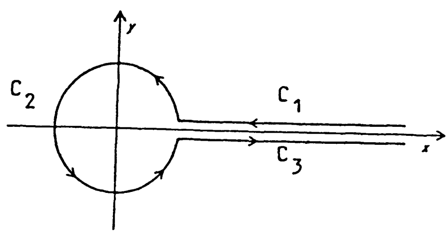
$$L(s) = -\Gamma(1-s)I(s) \qquad , s \in \mathbb{C}\ .$$

PROOF. Let $0 < r < 1$, and let C_r be the contour consisting of the paths C_1, C_2 and C_3, where

$C_1 = (\infty, r]$

$C_2 = \partial_+ D_r(0)$ is a circle of radius r and the center at the origin oriented in the positive direction.

$C_3 = [r, \infty)$.



Define the function I_r by

$$I_r(s) = \frac{1}{2\pi i}\int_{C_r} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt$$

We prove now that I_r is independent of r. We have

$$I_r(s) - I_{r'}(s) = \frac{1}{2\pi i}\int_{C_0} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1} \quad ,$$

where C_0 is the contour shown in figure (a). Now

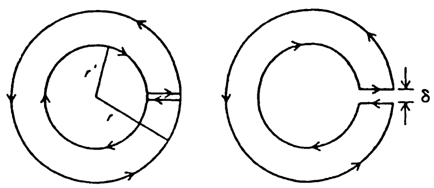
$$\int_{C_0} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt = \lim_{\delta \to 0} \int_C \frac{(-t)^{s-1}}{e^t + e^{-t} + 1} dt ,$$

where C is the contour in figure (b).

According to Cauchy's theorem, the integral around C is zero. Thus

$$\int_{C_0} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt = 0$$

It follows that I_r is independent of r.



(a) (b)

Now,

$$I_r(s) = \frac{1}{2\pi i} \int_\infty^r \frac{e^{(\log t - \pi i)(s-1)}}{e^t + e^{-t} + 1}\, dt +$$

$$\frac{1}{2\pi i} \int_{C_2} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1} dt + \frac{1}{2\pi i} \int_r^\infty \frac{e^{(\log t + \pi i)(s-1)}}{e^t + e^{-t} + 1}\, dt .$$

The middle term approaches zero as $r \to 0$ provided $\mathrm{Re}(s) > 0$, since

$$\left| \int_{C_2} \frac{(-t)^{s-1}}{e^t + e^{-t} + 1}\, dt \right| < M \int_0^{2\pi} r^{\mathrm{Re}(s)-1} e^{-(\pi+\theta)\mathrm{Im}(s)} r\, d\theta$$

$$< M' r^{\mathrm{Re}(s)} \quad .$$

Hence

$$\lim_{r \to 0} I_r(s) : \frac{-e^{-\pi i(s-1)} + e^{\pi i(s-1)}}{2\pi i} \int_0^\infty \frac{t^{s-1}}{e^t + e^{-t} + 1} dt.$$

Define the function I by

$$I(s) = \lim_{r \to 0} I_r(s)$$

Thus we have

$$I(s) = -\frac{\sin(\pi s)}{\pi} \int_0^\infty \frac{t^{s-1}}{e^t + e^{-t} + 1} dt.$$

We have seen in the proof of theorem 2.1 that the function defined by the integral

$$\int_0^\infty \frac{t^{s-1}}{e^t + e^{-t} + 1}$$

is a meromorphic function with simple poles at the points 0,-2,-4,... . Since the function $\sin(\pi s)$ has simple zeros at 0,-2,-4,... it follows that I is regular for

all values of s in the complex plane.
Moreover we have

$$I(s) = -\frac{\Gamma(s)\sin(\pi s)}{\pi} L(s)$$

Thus

$$I(s)\,\Gamma(1-s) = -L(s)$$

THEOREM 3.1. The function L satisfies the functional equation

$$L(s) = \frac{2}{\sqrt{3}}\left(\frac{2}{3}\pi\right)^{s-1}\Gamma(1-s)\cos\left(\frac{1}{2}\pi s\right)L(1-s)$$

PROOF. Let $R_n = n+\frac{1}{2}$, $n = 1,2,3,\ldots$, and let $C_{n,r}$ $(0<r<1)$ be the contour consisting of the positive real axis from R_n to r, a circle radius r and center at the origin oriented in the positive direction, the positive real axis from r to R_n, and finally a circle of radius R_n with center at the origin oriented in the negative direction.
i.e.

$$C_{n,r} = \left[R_n, r\right] + \underset{+}{\partial} D_r(o) + \left[r, R_n\right] + \underset{-}{\partial} D_{R_n}(o) \quad .$$

To deduce the functional equation of L we evaluate the integral

$$\frac{1}{2\pi i}\int_{C_{r,n}} \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1}dt$$

If we assume $s = x$ is a negative real number, then we have

$$(-t)^{x-1} = e^{(x-1)\log(-t)}$$

It follows that

$$|(-t)|^{x-1} = |t|^{x-1}$$

Since the function $(e^{t}+e^{-t}+1)^{-1}$ is bounded on the circle $\underset{-}{\partial} D_{R_n}(0)$,

$$\left|\int_{\underset{-}{\partial} D_{R_n}(o)} \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1}dt\right| < 2\,M^{*}R_n^{x} \quad ,$$

which goes to zero as n goes to infinity.
Thus we have

$$I(s) = \lim_{n\to\infty}\left(\frac{1}{2\pi i}\int_{C_{n,r}} \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1}\,dt\right) .$$

Now between $\partial D_{R_n}(0)$ and $D_r(0)$ the integrand has poles at the points

$$\pm\frac{2\pi i}{3}, \pm\frac{2\pi i}{3}(3m+1) \text{ and } \pm\frac{2\pi i}{3}(3m-1),\quad m=1,2,3,\ldots.$$

Denote

$$H(t) = \frac{(-t)^{s-1}}{e^{t}+e^{-t}+1}$$

Thus we have

$$\mathrm{Res}(H, \tfrac{2\pi i}{3}) = \frac{1}{\sqrt{3}}(\tfrac{2\pi}{3})^{s-1} e^{-\pi is/2} .$$

$$\mathrm{Res}(H, -\tfrac{2\pi i}{3}) = \frac{1}{\sqrt{3}} (\tfrac{2}{3}\pi)^{s-1} e^{\pi is/2} .$$

$$\mathrm{Res}(H, \tfrac{2\pi i}{3}(3m+1)) = \frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{-\pi is/2}(3m+1)^{s-1} .$$

$$\mathrm{Res}(H, -\tfrac{2\pi i}{3}(3m+1)) = \frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{\pi is/2}(3m+1)^{s-1} .$$

$$\mathrm{Res}(H, \tfrac{2\pi i}{3}(3m-1)) = -\frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{-\pi is/2}(3m-1)^{s-1} .$$

$$\mathrm{Res}(H, -\tfrac{2\pi i}{3}(3m-1) = -\frac{1}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} e^{\pi is/2}(3m-1)^{s-1} .$$

The sum of the residues between $\partial D_{R_n}(0)$ and $\partial D_r(0)$ equals

$$\frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1} \cos(\tfrac{1}{2}\pi s)\Big(1+\sum_{m=1}^{n}[(3m+1)^{s-1}-(3m-1)^{s-1}]\Big)$$

One can easily verify the identity

$$1+\sum_{m=1}^{n}[(3m+1)^{s-1}-(3m-1)^{s-1}] = \frac{2}{\sqrt{3}}\sum_{m=1}^{3n+1} \sin(\tfrac{2}{3}\pi m)m^{s-1}.$$

Thus the sum of the residues is

$$\frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)\Big(\frac{2}{\sqrt{3}}\sum_{m=1}^{3n+1} \sin(\tfrac{2}{3}\pi m)m^{s-1}\Big) .$$

It follows that

$$\begin{aligned} -I(s) &= \frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)(\frac{2}{\sqrt{3}}\sum_{m=1}^{\infty} \sin(\tfrac{2}{3}\pi m)m^{s-1}). \\ &= \frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)L(1-s) \qquad (3.1) \end{aligned}$$

We have seen that $-I(s)\Gamma(1-s) = L(s)$ for all $s \in C$, so by the identity theorem the formula (3.1) is true for all $s \in C$. Thus we have proved the functional equation

$$L(s) = \frac{2}{\sqrt{3}}(\tfrac{2}{3}\pi)^{s-1}\cos(\tfrac{1}{2}\pi s)\,\Gamma(1-s)L(1-s) .$$

LEMMA 3.2. The values of L at the points $s=2m+1, m=0,1,2,3,...$ are given by the formula

$$L(1+2m) = (-1)^m \frac{\sqrt{3}}{2}(\tfrac{2}{3}\pi)^{2m+1} a_m .$$

where a_m's are determined by (2.2).

PROOF. For $s = -2m$ the functional equation and the identity

$$L(-2m) = (2m)!\, a_m \quad , \quad m = 0,1,2,\ldots$$

of the previous section give the proof of the lemma.

REFERENCES

1. HARDY, G.H. and RIESZ, M. "The General Theory of Dirichlet Series," Cambridge University Press, 1952.

2. TITCHMARSH, E.C. "The Theory of the Riemann Zeta-Function," Oxford University Press, 1951.