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TWO-POINT BOUNDARY VALUE PROBLEMS INVOLVING REFLECTION OF THE ARGUMENT

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ABSTRACT. Two point boundary value problems involving reflection of the argument are studied. The nonlinearity involved is allowed to cross asymptotically any number of eigenvalues of the associated linear eigenvalue problem as long as those crossings take place in subsets of sufficiently small measure.

KEY WORDS AND PHRASES. Boundary value problems, reflection of the argument, Wirtinger's inequality, eigenvalues, asymptotic behaviour, Caratheodory's condition's sets of small measure.

1. INTRODUCTION.

This paper is devoted to the study of the two point boundary value problems

$$\begin{cases} x''(t) + f(x(t))x'(t) + g(t,x(t),x(-t)) = e(t), \ t \in [-1,1] \\ x(-1) = x(1) = 0; \end{cases} \tag{1.1}$$

and

$$\begin{cases} x''(t) + g(t,x(t),x(-t)) = e(t), \ t \in [-1,1] \\ x'(-1) = x'(1) = 0; \end{cases} \tag{1.2}$$

where $f: \mathbb{R} \to \mathbb{R}$ is a continuous function, $g:[-1,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function satisfying Caratheodory's conditions and $e \in L^1(-1,1)$ is given. It is assumed that

$$\limsup_{|x|\to\infty} \left| \frac{g(t,x,y)}{x} \right| \le \Gamma(t) \tag{1.3}$$

uniformly a.e. in $t \in [-1,1]$, $y \in \mathbb{R}$ where in the case of equation (1.1) $\Gamma(t) = \Gamma_0(t) + \Gamma_1(t) + \Gamma_\infty(t)$ where $\Gamma_0(t) \le \frac{\pi^2}{4}$ with strict inequality holding on a subset of $[-1,1]$ having positive measure, $\Gamma_1 \in L^1(-1,1)$, $\Gamma_\infty \in L^\infty(-1,1)$ and $|\Gamma_1|_{L^1}$ and $|\Gamma_\infty|_{L^\infty}$ sufficiently small and in the case of equation (1.2) $\Gamma(t)$ is such that $\Gamma \in L^1(-1,1)$ with $\int_{-1}^1 \Gamma(t)dt < 2$.

It should be observed that the linear eigen-value problem

$$\begin{cases} x'' + \lambda x = 0 \\ x(-1) = x(1) = 0 \end{cases} \tag{1.4}$$

has $\lambda = \frac{n^2\pi^2}{4}$, $n = 1,2,\ldots$ for eigen-values and the linear eigen-value problem

$$\begin{cases} x'' + \lambda x = 0 \\ x'(-1) = x'(1) = 0 \end{cases} \tag{1.5}$$

has $\lambda = \frac{n^2\pi^2}{4}$, $n = 0,1,2,\ldots$ for eigen-values. Accordingly, the asymptotic behavior of $\frac{g(t,x,y)}{x}$ is related to the eigen-value $\frac{\pi^2}{4}$ for the problem (1.1) and the first two-eigen-values 0 and $\frac{\pi^2}{4}$ for the problem (1.2). However, the expression $\limsup_{|x|\to\infty} \frac{g(t,x,y)}{x}$ is allowed to cross any number of eigen-values $\frac{n^2\pi^2}{4}$ as long as those crossings take place in subsets of $[-1,1]$ of sufficiently small measure.

Differential equations with reflection of the argument represent a particular case of functional differential equations whose arguments are involutions. Important in their own right, they have applications in the investigation of stability of differential difference equations. Initial value problems for equations with involutions have been considered in numerous papers. A survey of results in this direction is given in [1]. However, research on boundary value problems for such equations is developed yet insufficiently. Wiener and Aftabizadeh initiated the study of problem (1.1) in the case that $f \equiv 0$ and $g(t,x,y)$ is bounded on $[-1,1] \times \mathbb{R} \times \mathbb{R}$ in [2]. The methods used in this paper are similar to the ones used by Gupta-Mawhin [3] for periodic solutions of Lienard's differential equations. We mention that in addition to using the classical spaces $C([-1,1])$, $C^k([-1,1])$ and $L^k(-1,1)$ of continuous, k-times continuously differentiable or measurable real functions the k-th power of whose absolute value is Lebesgue integrable we use the space $H^1(-1,1)$ defined by

$$H^1(-1,1) = \{x\colon [-1,1] \to \mathbb{R} \mid x \text{ is abs. cont. on } [-1,1] \text{ and } x' \in L^2(-1,1)\}$$

with the usual inner-product and the corresponding norm $|\cdot|_{H^1}$.

2. SOME PRELIMINARY LEMMAS.

LEMMA 1. Let $\Gamma \in L^1(-1,1)$ be such that for a.e. $t \in [-1,1]$

$$\Gamma(t) \le \frac{\pi^2}{4} \tag{2.1}$$

with strict inequality holding on a subset of $[-1,1]$ of positive measure. Then, there exists a $\delta = \delta(\Gamma) > 0$ such that for all $x \in H^1(-1,1)$, $x(1) = x(-1) = 0$ one has

$$B_\Gamma(x) \equiv \int_{-1}^{1} ([x'(t)]^2 - \Gamma(t)x^2(t))dt \ge \delta |x|_{H^1}^2$$

PROOF. We first see, using (2.1) and the Wirtinger's inequality, [4],

$$\int_{-1}^{1} x^2(t)dt \le \frac{4}{\pi^2} \int_{-1}^{1} [x'(t)]^2 dt$$

for all $x \in H^1(-1,1)$ with $x(-1) = x(1) = 0$, that

$$B_\Gamma(x) \ge 0. \tag{2.2}$$

Moreover, $B_\Gamma(x) = 0$ if and only if

$$x(t) = A \cos \frac{\pi t}{2}$$

for some $A \in \mathbb{R}$, but, then

$$0 \ge \int_{-1}^{1} (\frac{\pi^2}{4} - \Gamma(t))x^2(t)dt = A^2 \int_{-1}^{1} (\frac{\pi^2}{4} - \Gamma(t))\cos^2 \frac{\pi}{2} tdt \ge 0$$

so that by our assumption that $\Gamma(t) < \frac{\pi^2}{4}$ on a subset of $[-1,1]$ of positive measure we have $A = 0$ and hence $x(t) \equiv 0$ for $t \in [-1,1]$.

Assume, now, that the conclusion of the lemma is not true. Then we can find a sequence $\{x_n\}$ in $H^1(-1,1)$ and x in $H^1(-1,1)$ such that

$$|x_n|_{H^1} = 1, \quad \text{for all } n, \quad x_n \to x \text{ in } C([-1,1]), \quad x_n \rightharpoonup x \tag{2.3}$$

in $H^1(-1,1)$ with $x_n(-1) = x_n(1) = 0$ for all n and

$$0 \leq B_\Gamma(x_n) \leq \frac{1}{n} \text{ for all } n = 1,2,\ldots . \tag{2.4}$$

Now, we have from Schwarz's inequality in $H^1(-1,1)$ that

$$[(x_n,x)_{H^1}]^2 \leq |x_n|^2_{H^1} \cdot |x|^2_{H^1} \quad \text{for all } n = 1,2,\ldots$$

and hence

$$|x|^2_{H^1} \leq \lim_{n\to\infty} |x_n|^2_{H^1}.$$

Also (2.3), (2.4) imply that

$$|x_n|^2_{H^1} \to \int_{-1}^1 (1+\Gamma(t))x^2(t)dt$$

and so,

$$|x|^2_{H^1} \leq \int_{-1}^1 (1+\Gamma(t))x^2(t)dt$$

which gives

$$B_\Gamma(x) \leq 0,$$

and hence $x(t) \equiv 0$ for $t \in [-1,1]$ by the first part of the proof of lemma. Thus, $|x_n|_{H^1} \to 0$, a contradiction to the first equality in (2.3). Hence the lemma is proved.

LEMMA 2. Let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty(-1,1)$, $\Gamma_1 \in L^1(-1,1)$ and $\Gamma_0(t) \leq \frac{\pi^2}{4}$ for a.e. t in $[-1,1]$ with strict inequality holding on a subset of $[-1,1]$ of positive measure. Let $\delta(\Gamma_0)$ be given by lemma 1. Then for all $x \in H^1(-1,1)$ with $x(-1) = x(1) = 0$

$$B_\Gamma(x) \geq [\delta(\Gamma_0) - \frac{1}{2}|\Gamma_1|_{L^1} - \frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty}]|x|^2_{H^1} .$$

PROOF. We have

$$B_\Gamma(x) = \int_{-1}^1 ([x'(t)]^2 - \Gamma_0(t)x^2(t))dt - \int_{-1}^1 \Gamma_1(t)x^2(t)dt - \int_{-1}^1 \Gamma_\infty(t)x^2(t)dt.$$

Using the fact that $H^1(-1,1) \subset C([-1,1])$ and the Wirtinger inequalities, [4],

$$|x|_{L^2(-1,1)} \leq \frac{4}{\pi^2}|x'|_{L^2(-1,1)}, \quad |x|_{L^\infty(-1,1)} \leq \frac{1}{\sqrt{2}}|x'|_{L^2(-1,1)} \leq \frac{1}{\sqrt{2}}|x|_{H^1}$$

as well as lemma 1, we get

$$\begin{aligned} B_\Gamma(x) &\geq \delta(\Gamma_0)|x|^2_{H^1} - \frac{1}{2}|x|^2_{H^1} \cdot |\Gamma_1|_{L^1} - \frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty}|x|^2_{H^1} \\ &= [\delta(\Gamma_0) - \frac{1}{2}|\Gamma_1|_{L^1} - \frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty}]|x|^2_{H^1} . \end{aligned}$$

REMARK 1. Clearly the best value for $\delta(0)$ is $\frac{\pi^2}{4}$, so that when $\Gamma_0 = 0$, $\Gamma_\infty = 0$, we have

$$B_\Gamma(x) \geq (\frac{\pi^2}{4} - \frac{1}{2}|\Gamma|_{L^1})|x|^2_{H^1}$$

for all $x \in H(-1,1)$ with $x(-1) = x(1) = 0$.

LEMMA 3. Let $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ be as in Lemma 2 and $\delta(\Gamma_0)$ be given by Lemma 1. Then for all measurable real functions p on $[-1,1]$ such that $p(t) \le \Gamma(t)$ a.e. on $[-1,1]$, all continuous $f: \mathbb{R} \to \mathbb{R}$ and all $x \in H^1(-1,1)$ with x' absolutely continuous on $[-1,1]$ and $x(-1) = x(1) = 0$ we have

$$-\int_{-1}^{1} x(t)(x''(t) + f(x(t))x'(t) + p(t)x(t))dt$$
$$\ge (\delta(\Gamma_0) - \frac{1}{2}|\Gamma_1|_{L^1} - \frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty})|x|^2_{H^1}.$$

PROOF. Since $x \in H^1(-1,1)$ with x' absolutely-continuous and $x(-1) = x(1) = 0$, we get on integrating by parts that

$$-\int_{-1}^{1} x(t)(x''(t) + f(x(t))x'(t) + p(t)x(t))dt$$
$$= \int_{-1}^{1} [(x'(t))^2 - p(t)x^2(t)]dt$$
$$\ge \int_{-1}^{1} [(x'(t))^2 - \Gamma(t)x^2(t)]dt$$
$$\ge [\delta(\Gamma_0) - \frac{1}{2}|\Gamma_1|_{L^1} - \frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty}]|x|^2_{H^1},$$

in view of Lemma 2.

REMARK 2. We observe that the main ingredient in Lemmas 1,2, and 3 is that the $x \in H^1(-1,1)$ satisfy Wirtinger inequalities. Now, since it is easy to see that for $x \in H^1(-1,1)$ with $x(-1) = 0$, $x'(1) + kx(1) = 0$, where $k \ge 0$ is given, (or $x'(-1) - hx(-1) = 0$, where $h \ge 0$ is given, $x(1) = 0$) Wirtinger type inequalities hold, analogues of Lemma 1, 2, 3, with $f \equiv 0$, can be obtained.

3. EXISTENCE THEOREMS

Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g: [-1,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be such that $g(\cdot,x,y)$ is measurable for each $x, y \in \mathbb{R}$ and $g(t,\cdot,\cdot)$ is continuous on $\mathbb{R} \times \mathbb{R}$ for a.e. $t \in [-1,1]$. Assume, moreover, that for each $r > 0$ there exists $\alpha_r \in L^1(-1,1)$ such that $|g(t,x,y)| \le \alpha_r(t)$ for a.e. t in $[-1,1]$, x in $[-r,r]$ and all $y \in \mathbb{R}$. We say that such a g satisfies Caratheodory's conditions.

We consider the following boundary value problem

$$\begin{cases} x''(t) + f(x(t))x'(t) + g(t,x(t),x(-t)) = e(t), \ t \in [-1,1] \\ x(-1) = x(1) = 0 \end{cases} \tag{3.1}$$

We prove the following existence theorem for (3.1).

THEOREM 1. Assume that there exists $\Gamma \in L^1(-1,1)$ such that

$$\limsup_{|x|\to\infty} \frac{g(t,x,y)}{x} \le \Gamma(t) \tag{3.2}$$

uniformly a.e. in $t \in [-1,1]$ and $y \in \mathbb{R}$. Suppose that $\Gamma = \Gamma_0 + \Gamma_1 + \Gamma_\infty$ where $\Gamma_\infty \in L^\infty(-1,1)$, $\Gamma_1 \in L^1(-1,1)$ and $\Gamma_0 \in L^1(-1,1)$ are such that $\Gamma_0(t) \le \frac{\pi^2}{4}$ for a.e. t in $[-1,1]$ with strict inequality on a subset of $[-1,1]$ of positive measure and $\frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty} + \frac{1}{2}|\Gamma_1|_{L^1} < \delta(\Gamma_0)$, where $\delta(\Gamma_0)$ is as determined in Lemma 1.

Then (3.1) has at least one solution for each $e \in L^1(-1,1)$.

PROOF. Let $\eta = \frac{1}{2}[\delta(\Gamma_0) - \frac{1}{2}|\Gamma_1|_{L^1} - \frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty}]$. Then, there exists $r_1 > 0$ such that for a.e. t in $[-1,1]$ all x with $|x| \ge r_1$ and $y \in \mathbb{R}$.

$$\frac{g(t,x,y)}{x} \leqq \Gamma(t) + \eta \tag{3.3}$$

Define $\gamma_1\colon [-1,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$\gamma_1(t,x,y) = \begin{cases} \dfrac{g(t,x,y)}{x}, & \text{if } |x| \geq r_1 \\ \dfrac{g(t,r_1,y)}{r_1}, & \text{if } 0 < x < r_1 \\ \Gamma(t), & \text{if } x = 0 \\ \dfrac{g(t,-r_1,y)}{-r_1}, & \text{if } -r_1 < x < 0. \end{cases}$$

Then, by (3.3)

$$\gamma_1(t,x,y) \leq \Gamma(t) + \eta$$

for a.e. t in $[-1,1]$ and x,y in $\mathbb{R}$. Now, the function

$$(t,x,y) \in [-1,1] \times \mathbb{R} \times \mathbb{R} \to \gamma_1(t,x,y)x \in \mathbb{R}$$

satisfies the Caratheodory's conditions and the function $h\colon [-1,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined by

$$h(t,x,y) = g(t,x,y) - \gamma_1(t,x,y)x$$

is such that there is an $\alpha \in L^1(-1,1)$ satisfying

$$|h(t,x,y)| \leq \alpha(t) \tag{3.4}$$

for a.e. t in $[-1,1]$ and all x,y in $\mathbb{R}$.

Now equation (3.1) can be written as

$$\begin{cases} x''(t) + f(x(t))x'(t) + \gamma_1(t,x,(t),x(-t))x(t) + h(t,x(t),x(-t)) = e(t), \\ x(-1) = x(1) = 0 \end{cases} \tag{3.5}$$

We next apply Theorem IV.5 of Mawhin [5] to (3.5) in the manner applied by Gupta-Mawhin in [3] (see also Mawhin-Ward [6]). To do this we need to verify that all possible solutions of the family of equations

$$\begin{cases} x''(t) + \lambda f(x(t))x'(t) + \{(1-\lambda)(\Gamma(t)+\eta) + \lambda\gamma_1(t,x(t),x(-t))\}x(t) \\ \qquad + \lambda h(t,x(t),x(-t)) = \lambda e(t) \\ x(-1) = x(1) = 0 \end{cases} \tag{3.6}$$

are, a priori, bounded by a constant independent of $\lambda \in [0,1]$ in $C^1([-1,1])$.

Let, now, $x(t)$ be a possible solution of (3.6) for some $\lambda \in [0,1]$. Since, now,

$$(1-\lambda)(\Gamma(t)+\eta) + \lambda\gamma_1(t,x(t),x(-t)) \leq \Gamma(t) + \eta$$

for a.e. t in $[-1,1]$ we have on integrating by parts the equation obtained by multiplying the equation in (3.6) by $-x(t)$ and applying Lemma 3 with Γ_∞ replaced by $\Gamma_\infty + \eta$

$$\begin{aligned} 0 &= -\int_{-1}^{1} x(t)[x''(t) + f(x(t))x'(t) + \{(1-\lambda)(\Gamma(t)+\eta) + \lambda\gamma_1(t,x(t),x(-t))\}x(t) \\ &\qquad + \lambda h(t,x(t),x(-t)) - \lambda e(t)]dt \\ &\geq \int_{-1}^{1} [x'^2(t) - \{(1-\lambda)(\Gamma(t)+\eta) + \lambda\gamma_1(t,x(t),x(-t))\}x^2(t) \\ &\qquad -\lambda\{h(t,x(t),x(-t)) - e(t)\}x(t)]dt \end{aligned}$$

$$\geq [\delta(\Gamma_0) - \frac{1}{2}|\Gamma_1|_{L^1} - \frac{4}{\pi^2}|\Gamma_\infty|_{L^\infty} - \frac{4}{\pi^2}\eta]\ |x|^2_{H^1} - (|\alpha|_{L^1} + |e|_{L^1})|x|_{L^\infty}$$

$$\geq \eta\ |x|^2_{H^1} - \frac{1}{\sqrt{2}}(|\alpha|_{L^1} + |e|_{L^1})\ |x|_{H^1}$$

It follows that there exists a constant C independent of $\lambda \in [0,1]$ such that

$$|x|_{H^1} \leq C, \quad |x|_{L^\infty} \leq C$$

and from (3.6) there is still another constant C_1 independent of λ such that

$$|x''|_{L^1} \leq C_1$$

and hence $|x|_{C^1} \leq C_2$, where C_2 is some constant independent of $\lambda \in [0,1]$.

Thus equation (3.1) has at least one solution for each $e \in L^1(-1,1)$.

The following result of Wiener-Aftabizadeh ([2], Theorem 3.4) is an immediate corollary of Theorem 1.

COROLLARY 1. Let $g: [-1,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be continuous and bounded on $[-1,1] \times \mathbb{R} \times \mathbb{R}$. Then the problem

$$\begin{cases} x''(t) = g(t,x(t),x(-t)) \\ x(-1) = x_0,\ x(1) = x_1 \end{cases} \tag{3.7}$$

has at least one solution.

PROOF. Define $h: [-1,1] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ by

$$h(t,x,y) = g(t,x + \frac{1+t}{2}x_1 + \frac{1-t}{2}x_2,\ y + \frac{1-t}{2}x_1 + \frac{1+t}{2}x_0).$$

Then (3.7) is equivalent to

$$\begin{cases} x''(t) = h(t,x(t),x(-t)) \\ x(-1) = x(1) = 0 \end{cases}$$

and (3.8) has at least one solution by Theorem 1, since $\lim\limits_{|x|\to\infty} \frac{h(t,x,y)}{x} = 0$ uniformly for a.e. t in $[-1,1]$ and all y in $\mathbb{R}$.

COROLLARY 2. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous function and let $g: [-1,1] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory conditions and let $\Gamma \in L^1(-1,1)$ be as in Theorem 1 and be such that

$$\limsup_{|x|\to\infty} \frac{g(t,x)}{x} \leq \Gamma(t)$$

uniformly for a.e. t in $[-1,1]$. Then the boundary value problem

$$x''(t) + f(x(t))x'(t) + g(t,x(t)) = e(t)$$
$$x(-1) = x(1) = 0$$

has at least one solution for each given $e \in L^1(-1,1)$.

REMARK 3. Existence of solutions for boundary value problems

$$\begin{cases} x''(t) + g(t,x(t),x(-t)) = e(t),\ t \in [-1,1] \\ x(-1) = 0 \\ x'(1) + kx(1) = 0,\ k \geq 0 \end{cases}$$

or

$$\begin{cases} x''(t) + g(t,x(t),x(-t)) = e(t), \ t \in [-1,1] \\ x'(-1) - hx(-1) = 0, \ h \geq 0 \\ x(1) = 0 \end{cases}$$

can be proved in view of Remark 2; for then Lemmas 1, 2, 3 remain valid (with different constants involved, of course) and they are the main ingredients in the proof of Theorem 1 and Theorem IV.5 of [5] continues to be applicable.

We next study the boundary value problem

$$\begin{cases} x''(t) + g(t,x(t),x(-t)) = e(t) \\ x'(-1) = x'(1) = 0. \end{cases} \tag{3.9}$$

Now, for $x \in H^1(-1,1)$ for which x and x' are absolutely-continuous on $[-1,1]$ with $x'(-1) = x'(1) = 0$, Wirtinger type inequalities are no longer available for x and x' as are needed in the proofs of Lemma 2. However, in this case Wirtinger type inequalities are available for x' and x''. This fact is exploited next to obtain an existence theorem for (3.9).

LEMMA 4. Let $e \in L^1(-1,1)$, $\Gamma \in L^1(-1,1)$ with $\overline{\Gamma} = \int_{-1}^1 \Gamma(t)dt \geq 0$. Then for every possible solution $x(t)$ of the boundary value problem

$$\begin{cases} x''(t) + p(t)x(t) = e(t) \\ x'(-1) = x'(1) = 0 \end{cases} \tag{3.10}$$

where $p \in L^1(-1,1)$ with $\overline{p} = \int_{-1}^1 p(t)dt \leq \overline{\Gamma}$, $p(t) \geq 0$ a.e. in $[-1,1]$ satisfies the inequality

$$(1 - \tfrac{1}{2}\overline{\Gamma})\ |x''|_{L^1}^2 \leq 2|e|_{L^1}|x''|_{L^1} + \overline{\Gamma}|e|_{L^1}|x|_{L^\infty}.$$

PROOF. Let $p(t)$ be as in the statement of lemma above and $x(t)$ be a possible solution of (3.10). Then multiplying the equation in (3.10) by $x(t)$ and integrating by parts we have

$$-\int_{-1}^1 x'^2(t)dt + \int_{-1}^1 p(t)x^2(t)dt = \int_{-1}^1 e(t)x(t)dt. \tag{3.11}$$

Since, by our assumptions $p^{1/2}(t)x(t)$ and $p^{1/2}(t)$ belong to $L^2(-1,1)$, we have by Schwarz's inequality

$$\begin{aligned}(\int_{-1}^1 |p(t)x(t)|dt)^2 &\leq (\int_{-1}^1 p(t)dt)(\int_{-1}^1 p(t)x^2(t)dt) \\ &\leq \overline{\Gamma} \int_{-1}^1 p(t)x^2(t)dt\end{aligned}$$

and hence using (3.10) we have

$$(\int_{-1}^1 |e(t) - x''(t)|dt)^2 \leq \overline{\Gamma} \int_{-1}^1 p(t)x^2(t)dt.$$

On the other hand, since $x'(-1) = x'(1) = 0$ we have

$$x'(t) = \int_{-1}^t x''(s)ds = -\int_t^1 x''(s)ds$$

so that

$$2|x'(t)| \leq \int_{-1}^t |x''(s)|ds + \int_t^1 |x''(s)|ds = \int_{-1}^1 |x''(s)|ds.$$

It follows that

$$\int_{-1}^1 |x'(t)|^2 dt \leq \frac{1}{2} (\int_{-1}^1 |x''(s)|ds)^2.$$

Using this in (3.11) we get

$$-\frac{1}{2}\,|x''|^2_{L^1} + \overline{\Gamma}^{\,-1}|e(t) - x''(t)|^2_{L^1} \le |e|_{L^1}\cdot |x|_{L^\infty}. \tag{3.12}$$

Now,

$$\begin{aligned} |e(t) - x''(t)|^2_{L^1} &\ge (|e|_{L^1} - |x''|_{L^1})^2 \\ &= |e|^2_{L^1} - 2|e|_{L^1}\cdot |x''|_{L^1} + |x''|^2_{L^1} \\ &\ge |x''|^2_{L^1} - 2|e|_{L^1}\cdot |x''|_{L^1}. \end{aligned}$$

Using this in (3.12) we finally get

$$(1 - \tfrac{1}{2}\,\overline{\Gamma})|x''|^2_{L^1} \le 2|e|_{L^1}\,|x''|_{L^1} + \overline{\Gamma}|e|_{L^1}|x|_{L^\infty}.$$

THEOREM 2. Assume that there exists $\Gamma(t) \in L^1(-1,1)$ such that $\limsup_{|x|\to\infty} \frac{g(t,x,y)}{x} \le \Gamma(t)$ uniformly a.e. in $t \in [-1,1]$ and all y in $\mathbb{R}$. Suppose that $\overline{\Gamma} < 2$ and that there exist real numbers a, A, r and R with $a \le R$, $r < 0 < R$ such that for a.e. t in $[-1,1]$ and y in $\mathbb{R}$, $g(t,x,y) \ge A$ when $x \ge R$ and $g(t,x,y) \le a$ when $x \le r$. Then the boundary value problem (3.9) has at least one solution for each $e \in L^1(-1,1)$ verifying the relation $2a \le \int_{-1}^{1} e(t)dt = \overline{e} \le 2A$.

PROOF. Define $g_1\colon [-1,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ by

$$g_1(t,x,y) = g(t,x,y) - \frac{1}{2}(a + A)$$

and $e_1\colon [-1,1]\to\mathbb{R}$ by

$$e_1(t) = e(t) - \frac{1}{2}(a + A)$$

so that for a.e. t in $[-1,1]$

$$\begin{aligned} g_1(t,x,y) &\ge \frac{1}{2}(A - a) \ge 0 \quad \text{for all } x \ge R \text{ and } y \text{ in } \mathbb{R} \\ g_1(t,x,y) &\le \frac{1}{2}(a - A) \le 0 \quad \text{for all } x \le r \text{ and } y \text{ in } \mathbb{R} \end{aligned} \tag{3.13}$$

and

$$(a - A) \le \overline{e}_1 \le (A - a). \tag{3.14}$$

Clearly the equation in (3.9) is equivalent to the equation

$$x'' + g_1(t,x(t),\, x(-t)) = e_1(t). \tag{3.15}$$

Moreover, we have $\limsup_{|x|\to\infty} \frac{g_1(t,x,y)}{x} \le \Gamma(t)$ uniformly a.e. for t in $[-1,1]$ and all y in $\mathbb{R}$ and if $|x| \ge \max(R,-r)$ then for a.e. t in $[-1,1]$ and all y in $\mathbb{R}$ we also have $\frac{g_1(t,x,y)}{x} \ge 0$ so that $\Gamma(t) \ge 0$ a.e. t in $[-1,1]$.

Let $\eta = \frac{1}{2}[2-\overline{\Gamma}]$ so that $\overline{\Gamma} + \eta < 2$ and let $r_1 > 0$ be such that $0 \le \frac{g_1(t,x,y)}{x} \le \Gamma(t) + \eta$ for all x with $|x| \ge r_1$, all y in $\mathbb{R}$ and a.e. t in $[-1,1]$. Proceeding as in the proof of Theorem 1 we can write the equation (3.15) in the form

$$x''(t) + \gamma_1(t,x(t),x(-t))x(t) + h(t,x(t),x(-t)) = e_1(t) \tag{3.16}$$

where

$$0 \leq \gamma_1(t,x,y) \leq \Gamma(t) + \eta$$

and there is an $\alpha \in L^1(-1,1)$ such that

$$|h(t,x,y)| \leq \alpha(t)$$

for a.e. t in $[1,1]$ and all x, y in $\mathbb{R}$.

Once again we apply Theorem IV.5 of [5] to conclude existence of a solution for (3.16) with the boundary conditions $x'(-1) = x'(1) = 0$. To do that we need to show that the set of all possible solutions of the family of equations

$$\begin{cases} x''(t) + [(1-\lambda)(\Gamma(t) + \eta) + \lambda\gamma_1(t,x(t),x(-t))]x(t) + \\ \qquad \lambda h(t,x(t), x(-t)) = \lambda e_1(t) \\ x'(-1) = x'(1) = 0 \end{cases} \tag{3.17}$$

is a priori bounded independently of $\lambda \in [0,1]$ in $C^1[-1,1]$.

Let, now, $x(t)$ be a solution of (3.17) for some $\lambda \in [0,1]$. Since,

$$0 \leq (1-\lambda)(\Gamma(t) + \eta) + \lambda\gamma_1(t,x(t),x(-t)) \leq \Gamma(t) + \eta$$

for a.e. t in $[-1,1]$ with $\overline{\Gamma} + \eta < 2$ and since

$$|e_1 - h(t,x(t),x(-t))|_{L^1} \leq |e_1|_{L^1} + |\alpha|_{L^1}$$

we see from Lemma 4 that

$$\begin{aligned} [1 - \tfrac{1}{2}(\overline{\Gamma}+\eta)]|x''|^2_{L^1} &\leq 2(|e_1|_{L^1} + |\alpha|_{L^1})|x''|_{L^1} \\ &\quad + (\overline{\Gamma}+\eta)(|e_1|_{L^1} + |\alpha|_{L^1})|x|_{L^\infty}. \end{aligned} \tag{3.18}$$

Also integrating the equation in (3.17) we have

$$\int_{-1}^{1} (1-\lambda)(\Gamma(t)+\eta)x(t)dt + \lambda\int_{-1}^{1} [g_1(t,x(t),x(-t)) - e_1(t)]dt = 0.$$

If, now, $x(t) \geq R$ for all t in $[-1,1]$ we have using (3.13), (3.14) that

$$(1-\lambda)(\overline{\Gamma}+\eta)R \leq 0$$

which contradicts the assumption that $R > 0$. Similarly, $x(t) \leq r$ for all t in $[-1,1]$ leads to a contradiction. Hence, there exists a τ in $[-1,1]$ such that

$$r < x(\tau) < R. \tag{3.19}$$

Next, it is easy to write explicitly the solution $x(t)$, with $\int_{-1}^{1} x(t)dt = 0$, of the boundary value problem $x''(t) = y(t)$, $x'(-1) = x'(1) = 0$ for $y \in L^1(-1,1)$ with $\int_{-1}^{1} y(t)dt = 0$. From this it is easy to deduce the existence of a $\delta > 0$ such that for every $x \in C^1[-1,1]$ with $\tilde{x}(t) = x(t) - \frac{1}{2}\int_{-1}^{1} x(t)dt$ and $x'(-1) = x'(1) = 0$ that

$$|\tilde{x}|_{L^\infty} \leq \delta|x''|_{L^1} \tag{3.20}$$

and

$$|\tilde{x}'|_{L^\infty} \leq \delta|x''|_{L^1}. \tag{3.21}$$

Noting that $x = \tilde{x}(t) + \frac{1}{2}\int_{-1}^{1} x(t)dt = \tilde{x} + \frac{1}{2}\overline{x}$ and inserting (3.20) in (3.18) we get

$$\begin{aligned} [1 - \tfrac{1}{2}(\overline{\Gamma} + \eta)]|x''|^2_{L^1} &\leq (|e_1|_{L^1}+|\alpha|_{L^1})(2 + \delta(\overline{\Gamma} + \eta))|x''|_{L^1} \\ &\quad + \tfrac{1}{2}(\overline{\Gamma} + \eta)(|e_1|_{L^1}+|\alpha|_{L^1})|\overline{x}|. \end{aligned} \tag{3.22}$$

Now, by the fact that there exists a $\tau \subset [-1,1]$ with $r < x(\tau) < R$ we have that

$$|x(t)| = |x(\tau) + \int_\tau^t x'(s)ds| \leq \max(-r,R) + 2|x'|_{L^\infty}$$
$$\leq \max(-r,R) + 2\delta|x''|_{L^1}$$

using (3.21). Hence

$$|\overline{x}| = |\int_{-1}^1 x(t)dt| \leq \int_{-1}^1 |(x(t)|dt \leq 2\max(-r,R) + 4\delta|x''|_{L^1}. \tag{3.23}$$

Finally inserting (3.23) in (3.22) we get that there exists a constant $\rho_1 > 0$ independent of $\lambda \in [0,1]$ such that

$$|x''|_{L^1} \leq \rho_1$$

Using this in (3.20) we then deduce the existence of a constant ρ independent of $\lambda \in [0,1]$ such that

$$|x|_{C^1[-1,1]} \leq \rho.$$

This completes the proof of the theorem.

COROLLARY 3. Let $g: [-1,1] \times \mathbb{R} \to \mathbb{R}$ satisfy Caratheodory's conditions and assume that there exists $\Gamma(t) \in L^1(-1,1)$ such that $\limsup_{|x|\to\infty} \frac{g(t,x)}{x} \leq \Gamma(t)$ uniformly a.e. in $t \in [-1,1]$. Suppose that $\overline{\Gamma} = \int_{-1}^1 \Gamma(t)dt < 2$ and that there exist real numbers a, A, r, R with $a \leq A$, $r < 0 < R$ such that for a.e. t in $[-1,1]$, $g(t,x) \geq A$ when $x \geq R$ and $g(t,x) \leq a$ when $x \leq r$.

Then the boundary value problem

$$\begin{cases} x''(t) + g(t,x(t)) = e(t) \\ x'(-1) = x'(1) = 0 \end{cases} \tag{3.24}$$

has at least one solution for each $e \in L^2(-1,1)$ with $2a \leq \overline{e} = \int_{-1}^1 e(t)dt \leq 2A$.

REMARK 4. In case $g(t,\cdot)$ is monotonically-increasing for a.e. t in $[-1,1]$ see Mawhin [7] for the boundary value problem (3.24) for a result similar to Corollary 3, above.

The existence of a solution for the boundary value problem

$$\begin{cases} x''(t) + g(t,x(t),x(-t)) = e(t) \\ x'(-1) = 0,\ x'(1) + hx(1) = 0, \quad (h > 0 \text{ given}) \end{cases} \tag{3.25}$$

can be obtained in a similar manner as for (3.9). In fact the following theorem is true, whose proof we omit as it is very similar to the proof of Theorem 2.

THEOREM 3. Assume that there exists $\Gamma(t) \in L^1(-1,1)$ such that $\limsup_{|x|\to\infty} \frac{g(t,x,y)}{x} \leq \Gamma(t)$ uniformly a.e. in $t \in [-1,1]$ and all y in $\mathbb{R}$. Suppose that $\overline{\Gamma} < \frac{1}{2}$ and that there exist real numbers a, A, r, R with $a \leq A$, $r < o < R$ such that for a.e. t in $[-1,1]$ and y in $\mathbb{R}$, $g(t,x,y) \geq A$ when $x \geq R$ and $g(t,x,y) \leq a$ when $x \leq r$.

Then the boundary value problem (3.25) has at least one solution for each $e \in L^1(-1,1)$.

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