Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2011, Article ID 432738, 12 pages
doi:10.1155/2011/432738

Research Article

Identities of Symmetry for Generalized Euler Polynomials

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Received 10 January 2011; Accepted 15 February 2011

Academic Editor: Chính T. Hoang

We derive eight basic identities of symmetry in three variables related to generalized Euler polynomials and alternating generalized power sums. All of these are new, since there have been results only about identities of symmetry in two variables. The derivations of identities are based on the p-adic fermionic integral expression of the generating function for the generalized Euler polynomials and the quotient of integrals that can be expressed as the exponential generating function for the alternating generalized power sums.

1. Introduction and Preliminaries

Let p be a fixed odd prime. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of the algebraic closure of $\mathbb{Q}_p$. Let d be a fixed odd positive integer. Then we let

$$X = X_d = \varprojlim_{N} \frac{\mathbb{Z}}{dp^N\mathbb{Z}}, \tag{1.1}$$

and let $\pi : X \to \mathbb{Z}_p$ be the map given by the inverse limit of the natural maps

$$\frac{\mathbb{Z}}{dp^N\mathbb{Z}} \longrightarrow \frac{\mathbb{Z}}{p^N\mathbb{Z}}. \tag{1.2}$$

If g is a function on $\mathbb{Z}_p$, then we will use the same notation to denote the function $g \circ \pi$. Let $\chi : (\mathbb{Z}/d\mathbb{Z})^* \to \overline{\mathbb{Q}}^*$ be a (primitive) Dirichlet character of conductor d. Then it will be pulled back to X via the natural map $X \to \mathbb{Z}/d\mathbb{Z}$. Here we fix, once and for all, an imbedding $\overline{\mathbb{Q}} \to \mathbb{C}_p$, so that χ is regarded as a map of X to $\mathbb{C}_p$ (cf. [1]).

For a continuous function $f : X \to \mathbb{C}_p$, the p-adic fermionic integral of f is defined by

$$\int_X f(z)d\mu_{-1}(z) = \lim_{N\to\infty} \sum_{j=0}^{dp^N-1} f(j)(-1)^j. \tag{1.3}$$

Then it is easy to see that

$$\int_X f(z+1)d\mu_{-1}(z) + \int_X f(z)d\mu_{-1}(z) = 2f(0). \tag{1.4}$$

More generally, we deduce from (1.4) that, for any odd positive integer n,

$$\int_X f(z+n)d\mu_{-1}(z) + \int_X f(z)d\mu_{-1}(z) = 2\sum_{a=0}^{n-1}(-1)^a f(a) \tag{1.5}$$

and that, for any even positive integer n,

$$\int_X f(z+n)d\mu_{-1}(z) - \int_X f(z)d\mu_{-1}(z) = 2\sum_{a=0}^{n-1}(-1)^{a-1} f(a). \tag{1.6}$$

Let $|\cdot|_p$ be the normalized absolute value of $\mathbb{C}_p$, such that $|p|_p = 1/p$, and let

$$E = \left\{t \in \mathbb{C}_p \mid |t|_p < p^{-1/(p-1)}\right\}. \tag{1.7}$$

Then, for each fixed $t \in E$, the function e^{zt} is analytic on $\mathbb{Z}_p$ and hence considered as a function on X, and, by applying (1.5) to f with $f(z) = \chi(z)e^{zt}$, we get the p-adic integral expression of the generating function for the generalized Euler numbers $E_{n,\chi}$ attached to χ:

$$\int_X \chi(z)e^{zt}d\mu_{-1}(z) = \frac{2}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at} = \sum_{n=0}^{\infty}E_{n,\chi}\frac{t^n}{n!} \quad (t \in E). \tag{1.8}$$

So we have the following p-adic integral expression of the generating function for the generalized Euler polynomials $E_{n,\chi}(x)$ attached to χ:

$$\int_X \chi(z)e^{(x+z)t}d\mu_{-1}(z) = \frac{2e^{xt}}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at} = \sum_{n=0}^{\infty}E_{n,\chi}(x)\frac{t^n}{n!} \quad \left(t \in E,\ x \in \mathbb{Z}_p\right). \tag{1.9}$$

Also, from (1.4), we have

$$\int_X e^{zt}d\mu_{-1}(z) = \frac{2}{e^t+1} \quad (t \in E). \tag{1.10}$$

Let $T_k(n,\chi)$ denote the kth alternating generalized power sum of the first $n+1$ nonnegative integers attached to χ, namely,

$$T_k(n,\chi)=\sum_{a=0}^{n}(-1)^a\chi(a)a^k=(-1)^0\chi(0)0^k+(-1)^1\chi(1)1^k+\cdots+(-1)^n\chi(n)n^k. \tag{1.11}$$

From (1.8), (1.10), and (1.11), one easily derives the following identities: for any odd positive integer w,

$$\frac{\int_X \chi(x)e^{xt}d\mu_{-1}(x)}{\int_X e^{wdyt}d\mu_{-1}(y)}=\frac{e^{wdt}+1}{e^{dt}+1}\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{at} \tag{1.12}$$

$$=\sum_{a=0}^{wd-1}(-1)^a\chi(a)e^{at} \tag{1.13}$$

$$=\sum_{k=0}^{\infty}T_k(wd-1,\chi)\frac{t^k}{k!}\quad (t\in E). \tag{1.14}$$

In what follows, we will always assume that the p-adic integrals of the various (twisted) exponential functions on X are defined for $t\in E$ (cf. (1.7)), and therefore it will not be mentioned.

References [2–6] are some of the previous works on identities of symmetry in two variables involving Bernoulli polynomials and power sums. On the other hand, for the first time we were able to produce in [7] some identities of symmetry in three variables related to Bernoulli polynomials and power sums and to extend in [8] to the case of generalized Bernoulli polynomials and generalized power sums. Also, [4] is about identities of symmetry in two variables for Euler polynomials and alternating power sums, and [9] is about those in three variables for them.

In this paper, we will be able to produce 8 identities of symmetry in three variables regarding generalized Euler polynomials and alternating generalized power sums. The case of two variables was treated in [10].

The following is stated as Theorem 4.2 and an example of the full six symmetries in w_1, w_2, w_3:

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_2y_2)T_m(w_3d-1,\chi)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_3y_2)T_m(w_2d-1,\chi)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)E_{l,\chi}(w_1y_2)T_m(w_3d-1,\chi)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)E_{l,\chi}(w_3y_2)T_m(w_1d-1,\chi)w_2^{l+m}w_3^{k+m}w_1^{k+l}
\end{aligned}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,\chi}(w_3 y_1) E_{l,\chi}(w_2 y_2) T_m(w_1 d - 1, \chi) w_3^{l+m} w_2^{k+m} w_1^{k+l}$$

$$= \sum_{k+l+m=n} \binom{n}{k,l,m} E_{k,\chi}(w_3 y_1) E_{l,\chi}(w_1 y_2) T_m(w_2 d - 1, \chi) w_3^{l+m} w_1^{k+m} w_2^{k+l}. \quad (1.15)$$

The derivations of identities are based on the p-adic integral expression of the generating function for the generalized Euler polynomials in (1.9) and the quotient of integrals in (1.12) that can be expressed as the exponential generating function for the alternating generalized power sums. This abundance of symmetries would not be unearthed if such p-adic integral representations had not been available. We indebted this idea to paper [10].

2. Several Types of Quotients of p-Adic Fermionic Integrals

Here we will introduce several types of quotients of p-adic fermionic integrals on X or X^3 from which some interesting identities follow owing to the built-in symmetries in w_1, w_2, w_3. In the following, w_1, w_2, w_3 are all positive integers, and all of the explicit expressions of integrals in (2.2), (2.4), (2.6), and (2.8) are obtained from the identities in (1.8) and (1.10). To ease notations, from now on, we will suppress μ_{-1} and denote, for example, $d\mu_{-1}(x)$ simply by dx.

(a) Type Λ_{23}^i (for $i = 0, 1, 2, 3$):

$$I\left(\Lambda_{23}^i\right) = \frac{\int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3) e^{(w_2 w_3 x_1 + w_1 w_3 x_2 + w_1 w_2 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} dx_1 dx_2 dx_3}{\left(\int_X e^{d w_1 w_2 w_3 x_4 t} dx_4\right)^i} \quad (2.1)$$

$$= \frac{2^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} \left(e^{d w_1 w_2 w_3 t} + 1\right)^i}{\left(e^{d w_2 w_3 t} + 1\right)\left(e^{d w_1 w_3 t} + 1\right)\left(e^{d w_1 w_2 t} + 1\right)}$$
$$\times \left(\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_2 w_3 t}\right)\left(\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_1 w_3 t}\right)\left(\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_1 w_2 t}\right). \quad (2.2)$$

(b) Type Λ_{13}^i (for $i = 0, 1, 2, 3$):

$$I\left(\Lambda_{13}^i\right) = \frac{\int_{X^3} \chi(x_1)\chi(x_2)\chi(x_3) e^{(w_1 x_1 + w_2 x_2 + w_3 x_3 + w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j))t} dx_1 dx_2 dx_3}{\left(\int_X e^{d w_1 w_2 w_3 x_4 t} dx_4\right)^i} \quad (2.3)$$

$$= \frac{2^{3-i} e^{w_1 w_2 w_3 (\sum_{j=1}^{3-i} y_j)t} \left(e^{d w_1 w_2 w_3 t} + 1\right)^i}{\left(e^{d w_1 t} + 1\right)\left(e^{d w_2 t} + 1\right)\left(e^{d w_3 t} + 1\right)}$$
$$\times \left(\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_1 t}\right)\left(\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_2 t}\right)\left(\sum_{a=0}^{d-1} (-1)^a \chi(a) e^{a w_3 t}\right). \quad (2.4)$$

(c-0) Type Λ_{12}^{0}:

$$I\left(\Lambda_{12}^{0}\right)=\int_{X^3}\chi(x_1)\chi(x_2)\chi(x_3)e^{(w_1x_1+w_2x_2+w_3w_3+w_2w_3y+w_1w_3y+w_1w_2y)t}dx_1dx_2dx_3 \tag{2.5}$$

$$\begin{aligned}&=\frac{8e^{(w_2w_3+w_1w_3+w_1w_2)yt}}{\left(e^{dw_1t}+1\right)\left(e^{dw_2t}+1\right)\left(e^{dw_3t}+1\right)}\\&\quad\times\left(\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{aw_1t}\right)\left(\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{aw_2t}\right)\left(\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{aw_3t}\right).\end{aligned} \tag{2.6}$$

(c-1) Type Λ_{12}^{1}:

$$I\left(\Lambda_{12}^{1}\right)=\frac{\int_{X^3}\chi(x_1)\chi(x_2)\chi(x_3)e^{(w_1x_1+w_2x_2+w_3x_3)t}dx_1dx_2dx_3}{\int_{X^3}e^{d(w_2w_3z_1+w_1w_3z_2+w_1w_2z_3)t}dz_1z_2z_3} \tag{2.7}$$

$$\begin{aligned}&=\frac{\left(e^{dw_2w_3t}+1\right)\left(e^{dw_1w_3t}+1\right)\left(e^{dw_1w_2t}+1\right)}{\left(e^{dw_1t}+1\right)\left(e^{dw_2t}+1\right)\left(e^{dw_3t}+1\right)}\\&\quad\times\left(\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{aw_1t}\right)\left(\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{aw_2t}\right)\left(\sum_{a=0}^{d-1}(-1)^a\chi(a)e^{aw_3t}\right).\end{aligned} \tag{2.8}$$

All of the above p-adic integrals of various types are invariant under all permutations of w_1, w_2, w_3, as one can see either from p-adic integral representations in (2.1), (2.3), (2.5), and (2.7) or from their explicit evaluations in (2.2), (2.4), (2.6), and (2.8).

3. Identities for Generalized Euler Polynomials

In the following, w_1, w_2, w_3 are all odd positive integers except for (a-0) and (c-0), where they are any positive integers. First, let's consider Type Λ_{23}^{i}, for each $i=0,1,2,3$. The following results can be easily obtained from (1.9) and (1.12):

(a-0)

$$\begin{aligned}&I\left(\Lambda_{23}^{0}\right)\\&=\int_X\chi(x_1)e^{w_2w_3(x_1+w_1y_1)t}dx_1\int_X\chi(x_2)e^{w_1w_3(x_2+w_2y_2)t}dx_2\int_X\chi(x_3)e^{w_1w_2(x_3+w_3y_3)t}dx_3\\&=\left(\sum_{k=0}^{\infty}\frac{E_{k,\chi}(w_1y_1)}{k!}(w_2w_3t)^k\right)\left(\sum_{l=0}^{\infty}\frac{E_{l,\chi}(w_2y_2)}{l!}(w_1w_3t)^l\right)\left(\sum_{m=0}^{\infty}\frac{E_{m,\chi}(w_3y_3)}{m!}(w_1w_2t)^m\right)\\&=\sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_2y_2)E_{m,\chi}(w_3y_3)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!},\end{aligned} \tag{3.1}$$

where the inner sum is over all nonnegative integers k, l, m with $k+l+m=n$ and

$$\binom{n}{k,l,m}=\frac{n!}{k!l!m!}. \tag{3.2}$$

(a-1) Here we write $I(\Lambda_{23}^{1})$ in two different ways:

(1)

$$\begin{aligned}
I\left(\Lambda_{23}^{1}\right) &= \int_X \chi(x_1)e^{w_2w_3(x_1+w_1y_1)t}dx_1\int_X \chi(x_2)e^{w_1w_3(x_2+w_2y_2)t}dx_2\frac{\int_X \chi(x_3)e^{w_1w_2x_3t}dx_3}{\int_X e^{dw_1w_2w_3x_4t}dx_4}\\
&= \left(\sum_{k=0}^{\infty}E_{k,\chi}(w_1y_1)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}E_{l,\chi}(w_2y_2)\frac{(w_1w_3t)^l}{l!}\right)\left(T_m(w_3d-1,\chi)\frac{(w_1w_2t)^m}{m!}\right)
\end{aligned} \tag{3.3}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_2y_2)T_m(w_3d-1,\chi)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!}. \tag{3.4}$$

(2) Invoking (1.13), (3.3) can also be written as

$$\begin{aligned}
I\left(\Lambda_{23}^{1}\right) &= \sum_{a=0}^{w_3d-1}(-1)^a\chi(a)\int_X \chi(x_1)e^{w_2w_3(x_1+w_1y_1)t}dx_1\int_X \chi(x_2)e^{w_1w_3(x_2+w_2y_2+w_2/w_3a)t}dx_2\\
&= \sum_{a=0}^{w_3d-1}(-1)^a\chi(a)\left(\sum_{k=0}^{\infty}E_{k,\chi}(w_1y_1)\frac{(w_2y_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}E_{l,\chi}\left(w_2y_2+\frac{w_2}{w_3}a\right)\frac{(w_1y_3t)^l}{l!}\right)\\
&= \sum_{n=0}^{\infty}\left(w_3^n\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}(w_1y_1)\sum_{a=0}^{w_3d-1}(-1)^a\chi(a)E_{n-k,\chi}\left(w_2y_2+\frac{w_2}{w_3}a\right)w_1^{n-k}w_2^k\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.5}$$

(a-2) Here we write $I(\Lambda_{23}^{2})$ in three different ways:

(1)

$$\begin{aligned}
I\left(\Lambda_{23}^{2}\right) &= \int_X \chi(x_1)e^{w_2w_3(x_1+w_1y_1)t}dx_1\frac{\int_X \chi(x_2)e^{w_1w_3x_2t}dx_2}{\int_X e^{dw_1w_2w_3x_4t}dx_4}\frac{\int_X \chi(x_3)e^{w_1w_2x_3t}dx_3}{\int_X e^{dw_1w_2w_3x_4t}dx_4}\\
&= \left(\sum_{k=0}^{\infty}E_{k,\chi}(w_1y_1)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}T_l(w_2d-1,\chi)\frac{(w_1w_3t)^l}{l!}\right)\\
&\quad\times\left(\sum_{m=0}^{\infty}T_m(w_3d-1,\chi)\frac{(w_1w_2t)^m}{m!}\right)
\end{aligned} \tag{3.6}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)T_l(w_2d-1,\chi)T_m(w_3d-1,\chi)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!}. \tag{3.7}$$

(2) Invoking (1.13), (3.6) can also be written as

$$I\left(\Lambda_{23}^{2}\right) = \sum_{a=0}^{w_2 d-1}(-1)^a\chi(a)\int_X \chi(x_1)e^{w_2w_3(x_1+w_1y_1+(w_1/w_2)a)t}dx_1 \times \frac{\int_X \chi(x_3)e^{w_1w_2x_3t}dx_3}{\int_X e^{dw_1w_2w_3x_4t}dx_4}$$

$$= \sum_{a=0}^{w_2 d-1}(-1)^a\chi(a)\left(\sum_{k=0}^{\infty}E_{k,\chi}\left(w_1y_1+\frac{w_1}{w_2}a\right)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}T_l(w_3d-1,\chi)\frac{(w_1w_2t)^l}{l!}\right) \tag{3.8}$$

$$= \sum_{n=0}^{\infty}\left(w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{w_2d-1}(-1)^a\chi(a)E_{k,\chi}\left(w_1y_1+\frac{w_1}{w_2}a\right)T_{n-k}(w_3d-1,\chi)w_1^{n-k}w_3^k\right)\frac{t^n}{n!}. \tag{3.9}$$

(3) Invoking (1.13) once again, (3.8) can be written as

$$I\left(\Lambda_{23}^{2}\right) = \sum_{a=0}^{w_2 d-1}(-1)^a\chi(a)\sum_{b=0}^{w_3 d-1}(-1)^b\chi(b)\int_X \chi(x_1)e^{w_2w_3(x_1+w_1y_1+(w_1/w_2a)+(w_1/w_3)b)t}dx_1$$

$$= \sum_{a=0}^{w_2 d-1}(-1)^a\chi(a)\sum_{b=0}^{w_3 d-1}(-1)^b\chi(b)\sum_{n=0}^{\infty}E_{n,\chi}\left(w_1y_1+\frac{w_1}{w_2}a+\frac{w_1}{w_3}b\right)\frac{(w_2w_3t)^n}{n!} \tag{3.10}$$

$$= \sum_{n=0}^{\infty}\left((w_2w_3)^n\sum_{a=0}^{w_2d-1}\sum_{b=0}^{w_3d-1}(-1)^{a+b}\chi(ab)E_{n,\chi}\left(w_1y_1+\frac{w_1}{w_2}a+\frac{w_1}{w_3}b\right)\right)\frac{t^n}{n!}. \tag{3.11}$$

(a-3)

$$I\left(\Lambda_{23}^{3}\right) = \frac{\int_X \chi(x_1)e^{w_2w_3x_1t}dx_1}{\int_X e^{dw_1w_2w_3x_4t}dx_4}\times\frac{\int_X \chi(x_2)e^{w_1w_3x_2t}dx_2}{\int_X e^{dw_1w_2w_3x_4t}dx_4}\times\frac{\int_X \chi(x_3)e^{w_1w_2x_3t}dx_3}{\int_X e^{dw_1w_2w_3x_4t}dx_4}$$

$$= \left(\sum_{k=0}^{\infty}T_k(w_1d-1,\chi)\frac{(w_2w_3t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}T_l(w_2d-1,\chi)\frac{(w_1w_3t)^l}{l!}\right)\times\left(\sum_{m=0}^{\infty}T_m(w_3d-1,\chi)\frac{(w_1w_2t)^m}{m!}\right) \tag{3.12}$$

$$= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_1d-1,\chi)T_l(w_2d-1,\chi)T_m(w_3d-1,\chi)w_1^{l+m}w_2^{k+m}w_3^{k+l}\right)\frac{t^n}{n!}. \tag{3.13}$$

(b) For Type Λ_{13}^{i} ($i = 0,1,2,3$), we may consider the analogous things to the ones in (a-0), (a-1), (a-2), and (a-3). However, these do not lead us to new identities. Indeed, if we substitute w_2w_3, w_1w_3, w_1w_2, respectively, for w_1, w_2, w_3 in (2.1), this amounts to replacing t by $w_1w_2w_3t$ in (2.3). So, upon replacing w_1, w_2, w_3, respectively, by w_2w_3, w_1w_3, w_1w_2 and dividing by $(w_1w_2w_3)^n$, in each of the expressions of (3.1), (3.4), (3.5), (3.7), (3.9)–(3.13), we will get the corresponding symmetric identities for Type Λ_{13}^{i} ($i = 0,1,2,3$).

(c-0)

$$\begin{aligned}
I\left(\Lambda_{12}^{0}\right) &= \int_X \chi(x_1)e^{w_1(x_1+w_2y)t}dx_1 \int_X \chi(x_2)e^{w_2(x_2+w_3y)t}dx_2 \int_X \chi(x_3)e^{w_3(x_3+w_1y)t}dx_3 \\
&= \left(\sum_{k=0}^{\infty}\frac{E_{k,\chi}(w_2y)}{k!}(w_1t)^k\right)\left(\sum_{l=0}^{\infty}\frac{E_{l,\chi}(w_3y)}{l!}(w_2t)^l\right)\left(\sum_{m=0}^{\infty}\frac{E_{m,\chi}(w_1y)}{m!}(w_3t)^m\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y)E_{l,\chi}(w_3y)E_{m,\chi}(w_1y)w_1^k w_2^l w_3^m\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.14}$$

(c-1)

$$\begin{aligned}
I\left(\Lambda_{12}^{1}\right) &= \frac{\int_X \chi(x_1)e^{w_1x_1t}dx_1}{\int_X e^{dw_1w_2z_3t}dz_3} \times \frac{\int_X \chi(x_2)e^{w_2x_2t}dx_2}{\int_X e^{dw_2w_3z_1t}dz_1} \times \frac{\int_X \chi(x_3)e^{w_3x_3t}dx_3}{\int_X e^{dw_3w_1z_2t}dz_2} \\
&= \left(\sum_{k=0}^{\infty}T_k(w_2d-1,\chi)\frac{(w_1t)^k}{k!}\right)\left(\sum_{l=0}^{\infty}T_l(w_3d-1,\chi)\frac{(w_2t)^l}{l!}\right)\left(\sum_{m=0}^{\infty}T_m(w_1d-1,\chi)\frac{(w_3t)^m}{m!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_2d-1,\chi)T_l(w_3d-1,\chi)T_m(w_1d-1,\chi)w_1^k w_2^l w_3^m\right)\frac{t^n}{n!}.
\end{aligned} \tag{3.15}$$

4. Main Theorems

As we noted earlier in the last paragraph of Section 2, the various types of quotients of p-adic fermionic integrals are invariant under any permutation of w_1, w_2, w_3. So the corresponding expressions in Section 3 are also invariant under any permutation of w_1, w_2, w_3. Thus, our results about identities of symmetry will be immediate consequences of this observation.

However, not all permutations of an expression in Section 3 yield distinct ones. In fact, as these expressions are obtained by permuting w_1, w_2, w_3 in a single one labeled by them, they can be viewed as a group in a natural manner, and hence it is isomorphic to a quotient of S_3. In particular, the number of possible distinct expressions is 1, 2, 3, or 6 (a-0), (a-1(1)), (a-1(2)), and (a-2(2)) give the full six identities of symmetry, (a-2(1)) and (a-2(3)) yield three identities of symmetry, and (c-0) and (c-1) give two identities of symmetry, while the expression in (a-3) yields no identities of symmetry.

Here we will just consider the cases of Theorems 4.4 and 4.8, leaving the others as easy exercises for the reader. As for the case of Theorem 4.4, in addition to (4.11)–(4.13), we get the following three ones:

$$\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)T_l(w_3d-1,\chi)T_m(w_2d-1,\chi)w_1^{l+m}w_3^{k+m}w_2^{k+l} \tag{4.1}$$

$$= \sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)T_l(w_1d-1,\chi)T_m(w_3d-1,\chi)w_2^{l+m}w_1^{k+m}w_3^{k+l} \tag{4.2}$$

$$= \sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_3y_1)T_l(w_2d-1,\chi)T_m(w_1d-1,\chi)w_3^{l+m}w_2^{k+m}w_1^{k+l}. \tag{4.3}$$

But, by interchanging l and m, we see that (4.1), (4.2), and (4.3) are, respectively, equal to (4.11), (4.12), and (4.13). As to Theorem 4.8, in addition to (4.17) and (4.18), we have:

$$\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_2d-1,\chi)T_l(w_3d-1,\chi)T_m(w_1d-1,\chi)w_1^k w_2^l w_3^m \tag{4.4}$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_3d-1,\chi)T_l(w_1d-1,\chi)T_m(w_2d-1,\chi)w_2^k w_3^l w_1^m \tag{4.5}$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_3d-1,\chi)T_l(w_2d-1,\chi)T_m(w_1d-1,\chi)w_1^k w_3^l w_2^m \tag{4.6}$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_2d-1,\chi)T_l(w_1d-1,\chi)T_m(w_3d-1,\chi)w_3^k w_2^l w_1^m. \tag{4.7}$$

However, (4.4) and (4.5) are equal to (4.17), as we can see by applying the permutations $k \to l, l \to m, m \to k$ for (4.4) and $k \to m, l \to k, m \to l$ for (4.5). Similarly, we see that (4.6) and (4.7) are equal to (4.18), by applying permutations $k \to l, l \to m, m \to k$ for (4.6) and $k \to m, l \to k, m \to l$ for (4.7).

Theorem 4.1. *Let w_1, w_2, w_3 be any positive integers. Then one has*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_2y_2)E_{m,\chi}(w_3y_3)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_3y_2)E_{m,\chi}(w_2y_3)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)E_{l,\chi}(w_1y_2)E_{m,\chi}(w_3y_3)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)E_{l,\chi}(w_3y_2)E_{m,\chi}(w_1y_3)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_3y_1)E_{l,\chi}(w_1y_2)E_{m,\chi}(w_2y_3)w_3^{l+m}w_1^{k+m}w_2^{k+l}\\
&\quad=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_3y_1)E_{l,\chi}(w_2y_2)E_{m,\chi}(w_1y_3)w_3^{l+m}w_2^{k+m}w_1^{k+l}.
\end{aligned}\tag{4.8}$$

Theorem 4.2. *Let w_1, w_2, w_3 be any odd positive integers. Then one has*

$$\begin{aligned}
&\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_2y_2)T_m(w_3d-1,\chi)w_1^{l+m}w_2^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)E_{l,\chi}(w_3y_2)T_m(w_2d-1,\chi)w_1^{l+m}w_3^{k+m}w_2^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)E_{l,\chi}(w_1y_2)T_m(w_3d-1,\chi)w_2^{l+m}w_1^{k+m}w_3^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)E_{l,\chi}(w_3y_2)T_m(w_1d-1,\chi)w_2^{l+m}w_3^{k+m}w_1^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_3y_1)E_{l,\chi}(w_2y_2)T_m(w_1d-1,\chi)w_3^{l+m}w_2^{k+m}w_1^{k+l}\\
&=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_3y_1)E_{l,\chi}(w_1y_2)T_m(w_2d-1,\chi)w_3^{l+m}w_1^{k+m}w_2^{k+l}.
\end{aligned}\tag{4.9}$$

Theorem 4.3. *Let w_1, w_2, w_3 be any odd positive integers. Then one has*

$$\begin{aligned}
&w_1^n\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}(w_3y_1)\sum_{a=0}^{w_1d-1}(-1)^a\chi(a)E_{n-k,\chi}\left(w_2y_2+\frac{w_2}{w_1}a\right)w_3^{n-k}w_2^k\\
&=w_1^n\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}(w_2y_1)\sum_{a=0}^{w_1d-1}(-1)^a\chi(a)E_{n-k,\chi}\left(w_3y_2+\frac{w_3}{w_1}a\right)w_2^{n-k}w_3^k\\
&=w_2^n\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}(w_3y_1)\sum_{a=0}^{w_2d-1}(-1)^a\chi(a)E_{n-k,\chi}\left(w_1y_2+\frac{w_1}{w_2}a\right)w_3^{n-k}w_1^k\\
&=w_2^n\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}(w_1y_1)\sum_{a=0}^{w_2d-1}(-1)^a\chi(a)E_{n-k,\chi}\left(w_3y_2+\frac{w_3}{w_2}a\right)w_1^{n-k}w_3^k\\
&=w_3^n\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}(w_2y_1)\sum_{a=0}^{w_3d-1}(-1)^a\chi(a)E_{n-k,\chi}\left(w_1y_2+\frac{w_1}{w_3}a\right)w_2^{n-k}w_1^k\\
&=w_3^n\sum_{k=0}^{n}\binom{n}{k}E_{k,\chi}(w_1y_1)\sum_{a=0}^{w_3d-1}(-1)^a\chi(a)E_{n-k,\chi}\left(w_2y_2+\frac{w_2}{w_3}a\right)w_1^{n-k}w_2^k.
\end{aligned}\tag{4.10}$$

Theorem 4.4. *Let w_1, w_2, w_3 be any odd positive integers. Then one has the following three symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y_1)T_l(w_2d-1,\chi)T_m(w_3d-1,\chi)w_1^{l+m}w_2^{k+m}w_3^{k+l} \tag{4.11}$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_2y_1)T_l(w_3d-1,\chi)T_m(w_1d-1,\chi)w_2^{l+m}w_3^{k+m}w_1^{k+l} \tag{4.12}$$

$$=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_3y_1)T_l(w_1d-1,\chi)T_m(w_2d-1,\chi)w_3^{l+m}w_1^{k+m}w_2^{k+l}. \tag{4.13}$$

Theorem 4.5. *Let w_1, w_2, w_3 be any odd positive integers. Then one has*

$$\begin{aligned}
&w_1^n\sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{w_1d-1}(-1)^a\chi(a)E_{k,\chi}\left(w_2y_1+\frac{w_2}{w_1}a\right)T_{n-k}(w_3d-1,\chi)w_2^{n-k}w_3^k\\
&=w_1^n\sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{w_1d-1}(-1)^a\chi(a)E_{k,\chi}\left(w_3y_1+\frac{w_3}{w_1}a\right)T_{n-k}(w_2d-1,\chi)w_3^{n-k}w_2^k\\
&=w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{w_2d-1}(-1)^a\chi(a)E_{k,\chi}\left(w_1y_1+\frac{w_1}{w_2}a\right)T_{n-k}(w_3d-1,\chi)w_1^{n-k}w_3^k\\
&=w_2^n\sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{w_2d-1}(-1)^a\chi(a)E_{k,\chi}\left(w_3y_1+\frac{w_3}{w_2}a\right)T_{n-k}(w_1d-1,\chi)w_3^{n-k}w_1^k\\
&=w_3^n\sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{w_3d-1}(-1)^a\chi(a)E_{k,\chi}\left(w_1y_1+\frac{w_1}{w_3}a\right)T_{n-k}(w_2d-1,\chi)w_1^{n-k}w_2^k\\
&=w_3^n\sum_{k=0}^{n}\binom{n}{k}\sum_{a=0}^{w_3d-1}(-1)^a\chi(a)E_{k,\chi}\left(w_2y_1+\frac{w_2}{w_3}a\right)T_{n-k}(w_1d-1,\chi)w_2^{n-k}w_1^k.
\end{aligned} \tag{4.14}$$

Theorem 4.6. *Let w_1, w_2, w_3 be any odd positive integers. Then one has the following three symmetries in w_1, w_2, w_3:*

$$\begin{aligned}
&(w_1w_2)^n\sum_{a=0}^{w_1d-1}\sum_{b=0}^{w_2d-1}(-1)^{a+b}\chi(ab)E_{n,\chi}\left(w_3y_1+\frac{w_3}{w_1}a+\frac{w_3}{w_2}b\right)\\
&=(w_2w_3)^n\sum_{a=0}^{w_2d-1}\sum_{b=0}^{w_3d-1}(-1)^{a+b}\chi(ab)E_{n,\chi}\left(w_1y_1+\frac{w_1}{w_2}a+\frac{w_1}{w_3}b\right)\\
&=(w_3w_1)^n\sum_{a=0}^{w_3d-1}\sum_{b=0}^{w_1d-1}(-1)^{a+b}\chi(ab)E_{n,\chi}\left(w_2y_1+\frac{w_2}{w_3}a+\frac{w_2}{w_1}b\right).
\end{aligned} \tag{4.15}$$

Theorem 4.7. *Let w_1, w_2, w_3 be any positive integers. Then one has the following two symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y)E_{l,\chi}(w_2y)E_{m,\chi}(w_3y)w_3^kw_1^lw_2^m$$
$$=\sum_{k+l+m=n}\binom{n}{k,l,m}E_{k,\chi}(w_1y)E_{l,\chi}(w_3y)E_{m,\chi}(w_2y)w_2^kw_1^lw_3^m. \tag{4.16}$$

Theorem 4.8. *Let w_1, w_2, w_3 be any odd positive integers. Then one has the following two symmetries in w_1, w_2, w_3:*

$$\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_1d-1,\chi)T_l(w_2d-1,\chi)T_m(w_3d-1,\chi)w_3^kw_1^lw_2^m, \tag{4.17}$$

$$\sum_{k+l+m=n}\binom{n}{k,l,m}T_k(w_1d-1,\chi)T_l(w_3d-1,\chi)T_m(w_2d-1,\chi)w_2^kw_1^lw_3^m. \tag{4.18}$$

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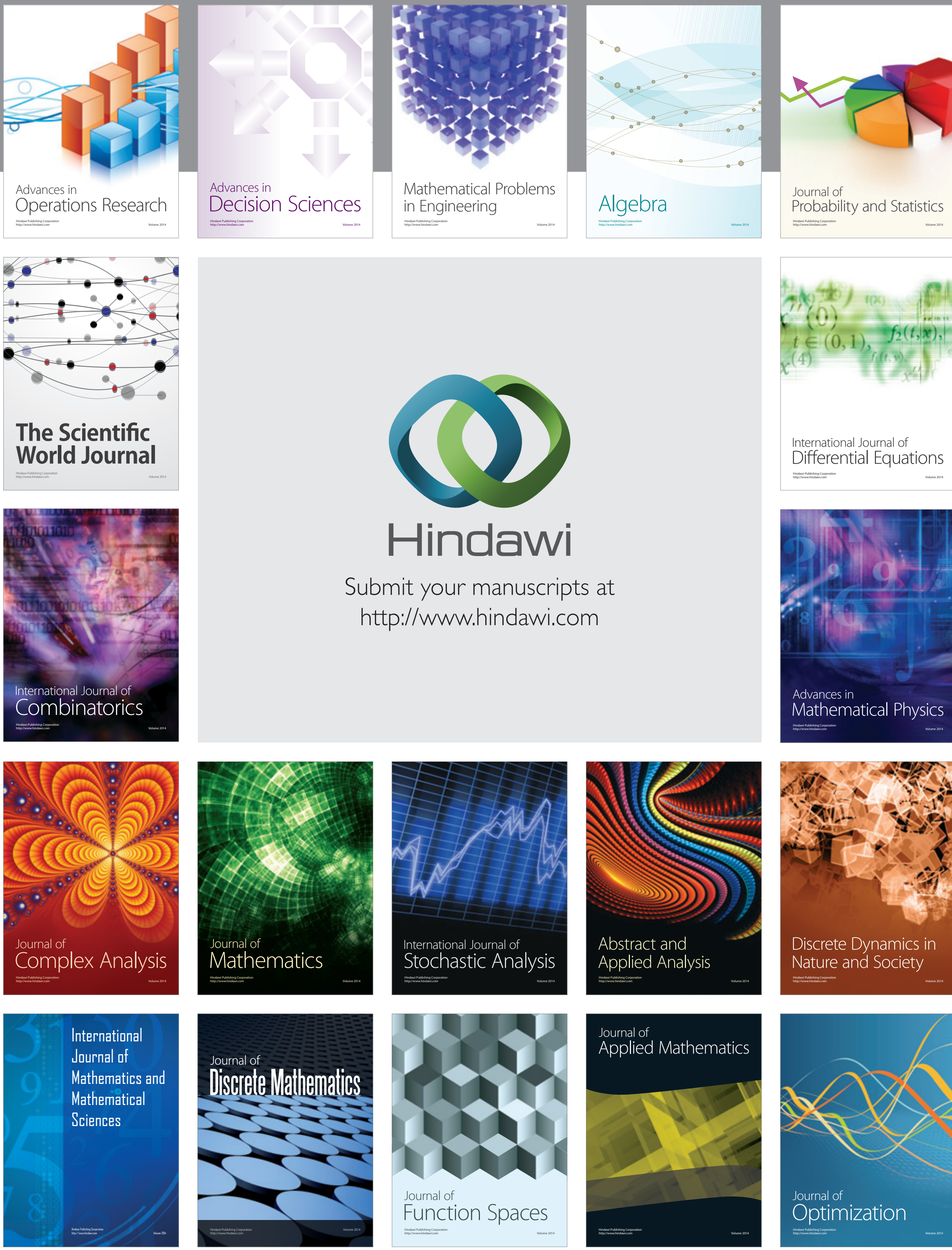

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