Hindawi Publishing Corporation
International Journal of Combinatorics
Volume 2011, Article ID 206930, 16 pages
doi:10.1155/2011/206930

Research Article

Cayley Graphs of Order *27p* Are Hamiltonian

Ebrahim Ghaderpour and Dave Witte Morris

Department of Mathematics and Computer Science, University of Lethbridge, Leth-Bridge, AB, Canada T1K 3M4

Correspondence should be addressed to Dave Witte Morris, dave.morris@uleth.ca

Received 22 January 2011; Accepted 18 April 2011

Academic Editor: Cai Heng Li

Suppose that G is a finite group, such that $|G| = 27p$, where p is prime. We show that if S is any generating set of G, then there is a Hamiltonian cycle in the corresponding Cayley graph $\mathrm{Cay}(G; S)$.

1. Introduction

Theorem 1.1. *If $|G| = 27p$, where p is prime, then every connected Cayley graph on G has a Hamiltonian cycle.*

Combining this with results in [1–3] establishes that

$$\begin{gathered}\text{Every Cayley graph on } G \text{ has a hamiltonian cycle}\\ \text{if } |G| = kp, \text{ where } p \text{ is prime, } 1 \leq k < 32, \text{ and } k \neq 24.\end{gathered} \tag{1.1}$$

The remainder of the paper provides a proof of the theorem. Here is an outline. Section 2 recalls known results on hamiltonian cycles in Cayley graphs; Section 3 presents the proof under the assumption that the Sylow p-subgroup of G is normal; Section 4 presents the proof under the assumption that the Sylow p-subgroups of G are not normal.

2. Preliminaries: Known Results on Hamiltonian Cycles in Cayley Graphs

For convenience, we record some known results that provide hamiltonian cycles in various Cayley graphs, after fixing some notation.

Notation 1 (see [4, Sections 1.1 and 5.1]). For any group G, we use the following notation:

(1) G' denotes the *commutator subgroup* $[G, G]$ of G,

(2) $Z(G)$ denotes the *center* of G,

(3) $\Phi(G)$ denotes the *Frattini subgroup* of G.

For $a, b \in G$, we use a^b to denote the *conjugate* $b^{-1}ab$.

Notation 2. If $(s_1, s_2, \ldots, s_n)$ *is any sequence, we use* $(s_1, s_2, \ldots, s_n)\#$ *to denote the sequence* $(s_1, s_2, \ldots, s_{n-1})$ *that is obtained by deleting the last term.*

Theorem 2.1 (Marušič, Durnberger, Keating-Witte [5]). *If G' is a cyclic group of prime-power order, then every connected Cayley graph on G has a hamiltonian cycle.*

Lemma 2.2 (see [3, Lemma 2.27]). *Let S generate the finite group G, and let $s \in S$. If*

(i) $\langle s \rangle \lhd G$,

(ii) $\mathrm{Cay}(G/\langle s \rangle; S)$ *has a hamiltonian cycle, and*

(iii) *either*

(1) $s \in Z(G)$, *or*

(2) $|s|$ *is prime,*

then $\mathrm{Cay}(G; S)$ *has a hamiltonian cycle.*

Lemma 2.3 (see [1, Lemma 2.7]). *Let S generate the finite group G, and let $s \in S$. If*

(i) $\langle s \rangle \lhd G$,

(ii) $|s|$ *is a divisor of pq, where p and q are distinct primes,*

(iii) $s^p \in Z(G)$,

(iv) $|G/\langle s \rangle|$ *is divisible by q, and*

(v) $\mathrm{Cay}(G/\langle s \rangle; S)$ *has a hamiltonian cycle,*

then there is a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

The following results are well known (and easy to prove).

Lemma 2.4 ("Factor Group Lemma"). *Suppose that*

(i) *S is a generating set of G,*

(ii) *N is a cyclic, normal subgroup of G,*

(iii) $(s_1 N, \ldots, s_n N)$ *is a hamiltonian cycle in* $\mathrm{Cay}(G/N; S)$, *and*

(iv) *the product $s_1 s_2 \cdots s_n$ generates N.*

Then $(s_1, \ldots, s_n)^{|N|}$ *is a hamiltonian cycle in* $\mathrm{Cay}(G; S)$.

Corollary 2.5. *Suppose that*

(i) *S is a generating set of G,*

(ii) *N is a normal subgroup of G, such that $|N|$ is prime,*

(iii) *$s \equiv t \pmod{N}$ for some $s, t \in S \cup S^{-1}$ with $s \neq t$, and*

(iv) *there is a hamiltonian cycle in $\mathrm{Cay}(G/N; S)$ that uses at least one edge labelled s.*

Then there is a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Definition 2.6. If H is any subgroup of G, then $H \backslash \mathrm{Cay}(G; S)$ denotes the multigraph in which

(i) the vertices are the right cosets of H, and

(ii) there is an edge joining Hg_1 and Hg_2 for each $s \in S \cup S^{-1}$, such that $g_1 s \in Hg_2$.

Thus, if there are two different elements s_1 and s_2 of $S \cup S^{-1}$, such that $g_1 s_1$ and $g_1 s_2$ are both in Hg_2, then the vertices Hg_1 and Hg_2 are joined by a double edge.

Lemma 2.7 (see [3, Corollary 2.9]). *Suppose that*

(i) *S is a generating set of G,*

(ii) *H is a subgroup of G, such that $|H|$ is prime,*

(iii) *the quotient multigraph $H \backslash \mathrm{Cay}(G; S)$ has a hamiltonian cycle C, and*

(iv) *C uses some double-edge of $H \backslash \mathrm{Cay}(G; S)$.*

Then there is a hamiltonian cycle in $\mathrm{Cay}(G; S)$.

Theorem 2.8 (see [6, Corollary 3.3]). *Suppose that*

(i) *S is a generating set of G,*

(ii) *N is a normal p-subgroup of G, and*

(iii) *$st^{-1} \in N$, for all $s, t \in S$.*

Then $\mathrm{Cay}(G; S)$ has a hamiltonian cycle.

Remark 2.9. In the proof of our main result, we may assume $p \geq 5$, for otherwise either

(i) $|G| = 54$ is of the form $18q$, where q is prime, and so [3, Propostion 9.1] applies, or

(ii) $|G| = 3^4$ is a prime power, and so the main theorem of [7] applies.

3. Assume the Sylow p-Subgroup of G Is Normal

Notation 3. Let

(i) G be a group of order $27p$, where p is prime, and $p \geq 5$ (see Remark 2.9),

(ii) S be a minimal generating set for G,

(iii) $P \cong \mathbb{Z}_p$ be a Sylow p-subgroup of G,

(iv) w be a generator of P, and

(v) Q be a Sylow 3-subgroup of G.

Assumption 3.1. *In this section, we assume that P is a normal subgroup of G.*

Therefore G is a semidirect product:

$$G = Q \ltimes P. \tag{3.1}$$

We may assume that G' is not cyclic of prime order (for otherwise Theorem 2.1 applies). This implies that Q is nonabelian and acts nontrivially on P; so

$$G' = Q' \times P \text{ is cyclic of order } 3p. \tag{3.2}$$

Notation 4. Since Q is a 3-group and acts nontrivially on $P \cong \mathbb{Z}_p$, we must have $p \equiv 1 \pmod 3$. Thus, one may choose $r \in \mathbb{Z}$, such that

$$r^3 \equiv 1 \pmod p, \text{ but } r \not\equiv 1 \pmod p. \tag{3.3}$$

Dividing $r^3 - 1$ by $r - 1$, we see that

$$r^2 + r + 1 \equiv 0 \pmod p. \tag{3.4}$$

3.1. A Lemma That Applies to Both of the Possible Sylow 3-Subgroups

There are only 2 nonabelian groups of order 27, and we will consider them as separate cases, but, first, we cover some common ground.

Note

Since Q is a nonabelian group of order 27, and $G = Q \ltimes P \cong Q \ltimes \mathbb{Z}_p$, it is easy to see that

$$Q' = \Phi(Q) = Z(Q) = Z(G) = \Phi(G). \tag{3.5}$$

Lemma 3.2. *Assume that*

(i) *$s \in (S \cup S^{-1}) \cap Q$, such that s does not centralize P, and*

(ii) *$c \in C_Q(P) \setminus \Phi(Q)$.*

Then we may assume that S is either $\{s, cw\}$ or $\{s, c^2w\}$ or $\{s, scw\}$ or $\{s, sc^2w\}$.

Proof. Since $G/P \cong Q$ is a 2-generated group of prime-power order, there must be an element a of S, such that $\{s, a\}$ generates G/P. We may write

$$a = s^i c^j z w^k, \quad \text{with } 0 \le i \le 2,\ 1 \le j \le 2,\ z \in Z(Q),\ \text{and } 0 \le k < p. \tag{3.6}$$

Note the following.

(i) By replacing a with its inverse if necessary, we may assume $i \in \{0, 1\}$.

(ii) By applying an automorphism of G that fixes s and maps c to cz^j, we may assume that z is trivial (since $(cz^j)^j = c^j z^{j^2} = c^j z$).

(iii) By replacing w with w^k if $k \neq 0$, we may assume $k \in \{0,1\}$.

Thus,

$$a = s^i c^j w^k \text{ with } i, k \in \{0,1\}, \text{ and } j \in \{1,2\}. \tag{3.7}$$

Case 1 (*Assume* $k = 1$). Then $\langle s, a \rangle = G$, and so $S = \{s, a\}$. This yields the four listed generating sets.

Case 2 (*Assume* $k = 0$). Then $\langle s, a \rangle = Q$, and there must be a third element b of S, with $b \notin Q$; after replacing w with an appropriate power, we may write $b = tw$ with $t \in Q$. We must have $t \in \langle s, \Phi(Q) \rangle$, for otherwise $\langle s, b \rangle = G$ (which contradicts the minimality of S). Therefore

$$t = s^{i'} z' \quad \text{with } 0 \leq i' \leq 2, \text{ and } z' \in \Phi(Q) = Z(G). \tag{3.8}$$

We may assume the following.

(i) $i' \neq 0$, for otherwise $b = z'w \in S \cap (Z(G) \times P)$; so Lemma 2.3 applies.

(ii) $i' = 1$, by replacing b with its inverse if necessary.

(iii) $z' \neq e$, for otherwise s and b provide a double edge in $\mathrm{Cay}(G/P; S)$; so Corollary 2.5 applies.

Then $s^{-1}b = z'w$ generates $Z(G) \times P$.

Consider the hamiltonian cycles

$$\left(a^{-1}, s^2\right)^3, \quad \left(\left(a^{-1}, s^2\right)^3 \#, b\right), \quad \left(\left(a^{-1}, s^2\right)^3 \#\#, b^2\right) \tag{3.9}$$

in $\mathrm{Cay}(G/\langle z, w \rangle; S)$. Letting $z'' = (a^{-1}s^2)^3 \in \langle z \rangle$, we see that their endpoints in G are (resp.)

$$z'', \quad z''\left(s^{-1}b\right) = z''z' \ w, \quad z''\left(s^{-1}b\right)^s\left(s^{-1}b\right) = z''\left(z'\right)^2 \ w^s w. \tag{3.10}$$

The final two endpoints both have a nontrivial projection to P (since s, being a 3-element, cannot invert w), and at least one of these two endpoints also has a nontrivial projection to $Z(G)$. Such an endpoint generates $Z(G) \times P = \langle z, w \rangle$, and so the Factor Group Lemma 2.4 provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

3.2. Sylow 3-Subgroup of Exponent 3

Lemma 3.3. *Assume that Q is of exponent 3; so*

$$Q = \left\langle x, y, z \mid x^3 = y^3 = z^3 = e, \ [x, y] = z, \ [x, z] = [y, z] = e \right\rangle. \tag{3.11}$$

Then one may assume the following:

(1) $w^x = w^r$, *but* y *and* z *centralize* P, *and*

(2) *either*

(a) $S = \{x, yw\}$, *or*

(b) $S = \{x, xyw\}$.

Proof. (1) Since Q acts nontrivially on P, and $\mathrm{Aut}(P)$ is cyclic, but $Q/\Phi(Q)$ is not cyclic, there must be elements a and b of $Q \setminus \Phi(Q)$, such that a centralizes P, but b does not. (And z must centralize P, because it is in Q'.) By applying an automorphism of Q, we may assume $a = y$ and $b = x$. Furthermore, we may assume $w^x = w^r$ by replacing x with its inverse if necessary.

(2) S must contain an element that does not centralize P; so we may assume $x \in S$. By applying Lemma 3.2 with $s = x$ and $c = y$, we see that we may assume that S is

$$\{x, yw\} \text{ or } \left\{x, y^2 w\right\} \text{ or } \{x, xyw\} \text{ or } \left\{x, xy^2 w\right\}. \tag{3.12}$$

But there is an automorphism of G that fixes x and w and sends y to y^2; so we need only consider two of these possibilities. □

Proposition 3.4. *Assume, as usual, that* $|G| = 27p$, *where* p *is prime, and that* G *has a normal Sylow* p*-subgroup. If the Sylow 3-subgroup* Q *is of exponent 3, then* $\mathrm{Cay}(G; S)$ *has a hamiltonian cycle.*

Proof. We write $^{-}$ for the natural homomorphism from G to $\overline{G} = G/P$. From Lemma 3.3(2), we see that we need only consider two possibilities for S.

Case 1 (*Assume* $S = \{x, yw\}$). For $a = x$ and $b = yw$, we have the following hamiltonian cycle in $\mathrm{Cay}(G/P; S)$:

$$\begin{array}{ccccccccccccc}
 & \overline{e} & \xrightarrow{a} & \overline{x} & \xrightarrow{a} & \overline{x^2} & \xrightarrow{b} & \overline{x^2y} & \xrightarrow{a^{-1}} & \overline{xyz} & \xrightarrow{a^{-1}} & \overline{yz^2} \\
\xrightarrow{b} & \overline{y^2z^2} & \xrightarrow{b} & \overline{z^2} & \xrightarrow{a} & \overline{xz^2} & \xrightarrow{a} & \overline{x^2z^2} & \xrightarrow{b} & \overline{x^2yz^2} & \xrightarrow{a} & \overline{yz} \\
\xrightarrow{a} & \overline{xy} & \xrightarrow{b} & \overline{xy^2} & \xrightarrow{a} & \overline{x^2y^2z} & \xrightarrow{b} & \overline{x^2z} & \xrightarrow{b} & \overline{x^2yz} & \xrightarrow{a^{-1}} & \overline{xyz^2} \\
\xrightarrow{b} & \overline{xy^2z^2} & \xrightarrow{a} & \overline{x^2y^2} & \xrightarrow{a} & \overline{y^2z} & \xrightarrow{b} & \overline{z} & \xrightarrow{a} & \overline{xz} & \xrightarrow{b^{-1}} & \overline{xy^2z} \\
\xrightarrow{a} & \overline{x^2y^2z^2} & \xrightarrow{a} & \overline{y^2} & \xrightarrow{b^{-1}} & \overline{y} & \xrightarrow{b^{-1}} & \overline{e}. & & & &
\end{array} \tag{3.13}$$

Its endpoint in G is

$$\begin{aligned}
&a^2ba^{-2}b^2a^2ba^2bab^2a^{-1}ba^2bab^{-1}a^2b^{-2} \\
&\quad = x^2ywx^{-2}(yw)^2x^2ywx^2ywx(yw)^2x^{-1}ywx^2ywx(yw)^{-1}x^2(yw)^{-2} \\
&\quad = x^2ywxy^2w^2x^2ywx^2ywxy^2w^2x^2ywx^2ywxy^2w^{-1}x^2yw^{-2}.
\end{aligned} \tag{3.14}$$

Since the walk is a hamiltonian cycle in G/P, we know that this endpoint is in $P = \langle w \rangle$. So all terms except powers of w must cancel. Thus, we need only calculate the contribution from each appearance of w in this expression. To do this, note that if a term w^i is followed by a net total of j appearances of x, then the term contributes a factor of w^{ir^j} to the product. So the endpoint in G is

$$w^{r^{13}} w^{2r^{12}} w^{r^{10}} w^{r^8} w^{2r^7} w^{r^5} w^{r^3} w^{-r^2} w^{-2}. \tag{3.15}$$

Since $r^3 \equiv 1 \pmod{p}$, this simplifies to

$$\begin{aligned} w^r w^2 w^r w^{r^2} w^{2r} w^{r^2} w w^{-r^2} w^{-2} &= w^{r+2+r+r^2+2r+r^2+1-r^2-2} \\ &= w^{r^2+4r+1} = w^{r^2+r+1} w^{3r} = w^0 w^{3r} = w^{3r}. \end{aligned} \tag{3.16}$$

Since $p \nmid 3r$, this endpoint generates P; so the Factor Group Lemma 2.4 provides a hamiltonian cycle in $\mathrm{Cay}(G;S)$.

Case 2 (*Assume* $S = \{x, xyw\}$). For $a = x$ and $b = xyw$, we have the hamiltonian cycle

$$\left(\left(a, b^2\right)^3 \#, a\right)^3 \tag{3.17}$$

in $\mathrm{Cay}(G/P;S)$. Its endpoint in G is

$$\begin{aligned} \left(\left(ab^2\right)^3 b^{-1} a\right)^3 &= \left(\left(x(xyw)^2\right)^3 (xyw)^{-1} x\right)^3 = \left(\left(x\left(x^2y^2w^{r+1}\right)\right)^3 \left(w^{-1}y^{-1}x^{-1}\right)x\right)^3 \\ &= \left(\left(y^2w^{r+1}\right)^3 \left(w^{-1}y^{-1}\right)\right)^3 = \left(w^{3(r+1)} \left(w^{-1}y^{-1}\right)\right)^3 = \left(y^{-1}w^{3r+2}\right)^3 \\ &= w^{3(3r+2)}. \end{aligned} \tag{3.18}$$

Since we are free to choose r to be either of the two primitive cube roots of 1 in $\mathbb{Z}_p$, and the equation $3r + 2 = 0$ has only one solution in $\mathbb{Z}_p$, we may assume that r has been selected to make the exponent nonzero. Then the Factor Group Lemma 2.4 provides a hamiltonian cycle in $\mathrm{Cay}(G;S)$. □

3.3. *Sylow 3-Subgroup of Exponent 9*

Lemma 3.5. *Assume that Q is of exponent 9; so*

$$Q = \left\langle x, y \mid x^9 = y^3 = e,\ [x, y] = x^3 \right\rangle. \tag{3.19}$$

There are two possibilities for G, depending on whether $C_Q(P)$ contains an element of order 9 or not.

(1) *Assume that $C_Q(P)$ does not contain an element of order 9. Then we may assume that y centralizes P, but $w^x = w^r$. Furthermore, we may assume that:*

(a) $S = \{x, yw\}$, *or*

(b) $S = \{x, xyw\}$.

(2) *Assume that $C_Q(P)$ contains an element of order 9. Then we may assume x centralizes P, but $w^y = w^r$. Furthermore, we may assume that:*

(a) $S = \{xw, y\}$,

(b) $S = \{xyw, y\}$,

(c) $S = \{xy, xw\}$, *or*

(d) $S = \{xy, x^2yw\}$.

Proof. (1) Since x has order 9, we know that it does not centralize P. But x^3 must centralize P (since x^3 is in G'). Therefore, we may assume $w^x = x^r$ (by replacing x with its inverse if necessary). Also, since $Q/C_Q(P)$ must be cyclic (because $\mathrm{Aut}(P)$ is cyclic), but $C_G(P)$ does not contain an element of order 9, we see that $C_Q(P)$ contains every element of order 3; so y must be in $C_Q(P)$.

Since S must contain an element that does not centralize P, we may assume $x \in S$. By applying Lemma 3.2 with $s = x$ and $c = y$, we see that we may assume that S is:

$$\{x, yw\} \text{ or } \left\{x, y^2w\right\} \text{ or } \{x, xyw\} \text{ or } \left\{x, xy^2w\right\}. \tag{3.20}$$

The second generating set need not be considered, because $(y^2w)^{-1} = yw^{-1} = yw'$; so it is equivalent to the first. Also, the fourth generating set can be converted into the third, since there is an automorphism of G that fixes y, but takes x to xyw and w to w^{-1}.

(2) We may assume $x \in C_Q(P)$; so $C_Q(P) = \langle x \rangle$.

We know that S must contain an element s that does not centralize P, and there are two possibilities: either

(I) s has order 3, or

(II) s has order 9.

We consider these two possibilities as separate cases.

Case I (*Assume that s has order* 3). We may assume $s = y$. Letting $c = x$, we see from Lemma 3.2 that we may assume S is either

$$\{y, xw\} \text{ or } \left\{y, x^2w\right\} \text{ or } \{y, yxw\} \text{ or } \left\{y, yx^2w\right\}. \tag{3.21}$$

The second and fourth generating sets need not be considered, because there is an automorphism of G that fixes y and w, but takes x to x^2. Also, the third generating set may be replaced with $\{y, xyw\}$, since there is an automorphism of G that fixes y and w, but takes x to $y^{-1}xy$.

Case II (*Assume that s has order* 9). We may assume $s = xy$. Letting $c = x$, we see from Lemma 3.2 that we may assume that S is either

$$\{xy, xw\} \text{ or } \left\{xy, x^2w\right\} \text{ or } \{xy, xyxw\} \text{ or } \left\{xy, xyx^2w\right\}. \tag{3.22}$$

The second generating set is equivalent to $\{xy, xw\}$, since the automorphism of G that sends x to x^4, y to $x^{-3}y$, and w to w^{-1} maps it to $\{xy, (xw)^{-1}\}$. The third generating set is mapped to $\{xy, x^2yw\}$ by the automorphism that sends x to $x[x,y]$ and y to $[x,y]^{-1}y$. The fourth generating set need not be considered, because xyx^2w is an element of order 3 that does not centralize P, which puts it in the previous case. □

Proposition 3.6. *Assume, as usual, that $|G| = 27p$, where p is prime, and that G has a normal Sylow p-subgroup. If the Sylow 3-subgroup Q is of exponent 9, then* $\mathrm{Cay}(G;S)$ *has a hamiltonian cycle.*

Proof. We will show that, for an appropriate choice of a and b in $S \cup S^{-1}$, the walk

$$\left(a^3, b^{-1}, a, b^{-1}, a^4, b^2, a^{-2}, b, a^2, b, a^3, b, a^{-1}, b^{-1}, a^{-1}, b^{-2}\right) \tag{3.23}$$

provides a hamiltonian cycle in $\mathrm{Cay}(G/P;S)$ whose endpoint in G generates P (so the Factor Group Lemma 2.4 applies).

We begin by verifying two situations in which (3.23) is a hamiltonian cycle.

(HC1) If $|\overline{a}| = 9$, $|\overline{b}| = 3$, and $\overline{a^b} = \overline{a^4}$ in $\overline{G} = G/P$, then we have the hamiltonian cycle:

$$\begin{array}{cccccccccccccc}
 & \overline{e} & \xrightarrow{a} & \overline{a} & \xrightarrow{a} & \overline{a^2} & \xrightarrow{a} & \overline{a^3} & \xrightarrow{b^{-1}} & \overline{a^3b^2} & \xrightarrow{a} & \overline{a^7b^2} & \xrightarrow{b^{-1}} & \overline{a^7b} \\
\xrightarrow{a} & \overline{a^5b} & \xrightarrow{a} & \overline{a^3b} & \xrightarrow{a} & \overline{ab} & \xrightarrow{a} & \overline{a^8b} & \xrightarrow{b} & \overline{a^8b^2} & \xrightarrow{b} & \overline{a^8} & \xrightarrow{a^{-1}} & \overline{a^7} \\
\xrightarrow{a^{-1}} & \overline{a^6} & \xrightarrow{b} & \overline{a^6b} & \xrightarrow{a} & \overline{a^4b} & \xrightarrow{a} & \overline{a^2b} & \xrightarrow{b} & \overline{a^2b^2} & \xrightarrow{a} & \overline{a^6b^2} & \xrightarrow{a} & \overline{ab^2} \\
\xrightarrow{a} & \overline{a^5b^2} & \xrightarrow{b} & \overline{a^5} & \xrightarrow{a^{-1}} & \overline{a^4} & \xrightarrow{b^{-1}} & \overline{a^4b^2} & \xrightarrow{a^{-1}} & \overline{b^2} & \xrightarrow{b^{-1}} & \overline{b} & \xrightarrow{b^{-1}} & \overline{e}.
\end{array} \tag{3.24}$$

(HC2) If $|\overline{a}| = 9$, $|\overline{b}| = 9$, $\overline{a^b} = \overline{a^7}$, and $\overline{b^3} = \overline{a^6}$ in $\overline{G} = G/P$, then we have the hamiltonian cycle:

$$\begin{array}{cccccccccccccc}
 & \overline{e} & \xrightarrow{a} & \overline{a} & \xrightarrow{a} & \overline{a^2} & \xrightarrow{a} & \overline{a^3} & \xrightarrow{b^{-1}} & \overline{a^6b^2} & \xrightarrow{a} & \overline{a^4b^2} & \xrightarrow{b^{-1}} & \overline{a^4b} \\
\xrightarrow{a} & \overline{a^8b} & \xrightarrow{a} & \overline{a^3b} & \xrightarrow{a} & \overline{a^7b} & \xrightarrow{a} & \overline{a^2b} & \xrightarrow{b} & \overline{a^2b^2} & \xrightarrow{b} & \overline{a^8} & \xrightarrow{a^{-1}} & \overline{a^7} \\
\xrightarrow{a^{-1}} & \overline{a^6} & \xrightarrow{b} & \overline{a^6b} & \xrightarrow{a} & \overline{ab} & \xrightarrow{a} & \overline{a^5b} & \xrightarrow{b} & \overline{a^5b^2} & \xrightarrow{a} & \overline{a^3b^2} & \xrightarrow{a} & \overline{ab^2} \\
\xrightarrow{a} & \overline{a^8b^2} & \xrightarrow{b} & \overline{a^5} & \xrightarrow{a^{-1}} & \overline{a^4} & \xrightarrow{b^{-1}} & \overline{a^7b^2} & \xrightarrow{a^{-1}} & \overline{b^2} & \xrightarrow{b^{-1}} & \overline{b} & \xrightarrow{b^{-1}} & \overline{e}.
\end{array} \tag{3.25}$$

To calculate the endpoint in G, fix $r_1, r_2 \in \mathbb{Z}_p$, with

$$w^a = w^{r_1}, \qquad w^b = w^{r_2}, \tag{3.26}$$

and write

$$a = \underline{a}w_1, \quad b = \underline{b}w_2, \text{ where } \underline{a}, \underline{b} \in Q, \quad w_1, w_2 \in P. \tag{3.27}$$

Note that if an occurrence of w_i in the product is followed by a net total of j_1 appearances of $\underline{a}$ and a net total of j_2 appearances of $\underline{b}$, then it contributes a factor of $w_i^{r_1^{j_1} r_2^{j_2}}$ to the product. (A similar occurrence of w_i^{-1} contributes a factor of $w_i^{-r_1^{j_1} r_2^{j_2}}$ to the product.) Furthermore, since $r_1^3 \equiv r_2^3 \equiv 1 \pmod{p}$, there is no harm in reducing j_1 and j_2 modulo 3.

We will apply these considerations only in a few particular situations.

(E1) Assume $w_1 = e$ (so $a \in Q$ and $\underline{a} = a$). Then the endpoint of the path in G is

$$\begin{aligned} &a^3 b^{-1} a b^{-1} a^4 b^2 a^{-2} b a^2 b a^3 b a^{-1} b^{-1} a^{-1} b^{-2} \\ &\quad = a^3 (\underline{b}w_2)^{-1} a (\underline{b}w_2)^{-1} a^4 (\underline{b}w_2)^2 a^{-2} (\underline{b}w_2) a^2 \\ &\qquad \times (\underline{b}w_2) a^3 (\underline{b}w_2) a^{-1} (\underline{b}w_2)^{-1} a^{-1} (\underline{b}w_2)^{-2} \\ &\quad = a^3 \left(w_2^{-1}\underline{b}^{-1}\right) a \left(w_2^{-1}\underline{b}^{-1}\right) a^4 (\underline{b}w_2\underline{b}w_2) a^{-2} (\underline{b}w_2) a^2 \\ &\qquad \times (\underline{b}w_2) a^3 (\underline{b}w_2) a^{-1} \left(w_2^{-1}\underline{b}^{-1}\right) a^{-1} \left(w_2^{-1}\underline{b}^{-1} w_2^{-1}\underline{b}^{-1}\right). \end{aligned} \tag{3.28}$$

By the above considerations, this simplifies to w_2^m, where

$$\begin{aligned} m &= -1 - r_1^2 r_2 + r_1 r_2 + r_1 + r_2^2 + r_1 r_2 + r_1 - r_1^2 - r_2 - r_2^2 \\ &= -r_1^2 r_2 - r_1^2 + 2r_1 r_2 + 2r_1 - r_2 - 1. \end{aligned} \tag{3.29}$$

Note the following.

(a) If $r_1 \neq 1$ and $r_2 = 1$, then m simplifies to $6r_1$, because $r_1^2 + r_1 + 1 \equiv 0 \pmod{p}$ in this case.

(b) If $r_1 \neq 1$ and $r_2 \neq 1$, then m simplifies to $3r_1(r_2+1)$, because $r_1^2+r_1+1 \equiv r_2^2+r_2+1 \equiv 0 \pmod{p}$ in this case.

(E2) Assume $w_2 = e$ (so $b \in Q$ and $\underline{b} = b$). Then the endpoint of the path in G is

$$\begin{aligned}
&a^3b^{-1}ab^{-1}a^4b^2a^{-2}ba^2ba^3ba^{-1}b^{-1}a^{-1}b^{-2} \\
&= (\underline{a}w_1)^3b^{-1}(\underline{a}w_1)b^{-1}(\underline{a}w_1)^4b^2(\underline{a}w_1)^{-2}b(\underline{a}w_1)^2b(\underline{a}w_1)^3b(\underline{a}w_1)^{-1}b^{-1}(\underline{a}w_1)^{-1}b^{-2} \\
&= (\underline{a}w_1\underline{a}w_1\underline{a}w_1)b^{-1}(\underline{a}w_1)b^{-1}(\underline{a}w_1\underline{a}w_1\underline{a}w_1\underline{a}w_1)b^2\left(w_1^{-1}\underline{a}^{-1}w_1^{-1}\underline{a}^{-1}\right) \\
&\quad \times b(\underline{a}w_1\underline{a}w_1)b(\underline{a}w_1\underline{a}w_1\underline{a}w_1)b\left(w_1^{-1}\underline{a}^{-1}\right)b^{-1}\left(w_1^{-1}\underline{a}^{-1}\right)b^{-2}.
\end{aligned} \tag{3.30}$$

By the above considerations, this simplifies to w_1^m, where

$$\begin{aligned}
m &= r_1^2 + r_1 + 1 + r_1^2r_2 + r_1r_2^2 + r_2^2 + r_1^2r_2^2 + r_1r_2^2 - r_1 \\
&\quad - r_1^2 + r_1^2r_2^2 + r_1r_2^2 + r_2 + r_1^2r_2 + r_1r_2 - r_1 - r_1^2r_2 \\
&= 2r_1^2r_2^2 + 3r_1r_2^2 + r_2^2 + r_1^2r_2 + r_1r_2 + r_2 - r_1 + 1.
\end{aligned} \tag{3.31}$$

Note the following.

(a) If $r_1 = 1$ and $r_2 \neq 1$, then m simplifies to $-3(r_2+2)$, because $r_2^2+r_2+1 \equiv 0 \pmod{p}$ in this case.

(b) If $r_1 \neq 1$ and $r_2 \neq 1$, then m simplifies to $-r_1r_2 - 2r_1 + r_2 + 2$, because $r_1^2 + r_1 + 1 \equiv r_2^2 + r_2 + 1 \equiv 0 \pmod{p}$ in this case.

Now we provide a hamiltonian cycle for each of the generating sets listed in Lemma 3.5.

(1a) If $C_Q(P)$ has exponent 3, and $S = \{x, yw\}$, we let $a = x$ and $b = yw$ in (HC1). In this case, we have $w_1 = e$, $r_1 = r$, and $r_2 = 1$; so (E1(a)) tells us that the endpoint in G is w_2^{6r}.

(1b) If $C_Q(P)$ has exponent 3, and $S = \{x, xyw\}$, we let $a = x$ and $b = (xyw)^{-1}$ in (HC2). In this case, we have $w_1 = e$, $r_1 = r$, and $r_2 = r^{-1} = r^2$; so (E1(b)) tells us that the endpoint in G is w_2^m, where

$$m = 3r_1(r_2+1) = 3r\left(r^2+1\right) = 3\left(r^3+r\right) \equiv 3(1+r) = 3(r+1) \pmod{p}. \tag{3.32}$$

(2a) If $C_Q(P)$ has exponent 9, and $S = \{xw, y\}$, we let $a = xw$ and $b = y$ in (HC1). In this case, we have $w_2 = e$, $r_1 = 1$, and $r_2 = r$; so (E2(a)) tells us that the endpoint in G is $w_1^{-3(r+2)}$.

(2b) If $C_Q(P)$ has exponent 9, and $S = \{xyw, y\}$, we let $a = xyw$ and $b = y$ in (HC1). In this case, we have $w_2 = e$ and $r_1 = r_2 = r$; so (E2(b)) tells us that the endpoint in G is w_2^m, where

$$m = -r_1r_2 - 2r_1 + r_2 + 2 = -r^2 - 2r + r + 2 = -\left(r^2 + r + 1\right) + 3 \equiv 3 \pmod{p}. \tag{3.33}$$

(2c) If $C_Q(P)$ has exponent 9, and $S = \{xy, xw\}$, we let $a = xw$ and $b = (xy)^{-1}$ in (HC2). In this case, we have $w_2 = e$, $r_1 = 1$, and $r_2 = r^{-1} = r^2$; so (E2(a)) tells us that the endpoint in G is w_1^m, where

$$m = -3(r_2 + 2) = -3\left(r^2 + 2\right) \equiv -3(-(r+1) + 2) = 3(r-1) \pmod{p}. \tag{3.34}$$

(2d) If $C_Q(P)$ has exponent 9, and $S = \{xy, x^2yw\}$, we let $a = xy$ and $b = x^2yw$ in (HC2). In this case, we have $w_1 = e$ and $r_1 = r_2 = r$; so (E1(b)) tells us that the endpoint in G is w_2^m, where

$$m = 3r_1(r_2 + 1) = 3r(r+1) = 3\left(r^2 + r\right) \equiv 3(-1) = -3 \pmod{p}. \tag{3.35}$$

In all cases, there is at most one nonzero value of r (modulo p) for which the exponent of w_i is 0. Since we are free to choose r to be either of the two primitive cube roots of 1 in $\mathbb{Z}_p$, we may assume that r has been selected to make the exponent nonzero. Then the Factor Group Lemma 2.4 provides a hamiltonian cycle in $\mathrm{Cay}(G; S)$. □

4. Assume the Sylow p-Subgroups of G Are Not Normal

Lemma 4.1. *Assume that*

(i) $|G| = 27p$, *where* p *is an odd prime, and*

(ii) *the Sylow* p*-subgroups of* G *are not normal.*

Then $p = 13$, *and* $G = \mathbb{Z}_{13} \ltimes (\mathbb{Z}_3)^3$, *where a generator* w *of* $\mathbb{Z}_{13}$ *acts on* $(\mathbb{Z}_3)^3$ *via multiplication on the right by the matrix*

$$W = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}. \tag{4.1}$$

Furthermore, we may assume that

$$S \text{ is of the form } \left\{w^i, w^j v\right\}, \tag{4.2}$$

where $v = (1, 0, 0) \in (\mathbb{Z}_3)^3$, *and*

$$(i, j) \in \{(1,0), (2,0), (1,2), (1,3), (1,5), (1,6), (2,5)\}. \tag{4.3}$$

Proof. Let P be a Sylow p-subgroup of G, and let Q be a Sylow 3-subgroup of G. Since no odd prime divides $3-1$ or 3^2-1, and 13 is the only odd prime that divides 3^3-1, Sylow's Theorem [8, Theorem 15.7, page 230] implies that $p = 13$, and that $N_G(P) = P$; so G must have a normal

p-complement [4, Theorem 7.4.3]; that is, $G = P \ltimes Q$. Since P must act nontrivially on Q (since P is not normal), we know that it must act nontrivially on $Q/\Phi(Q)$ [4, Theorem 5.3.5, page 180]. However, P cannot act nontrivially on an elementary abelian group of order 3 or 3^2, because $|P| = 13$ is not a divisor of $3-1$ or 3^2-1. Therefore, we must have $|Q/\Phi(Q)| = 3^3$; so Q must be elementary abelian (and the action of P is irreducible).

Let W be the matrix representing the action of w on $(\mathbb{Z}_3)^3$ (with respect to some basis that will be specified later). In the polynomial ring $\mathbb{Z}_3[X]$, we have the factorization:

$$\frac{X^{13}-1}{X-1} = \left(X^3 - X - 1\right) \cdot \left(X^3 + X^2 - 1\right) \cdot \left(X^3 + X^2 + X - 1\right) \cdot \left(X^3 - X^2 - X - 1\right). \tag{4.4}$$

Since $w^{13} = e$, the minimal polynomial of W must be one of the factors on the right-hand side. By replacing w with an appropriate power, we may assume that it is the first factor. Then, choosing any nonzero $v \in (\mathbb{Z}_3)^3$, the matrix representation of w with respect to the basis $\{v, v^w, v^{w^2}\}$ is W (the Rational Canonical Form).

Now, let ζ be a primitive 13th root of unity in the finite field GF(27). Then any Galois automorphism of GF(27) over GF(3) must raise ζ to a power. Since the subgroup of order 3 in $\mathbb{Z}_{13}^{\times}$ is generated by the number 3, we conclude that the orbit of ζ under the Galois group is $\{\zeta, \zeta^3, \zeta^9\}$. These must be the 3 roots of one of the irreducible factors on the right-hand side of (4.4). Thus, for any $k \in \mathbb{Z}_{13}^{\times}$, the matrices W^k, W^{3k}, and W^{9k} all have the same minimal polynomial; so they are conjugate under $\mathrm{GL}_3(3)$. That is,

$$\begin{array}{rl} & W, W^3, W^9 \\ \text{powers of } W \text{ in the same row of the} & W^2, W^5, W^6 \\ \text{following array are conjugate under } \mathrm{GL}_3(3): & W^4, W^{12}, W^{10} \\ & W^7, W^8, W^{11}. \end{array} \tag{4.5}$$

There is an element a of S that generates $G/Q \cong P$. Then a has order p; so, replacing it by a conjugate, we may assume $a \in P = \langle w \rangle$, and so $a = w^i$ for some $i \in \mathbb{Z}_{13}^{\times}$. From (4.5), we see that we may assume $i \in \{1, 2\}$ (perhaps after replacing a by its inverse).

Now let b be the second element of S; so we may assume $b = w^j v$ for some j. We may assume $0 \le j \le 6$ (by replacing b with its inverse, if necessary). We may also assume $j \ne i$, for otherwise $S \subset aQ$, and so Theorem 2.8 applies.

If $j = 0$, then (i, j) is either $(1, 0)$ or $(2, 0)$, both of which appear in the list; henceforth, let us assume $j \ne 0$.

Case 1 (*Assume* $i = 1$). Since $j \ne i$, we must have $j \in \{2, 3, 4, 5, 6\}$.

Note that since W^3 is conjugate to W under $\mathrm{GL}_3(3)$ (since they are in the same row of (4.5)), we know that the pair (w, w^4) is isomorphic to the pair $(w^3, (w^3)^4) = (w^3, w^{-1})$. By replacing b with its inverse, and then interchanging a and b, this is transformed to (w, w^3). So we may assume $j \ne 4$.

Case 2 (*Assume* $i = 2$). We may assume that W^j is in the second or fourth row of the table (for otherwise we could interchange a with b to enter the previous case. So $j \in \{2, 5, 6\}$. Since

$j \neq i$, this implies $j \in \{5, 6\}$. However, since W^5 is conjugate to W^2 (since they are in the same row of (4.5)), and we have $(w^2)^3 = w^6$ and $(w^5)^3 = w^2$, we see that the pair (w^2, w^6) is isomorphic to (w^2, w^5). So we may assume $j \neq 6$. $\square$

Proposition 4.2. *If $|G| = 27p$, where p is prime, and the Sylow p-subgroups of G are not normal, then* $\mathrm{Cay}(G; S)$ *has a hamiltonian cycle.*

Proof. From Lemma 4.1 (and Remark 2.9), we may assume $G = \mathbb{Z}_{13} \ltimes (\mathbb{Z}_3)^3$. For each of the generating sets listed in Lemma 4.1, we provide an explicit hamiltonian cycle in the quotient multigraph $P \setminus \mathrm{Cay}(G; S)$ that uses at least one double edge. So Lemma 2.7 applies.

To save space, we use $i_1 i_2 i_3$ to denote the vertex $P(i_1, i_2, i_3)$.

$(i, j) = (1, 0)$ $a = w,\ a^{-1} = w^{12},\ b = (1, 0, 0),$ and $b^{-1} = (-1, 0, 0)$
Double edge: $222 \to 022$ with a^{-1} and b:

$$\begin{aligned}
&000 \xrightarrow{b^{-1}} 200 \xrightarrow{a} 020 \xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{b^{-1}} 120 \xrightarrow{a} 012\\
&\xrightarrow{a} 221 \xrightarrow{a} 102 \xrightarrow{b} 202 \xrightarrow{a} 210 \xrightarrow{a} 021 \xrightarrow{a} 112 \xrightarrow{a} 201\\
&\xrightarrow{b^{-1}} 101 \xrightarrow{a^{-1}} 211 \xrightarrow{a^{-1}} 212 \xrightarrow{a^{-1}} 222 \xrightarrow{b} 022 \xrightarrow{b} 122 \xrightarrow{a^{-1}} 121\\
&\xrightarrow{a^{-1}} 111 \xrightarrow{b^{-1}} 011 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000.
\end{aligned} \tag{4.6}$$

$(i, j) = (2, 0)$ $a = w^2,\ a^{-1} = w^{11},\ b = (1, 0, 0),$ and $b^{-1} = (-1, 0, 0)$
Double edge: $020 \to 220$ with a and b^{-1}:

$$\begin{aligned}
&000 \xrightarrow{b^{-1}} 200 \xrightarrow{a} 002 \xrightarrow{a} 022 \xrightarrow{a} 212 \xrightarrow{b^{-1}} 112 \xrightarrow{a^{-1}} 210\\
&\xrightarrow{a^{-1}} 122 \xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 110 \xrightarrow{b^{-1}} 010 \xrightarrow{a^{-1}} 201 \xrightarrow{b^{-1}} 101 \xrightarrow{a} 012\\
&\xrightarrow{a} 102 \xrightarrow{a} 020 \xrightarrow{b^{-1}} 220 \xrightarrow{a} 222 \xrightarrow{a} 211 \xrightarrow{a} 120 \xrightarrow{a} 221\\
&\xrightarrow{b} 021 \xrightarrow{a^{-1}} 202 \xrightarrow{a^{-1}} 121 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000.
\end{aligned} \tag{4.7}$$

$(i, j) = (1, 2)$ $a = w,\ a^{-1} = w^{12},\ b = w^2(1, 0, 0),$ and $b^{-1} = w^{11}(-1, -1, 1)$
Double edge: $220 \to 022$ with a and b:

$$\begin{aligned}
&000 \xrightarrow{b^{-1}} 221 \xrightarrow{a^{-1}} 012 \xrightarrow{a^{-1}} 120 \xrightarrow{b^{-1}} 102 \xrightarrow{b^{-1}} 200 \xrightarrow{a} 020\\
&\xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{b} 022 \xrightarrow{a} 222 \xrightarrow{b} 011 \xrightarrow{a} 111 \xrightarrow{a} 121\\
&\xrightarrow{a} 122 \xrightarrow{a} 202 \xrightarrow{a} 210 \xrightarrow{a} 021 \xrightarrow{a} 112 \xrightarrow{b^{-1}} 101 \xrightarrow{a^{-1}} 211\\
&\xrightarrow{a^{-1}} 212 \xrightarrow{b} 201 \xrightarrow{b} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000.
\end{aligned} \tag{4.8}$$

$(i,j) = (1,3)$ $a = w$, $a^{-1} = w^{12}$, $b = w^3(1,0,0)$, and $b^{-1} = w^{10}(0,1,-1)$
Double edge: $200 \rightarrow 020$ with a and b:

$$\begin{aligned} &000 \xrightarrow{b^{-1}} 012 \xrightarrow{a^{-1}} 120 \xrightarrow{b^{-1}} 221 \xrightarrow{a} 102 \xrightarrow{a} 200 \xrightarrow{b} 020 \\ &\xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{a} 022 \xrightarrow{a} 222 \xrightarrow{a} 212 \xrightarrow{a} 211 \xrightarrow{a} 101 \\ &\xrightarrow{b^{-1}} 201 \xrightarrow{a^{-1}} 112 \xrightarrow{a^{-1}} 021 \xrightarrow{a^{-1}} 210 \xrightarrow{a^{-1}} 202 \xrightarrow{a^{-1}} 122 \xrightarrow{b} 121 \\ &\xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000. \end{aligned} \tag{4.9}$$

$(i,j) = (1,5)$ $a = w$, $a^{-1} = w^{12}$, $b = w^5(1,0,0)$, and $b^{-1} = w^{8}(1,0,1)$
Double edge: $220 \rightarrow 022$ with a and b^{-1}:

$$\begin{aligned} &000 \xrightarrow{b^{-1}} 101 \xrightarrow{a} 120 \xrightarrow{a} 012 \xrightarrow{a} 221 \xrightarrow{b^{-1}} 010 \xrightarrow{a} 001 \\ &\xrightarrow{a} 110 \xrightarrow{a} 011 \xrightarrow{a} 111 \xrightarrow{b} 121 \xrightarrow{a} 122 \xrightarrow{b^{-1}} 102 \xrightarrow{a} 200 \\ &\xrightarrow{a} 020 \xrightarrow{a} 002 \xrightarrow{a} 220 \xrightarrow{b^{-1}} 022 \xrightarrow{a} 222 \xrightarrow{a} 212 \xrightarrow{a} 211 \\ &\xrightarrow{b} 202 \xrightarrow{a} 210 \xrightarrow{a} 021 \xrightarrow{a} 112 \xrightarrow{a} 201 \xrightarrow{a} 100 \xrightarrow{b^{-1}} 000. \end{aligned} \tag{4.10}$$

$(i,j) = (1,6)$ $a = w$, $a^{-1} = w^{12}$, $b = w^6(1,0,0)$, and $b^{-1} = w^{7}(-1,1,1)$
Double edge: $021 \rightarrow 210$ with a^{-1} and b:

$$\begin{aligned} &000 \xrightarrow{b^{-1}} 211 \xrightarrow{b^{-1}} 201 \xrightarrow{a^{-1}} 112 \xrightarrow{a^{-1}} 021 \xrightarrow{b} 210 \xrightarrow{b} 101 \\ &\xrightarrow{b} 120 \xrightarrow{a} 012 \xrightarrow{a} 221 \xrightarrow{a} 102 \xrightarrow{a} 200 \xrightarrow{a} 020 \xrightarrow{a} 002 \\ &\xrightarrow{a} 220 \xrightarrow{a} 022 \xrightarrow{a} 222 \xrightarrow{a} 212 \xrightarrow{b} 202 \xrightarrow{a^{-1}} 122 \xrightarrow{a^{-1}} 121 \\ &\xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000. \end{aligned} \tag{4.11}$$

$(i,j) = (2,5)$ $a = w^2$, $a^{-1} = w^{11}$, $b = w^5(1,0,0)$, and $b^{-1} = w^{8}(1,0,1)$
Double edge: $112 \rightarrow 210$ with a^{-1} and b:

$$\begin{aligned} &000 \xrightarrow{b^{-1}} 101 \xrightarrow{a} 012 \xrightarrow{b} 102 \xrightarrow{a} 020 \xrightarrow{a} 220 \xrightarrow{a} 222 \\ &\xrightarrow{b} 112 \xrightarrow{b} 210 \xrightarrow{a^{-1}} 122 \xrightarrow{a^{-1}} 111 \xrightarrow{a^{-1}} 110 \xrightarrow{a^{-1}} 010 \xrightarrow{a^{-1}} 201 \\ &\xrightarrow{a^{-1}} 021 \xrightarrow{a^{-1}} 202 \xrightarrow{b^{-1}} 211 \xrightarrow{a} 120 \xrightarrow{a} 221 \xrightarrow{a} 200 \xrightarrow{a} 002 \\ &\xrightarrow{a} 022 \xrightarrow{a} 212 \xrightarrow{b^{-1}} 121 \xrightarrow{a^{-1}} 011 \xrightarrow{a^{-1}} 001 \xrightarrow{a^{-1}} 100 \xrightarrow{b^{-1}} 000. \end{aligned} \tag{4.12}$$

□

Acknowledgments

This work was partially supported by research grants from the Natural Sciences and Engineering Research Council of Canada.

References

[1] S. J. Curran, D. W. Morris, and J. Morris, "Hamiltonian cycles in Cayley graphs of order $16p$," Preprint.
[2] E. Ghaderpour and D. W. Morris, "Cayley graphs of order $30p$ are hamiltonian," Preprint.
[3] K. Kutnar, D. Marušič, J. Morris, D. W. Morris, and P. Šparl, "Hamiltonian cycles in Cayley graphs whose order has few prime factors," *Ars Mathematica Contemporanea*. In press.
[4] D. Gorenstein, *Finite Groups*, Chelsea, New York, NY, USA, 1980.
[5] K. Keating and D. Witte, "Hamilton cycles in Cayley graphs with cyclic commutator subgroup," *Annals of Discrete Mathematics*, vol. 27, pp. 89–102, 1985.
[6] D. W. Morris, "2-generated Cayley digraphs on nilpotent groups have hamiltonian paths," Preprint.
[7] D. Witte, "Cayley digraphs of prime-power order are hamiltonian," *Journal of Combinatorial Theory, Series B*, vol. 40, no. 1, pp. 107–112, 1986.
[8] T. W. Judson, *Abstract Algebra*, Virginia Commonwealth University, Richmond, Va, USA, 2009.