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Research Article

On Some Combinatorial Structures Constructed from the Groups $L(3,5)$, $U(5,2)$, and $S(6,2)$

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We describe a construction of primitive 2-designs and strongly regular graphs from the simple groups $L(3,5)$, $U(5,2)$, and $S(6,2)$. The designs and the graphs are constructed by defining incidence structures on conjugacy classes of maximal subgroups of $L(3,5)$, $U(5,2)$, and $S(6,2)$. In addition, from the group $S(6,2)$, we construct 2-designs with parameters $(28,4,4)$ and $(28,4,1)$ having the full automorphism group isomorphic to $U(3,3):Z_2$.

1. Introduction

Questions about combinatorial structures related to finite groups arose naturaly in studying of the groups. Studies on the interplay between finite groups and combinatorial structures have provided many useful and interesting results. From a geometric point of view, the most interesting designs are generally those admitting large automorphism groups. Famous Witt designs constructed from Mathieu groups have been discovered in 1930's (see [1, 2]), and for some further construction of combinatorial structures from finite groups, we refere the reader to [3–5]. A construction of 1-designs and regular graphs from primitive groups is described in [6] and corrected in [7]. The construction employed in this paper is described in [8], as a generalization of the one from [6, 7].

An incidence structure is an ordered triple $\mathfrak{D}=(\mathcal{P},\mathcal{B},\mathcal{I})$ where $\mathcal{P}$ and $\mathcal{B}$ are nonempty disjoint sets and $\mathcal{I}\subseteq\mathcal{P}\times\mathcal{B}$. The elements of the set $\mathcal{P}$ are called points, the elements of the set $\mathcal{B}$ are called blocks, and $\mathcal{I}$ is called an incidence relation. If $|\mathcal{P}|=|\mathcal{B}|$, then the incidence structure is called symmetric. The incidence matrix of an incidence structure is a $b\times v$ matrix $[m_{ij}]$ where b and v are the number of blocks and points, respectively, such that $m_{ij}=1$ if the point P_j and block x_i are incident, and $m_{ij}=0$ otherwise. An isomorphism from one incidence structure to another is a bijective mapping of points to points and blocks to blocks which preserves incidence. An isomorphism from an incidence structure $\mathfrak{D}$ onto itself is called an automorphism of $\mathfrak{D}$. The set of all automorphisms forms a group called the full automorphism group of $\mathfrak{D}$ and is denoted by $\mathrm{Aut}(\mathfrak{D})$.

A $t-(v,k,\lambda)$ design is a finite incidence structure $(\mathcal{P},\mathcal{B},\mathcal{I})$ satisfying the following requirements:

(1) $|\mathcal{P}|=v$,

(2) every element of $\mathcal{B}$ is incident with exactly k elements of $\mathcal{P}$,

(3) every t elements of $\mathcal{P}$ are incident with exactly λ elements of $\mathcal{B}$.

A $2-(v,k,\lambda)$ design is called a block design. A $2-(v,k,\lambda)$ design is called quasisymmetric if the number of points in the intersection of any two blocks takes only two values. If $|\mathcal{P}|=|\mathcal{B}|$, then the design is called symmetric. A symmetric $2-(v,k,1)$ design is called a projective plane.

An incidence structure has repeated blocks if there are two blocks incident with exactly the same points. An incidence structure that has no repeated blocks is called simple. All designs described in this paper are simple.

Let $\mathcal{G}=(\mathcal{V},\mathcal{E},\mathcal{I})$ be a finite incidence structure. $\mathcal{G}$ is a graph if each element of $\mathcal{E}$ is incident with exactly two elements of $\mathcal{V}$. The elements of $\mathcal{V}$ are called vertices, and the elements of $\mathcal{E}$ are called edges. Two vertices u and v are called adjacent or neighbors if they are incident with the same edge. The number of neighbors of a vertex v is called the degree of v. If all the vertices of the graph $\mathcal{G}$ have the same degree k, then $\mathcal{G}$ is called k-regular. Define a square $\{0,1\}$-matrix $A=(a_{uv})$ labelled with the vertices of $\mathcal{G}$ in such a way that $a_{uv}=1$ if and only if the vertices u and v are adjacent. The matrix A is called the adjacency matrix of the graph $\mathcal{G}$.

A graph $\mathcal{G}$ is called a strongly regular graph with parameters (n,k,λ,μ) and denoted by $\mathrm{SRG}(n,k,\lambda,\mu)$ if $\mathcal{G}$ is k-regular graph with n vertices and if any two adjacent vertices have λ common neighbors and any two nonadjacent vertices have μ common neighbors.

Let x and y $(x<y)$ be the two cardinalities of block intersections in a quasi-symmetric design $\mathfrak{D}$. The block graph of $\mathfrak{D}$ has as vertices the blocks of $\mathfrak{D}$ and two vertices are adjacent if and only if they intersect in y points. The block graph of a quasi-symmetric $2-(v,k,\lambda)$ design is strongly regular. In a $2-(v,k,1)$ design which is not a projective plane, two blocks intersect in 0 or 1 points; therefore, the block graph of this design is strongly regular (see [9]).

Let $\mathfrak{D}$ be a symmetric (v,k,λ) design which possesses a symmetric incidence matrix M with 1 everywhere on the diagonal. Then, the matrix $M-I$ is an adjacency matrix of a strongly regular graph $\mathcal{G}$ with parameters $(v,k-1,\lambda-2,\lambda)$ (see [9]) and $\mathrm{Aut}(\mathcal{G})\leq\mathrm{Aut}(\mathfrak{D})$.

Let G be a simple group and H be a maximal subgroup of G. The conjugacy class of H is denoted by $\mathrm{ccl}_G(H)$. Obviously, $N_G(H)=H$, so $|\mathrm{ccl}_G(H)|=[G:H]$. Denote the elements of the conjugacy class $\mathrm{ccl}_G(H)$ by $H^{g_1},H^{g_2},\ldots,H^{g_j}$, $j=[G:H]$.

In this paper, we consider block designs constructed from the linear group $L(3,5)$, strongly regular graphs constructed from the unitary group $U(5,2)$, and block designs and strongly regular graphs constructed from the symplectic group $S(6,2)$.

$L(3,5)$ is the simple group of order 372000, and it has five distinct classes of maximal subgroups: $H_1\cong H_2\cong(Z_5\times Z_5):GL(2,5)$, $H_3\cong S_5$, $H_4\cong(Z_4\times Z_4):S_3$, and $H_5\cong F_{93}$. We define incidence structures on the elements of conjugacy classes of the maximal subgroups of $L(3,5)$; that is, points and blocks are labelled by elements of conjugacy classes of the maximal subgroups of $L(3,5)$.

$U(5,2)$ is the simple group of order 13685760, and it has six distinct classes of maximal subgroups: $K_1\cong(E_{64}:Z_2)\cdot(E_9:Z_3)\cdot SL(2,3)$, $K_2\cong Z_3\times U(4,2)$, $K_3\cong(E_{16}:E_{16}):(Z_3\times A_5)$,

$K_4 \cong E_{81} : S_5$, $K_5 \cong (S_3 \times (E_9 : Z_3)) : SL(2,3)$, and $K_6 \cong L(2,11)$. We define strongly regular graphs whose vertices are labelled by elements of conjugacy classes of the maximal subgroups of $U(5,2)$.

$S(6,2)$ is the simple group of order 1451520, and it has eight distinct classes of maximal subgroups: $M_1 \cong U(4,2) : Z_2$, $M_2 \cong S_8$, $M_3 \cong E_{32} : S_6$, $M_4 \cong U(3,3) : Z_2$, $M_5 \cong E_{64} : L(3,2)$, $M_6 \cong ((E_{16} : Z_2) \times E_4) : (S_4 \times S_4)$, $M_7 \cong S_3 \times S_6$, and $M_8 \cong L(2,8) : Z_3$. We define incidence structures on the elements of conjugacy classes of the maximal subgroups of $S(6,2)$ and strongly regular graphs whose vertices are labelled by elements of conjugacy classes of the maximal subgroups of $S(6,2)$.

Generators of groups $L(3,5)$, $U(5,2)$, and $S(6,2)$ and their maximal subgroups are available on the Internet: http://brauer.maths.qmul.ac.uk/Atlas/.

In this paper, we describe a construction of primitive block designs with parameters $(28,12,11)$, $(28,4,5)$, $(28,10,40)$, $(31,6,1)$, $(31,6,100)$, $(31,10,300)$, $(31,15,700)$, $(31,3,25)$, $(31,12,550)$, $(31,15,875)$, $(36,16,12)$, $(36,8,6)$, $(36,12,33)$, $(36,6,8)$, and $(63,31,15)$ and strongly regular graphs with parameters $(63,30,13,15)$, $(120,56,28,24)$, $(135,64,28,24)$, $(165,36,3,9)$, $(176,40,12,8)$, $(297,40,7,5)$, and $(1408,567,246,216)$. The designs and the graphs are constructed by defining incidence structures on conjugacy classes of maximal subgroups of the simple groups $L(3,5)$, $U(5,2)$, and $S(6,2)$. In addition, from the group $S(6,2)$, we construct 2-designs with parameters $(28,4,4)$ and $(28,4,1)$ having the full automorphism group isomorphic to $U(3,3) : Z_2$.

The graphs described in this paper have been previously known, since they can be constructed as rank 3 graphs. For more details on rank 3 graphs, we refer the reader to [9]. Some of the constructed block designs on 31 points have large number of blocks, and we did not find a record that they have been previously studied, although these designs can be constructed from PG(2,5). The constructed designs with parameters $(28,12,11)$, $(28,4,5)$, $(36,16,12)$, $(36,8,6)$, and $(63,31,15)$ are isomorphic to the already known designs. However, we did not find an evidence that the constructed designs with parameters $(28,10,40)$, $(36,12,33)$, and $(36,6,8)$ are isomorphic to the already known objects.

The paper is organized as follows: in Section 2, we describe the method of construction of primitive designs and graphs used in this paper, Section 3 describes block designs on 31 points constructed from the group $L(3,5)$, Section 4 gives strongly regular graphs constructed from the group $U(5,2)$, and Section 5 describes the group $S(6,2)$ as block designs and strongly regular graphs. At the end of the paper, we give a list of the constructed designs and strongly regular graphs and their full automorphism groups.

For basic definitions and group theoretical notation, we refer the reader to [10, 11].

2. The Construction

The following construction of symmetric 1-designs and regular graphs is presented in [6, 12].

Theorem 2.1. *Let G be a finite primitive permutation group acting on the set Ω of size n. Let $\alpha \in \Omega$, and let $\Delta \neq \{\alpha\}$ be an orbit of the stabilizer G_α of α. If $\mathcal{B} = \{\Delta g : g \in G\}$ and, given $\delta \in \Delta$, $\mathcal{E} = \{\{\alpha,\delta\}g : g \in G\}$, then $\mathfrak{D} = (\Omega,\mathcal{B})$ forms a symmetric $1-(n,|\Delta|,|\Delta|)$ design. Further, if Δ is a self-paired orbit of G_α, then $\Gamma(\Omega,\mathcal{E})$ is a regular connected graph of valency $|\Delta|$, $\mathfrak{D}$ is self-dual, and G acts as an automorphism group on each of these structures, primitive on vertices of the graph and on points and blocks of the design.*

In [8], we introduced a generalization of the above construction. This generalization, presented below in Theorem 2.2, allows us to construct 1-designs that are not necessarily symmetric and stabilizers of a point and a block are not necessarily conjugate.

Theorem 2.2. *Let G be a finite permutation group acting primitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$, $\delta \in \Omega_2$, and let $\Delta_2 = \delta G_\alpha$ be the G_α-orbit of $\delta \in \Omega_2$ and $\Delta_1 = \alpha G_\delta$ be the G_δ-orbit of $\alpha \in \Omega_1$. If $\Delta_2 \neq \Omega_2$ and*

$$\mathcal{B} = \left\{\Delta_2 g : g \in G\right\}, \tag{2.1}$$

then $\mathfrak{D}(G, \alpha, \delta) = (\Omega_2, \mathcal{B})$ is a $1-(n, |\Delta_2|, |\Delta_1|)$ design with m blocks and G acts as an automorphism group, primitive on points and blocks of the design.

The construction of a design described in Theorem 2.2 can be interpreted in the following way:

(i) the point set is $\Omega_2 = \delta G$,

(ii) the block set is $\Omega_1 = \alpha G$,

(iii) the block $\alpha g'$ is incident with the set of points $\{\delta g : g \in G_\alpha g'\}$.

Let a point $\delta g \in \Omega_2$ be incident with a block $\alpha g' \in \Omega_1$. Then, $g \in G_\alpha g'$; hence, there exists $\overline{g} \in G_\alpha$ such that $g = \overline{g} g'$. Therefore,

$$\begin{aligned} G_{\alpha g'} \cap G_{\delta g} &= G_{\alpha g'} \cap G_{\delta \overline{g} g'} = G_\alpha^{g'} \cap G_{\delta \overline{g}}^{g'} = \left(G_\alpha \cap G_{\delta \overline{g}}\right)^{g'} = \left(G_\alpha \cap G_\delta^{\overline{g}}\right)^{g'} \\ &= \left(G_\alpha^{\overline{g}^{-1}} \cap G_\delta\right)^{\overline{g} g'} = \left(G_\alpha \cap G_\delta\right)^{\overline{g} g'} = \left(G_\alpha \cap G_\delta\right)^{g}. \end{aligned} \tag{2.2}$$

If a point $\delta g \in \Omega_2$ is incident with the block $\alpha \in \Omega_1$, then $G_\alpha \cap G_{\delta g} = (G_\alpha \cap G_\delta)^g$. If the set $\{G_\alpha \cap G_{\delta g} \mid g \in G\}$ contains $\mathrm{Orb}(G_\alpha, \Omega_2)$ G_α-conjugacy classes, where $\mathrm{Orb}(G_\alpha, \Omega_2)$ is the number of G_α-orbits on Ω_2, then each conjugacy class corresponds to one G_α-orbit and the incidence relation in the design $\mathfrak{D}(G, \alpha, \delta)$ can be defined as follows:

(i) the block $\alpha g'$ is incident with the point δg if and only if $G_{\alpha g'} \cap G_{\delta g}$ is conjugate to $G_\alpha \cap G_\delta$.

Similarly, if the set $\{G_\alpha \cap G_{\delta g} \mid g \in G\}$ contains $\mathrm{Orb}(G_\alpha, \Omega_2)$ isomorphism classes, then the incidence in the design $\mathfrak{D}(G, \alpha, \delta)$ can be defined as follows:

(ii) the block $\alpha g'$ is incident with the point δg if and only if $G_{\alpha g'} \cap G_{\delta g} \cong G_\alpha \cap G_\delta$.

In the construction of the design $\mathfrak{D}(G, \alpha, \delta)$ described in Theorem 2.2, instead of taking a single G_α-orbit, we can take Δ_2 to be any union of G_α-orbits.

Corollary 2.3. *Let G be a finite permutation group acting primitively on the sets Ω_1 and Ω_2 of size m and n, respectively. Let $\alpha \in \Omega_1$ and $\Delta_2 = \bigcup_{i=1}^{s} \delta_i G_\alpha$, where $\delta_1, \ldots, \delta_s \in \Omega_2$ are representatives of distinct G_α-orbits. If $\Delta_2 \neq \Omega_2$ and*

$$\mathcal{B} = \left\{\Delta_2 g : g \in G\right\}, \tag{2.3}$$

then $\mathfrak{D}(G, \alpha, \delta_1, \ldots, \delta_s) = (\Omega_2, \mathcal{B})$ is a $1-(n, |\Delta_2|, \sum_{i=1}^{s} |\alpha G_{\delta_i}|)$ design with m blocks and G acts as an automorphism group, primitive on points and blocks of the design.

Proof. Clearly, the number of points is $v = n$, since the point set is $\mathcal{P} = \Omega_2$. Further, each element of $\mathcal{B}$ consists of $k = |\Delta_2|$ elements of Ω_2.

The set Δ_2 is a union of G_α-orbits, so $G_\alpha \subseteq G_{\Delta_2}$, where G_{Δ_2} is the setwise stabilizer of Δ_2. Since G is primitive on Ω_1, G_α is a maximal subgroup of G, and therefore $G_{\Delta_2} = G_\alpha$. The number of blocks is

$$b = |\Delta_2 G| = \frac{|G|}{|G_{\Delta_2}|} = \frac{|G|}{|G_\alpha|} = |\Omega_1| = m. \tag{2.4}$$

Since G acts transitively on Ω_1 and Ω_2, the constructed structure is a 1-design, hence $bk = vr$, where each point is incident with r blocks. Therefore,

$$|\Omega_1||\Delta_2| = |\Omega_2|r, \tag{2.5}$$

and consequently

$$\frac{|G|}{|G_\alpha|}\sum_{i=1}^{s}\frac{|G_\alpha|}{\left|(G_\alpha)_{\delta_i}\right|} = \frac{|G|}{|G_{\delta_1}|}r. \tag{2.6}$$

It follows that

$$r = \sum_{i=1}^{s}\frac{|G_{\delta_1}|}{\left|(G_\alpha)_{\delta_i}\right|} = \sum_{i=1}^{s}\frac{|G_{\delta_i}|}{\left|(G_{\delta_i})_\alpha\right|} = \sum_{i=1}^{s}|\alpha G_{\delta_i}|. \tag{2.7}$$

□

Remark 2.4. In the construction of graphs described in Theorem 2.1, we can define the set of edges $\mathcal{E}$ as a union $\bigcup_{i=1}^{s}\{\{\alpha,\delta_i\}g : g \in G\}$.

The construction described in Corollary 2.3 gives us all designs that admit primitive action of the group G on points and blocks.

Corollary 2.5. *If the group G acts primitively on the points and the blocks of a 1-design $\mathfrak{D}$, then $\mathfrak{D}$ can be obtained as described in Corollary 2.3, that is, such that Δ_2 is a union of G_α-orbits.*

Proof. Let α be any block of the design $\mathfrak{D}$. G acts transitively on the block set $\mathcal{B}$ of the design $\mathfrak{D}$, hence $\mathcal{B} = \alpha G$. Since G acts primitively on $\mathcal{B}$, the stabilizer G_α is a maximal subgroup of G. G_α fixes α, so α is a union of G_α-orbits. □

We can obtain a 1-design by defining the incidence in such a way that the point δg is incident with the block $\alpha g'$ if and only if

$$G_{\alpha g'} \cap G_{\delta g} \cong G_i, \quad i = 1, \ldots, k, \tag{2.8}$$

where $\{G_1, \ldots, G_k\} \subset \{G_{\alpha x} \cap G_{\delta y} \mid x, y \in G\}$.

Let G be a simple group and H_1 and H_2 be maximal subgroups of G. The stabilizer of any element $H_i^x \in \mathrm{ccl}_G(H_i)$, $i = 1, 2$, is the maximal subgroup H_i^x, hence G acts primitively on the class $\mathrm{ccl}_G(H_i)$, $i = 1, 2$, by conjugation and

$$\begin{aligned} |\mathrm{ccl}_G(H_1)| &= [G : H_1] = m, \\ |\mathrm{ccl}_G(H_2)| &= [G : H_2] = n. \end{aligned} \tag{2.9}$$

Corollary 2.3 allows us to define a 1-design on conjugacy classes of the maximal subgroups H_1 and H_2 of a simple group G. Let us denote the elements of $\mathrm{ccl}_G(H_1)$ by $H_1^{g_1}$, $H_1^{g_2}, \ldots, H_1^{g_m}$ and the elements of $\mathrm{ccl}_G(H_2)$ by $H_2^{h_1}, H_2^{h_2}, \ldots, H_2^{h_n}$.

We can construct a 1-design such that

(i) the point set of the design is $\mathrm{ccl}_G(H_2)$,

(ii) the block set is $\mathrm{ccl}_G(H_1)$,

(iii) the block $H_1^{g_i}$ is incident with the point $H_2^{h_j}$ if and only if $H_2^{h_j} \cap H_1^{g_i} \cong G_i$, $i = 1, \ldots, k$, where $\{G_1, \ldots, G_k\} \subset \{H_2^x \cap H_1^y \mid x, y \in G\}$.

Let us denote a 1-design constructed in this way by $\mathfrak{D}(G, H_2, H_1; G_1, \ldots, G_k)$.

Similarly, from the conjugacy class of a maximal subgroup H of a simple group G, one can construct regular graph in the following way:

(i) the vertex set of the graph is $\mathrm{ccl}_G(H)$,

(ii) the vertex H^{g_i} is adjacent to the vertex H^{g_j} if and only if $H^{g_i} \cap H^{g_j} \cong G_i$, $i = 1, \ldots, k$, where $\{G_1, \ldots, G_k\} \subset \{H^x \cap H^y \mid x, y \in G\}$.

We denote a regular graph constructed in this way by $\mathcal{G}(G, H; G_1, \ldots, G_k)$.

Remark 2.6. Let ϕ be an automorphism of a finite group G. Then, the design $\mathfrak{D}(G, H_2, H_1; G_1, \ldots, G_k)$ is isomorphic to $\mathfrak{D}(G, (H_2)\phi, (H_1)\phi; G_1, \ldots, G_k)$, and the graph $\mathcal{G}(G, H; G_1, \ldots, G_k)$ is isomorphic to $\mathcal{G}(G, (H)\phi; G_1, \ldots, G_k)$.

3. Block Designs on 31 Points Constructed from the Group $L(3, 5)$

Let G be a group isomorphic to the linear group $L(3, 5)$.

Using GAP (see [13]), one can check that $H_2^x \cap H_1^y \cong (Z_2 \cdot A_5) : Z_4$ or $((Z_5 \times Z_5) \cdot Z_5) : (Z_4 \times Z_4)$ for all $x, y \in G$. Further, for every H_2^x,

$$\left| \left\{ H_1^y \mid y \in G,\ \ H_1^y \cap H_2^x \cong ((Z_5 \times Z_5) \cdot Z_5) : (Z_4 \times Z_4) \right\} \right| = 6. \tag{3.1}$$

Let us define sets $S_i^{(1)} = \{H_2^{g_j} \in \mathrm{ccl}_G(H_2) \mid H_1^{h_i} \cap H_2^{g_j} \cong ((Z_5 \times Z_5) \cdot Z_5) : (Z_4 \times Z_4)\}$, $1 \le i \le 31$. For every $1 \le i, k \le 31$, $i \ne k$, the set $S_i^{(1)} \cap S_k^{(1)}$ has exactly one element. That proves that the incidence structure $\mathfrak{D}_1 = \mathfrak{D}(L(3, 5), H_1, H_2; ((Z_5 \times Z_5) \cdot Z_5) : (Z_4 \times Z_4))$ is the unique symmetric $2 - (31, 6, 1)$ design (see [14]), that is, the projective plane $\mathrm{PG}(2, 5)$. The full automorphism group of the design $\mathfrak{D}_1$ has 372000 elements, and it is isomorphic to the group $L(3, 5)$.

Let $H_3 \cong S_5$ be a maximal subgroup in G. The conjugacy class $\mathrm{ccl}_G(H_3)$ has 3100 elements. One can verify using GAP that $H_1^x \cap H_3^y$ is isomorphic to D_8, D_{12}, or $Z_5 : Z_4$, for all $x, y \in G$. Using GAP (see [13]), we obtain the following:

(1) $|\{H_1^{h_i} \in \mathrm{ccl}_G(H_1) \mid H_1^{h_i} \cap H_3^{g_j} \cong Z_5 : Z_4\}| = 6$, for all $H_3^{g_j} \in \mathrm{ccl}_G(H_3)$, $1 \leq j \leq 3100$,

(2) $|S_i^{(2)} \cap S_k^{(2)}| = 100$, for all $i, k \in \{1, 2, \ldots, 31\}$, $i \neq j$, $S_i^{(2)} = \{H_3^{g_j}\mathrm{ccl}_G(H_3) \mid H_1^{h_i} \cap H_3^{g_j} \cong Z_5 : Z_4\}$,

(3) $|\{H_1^{h_i} \in \mathrm{ccl}_G(H_1) \mid H_1^{h_i} \cap H_3^{g_j} \cong D_{12}\}| = 10$, for all $H_3^{g_j} \in \mathrm{ccl}_G(H_3)$, $1 \leq j \leq 3100$,

(4) $|S_i^{(3)} \cap S_k^{(3)}| = 300$, for all $i, k \in \{1, 2, \ldots, 31\}$, $i \neq j$, $S_i^{(3)} = \{H_3^{g_j} \in \mathrm{ccl}_G(H_3) \mid H_1^{h_i} \cap H_3^{g_j} \cong D_{12}\}$,

(5) $|\{H_1^{h_i} \in \mathrm{ccl}_G(H_1) \mid H_1^{h_i} \cap H_3^{g_j} \cong D_8\}| = 15$, for all $H_3^{g_j} \in \mathrm{ccl}_G(H_3)$, $1 \leq j \leq 3100$,

(6) $|S_i^{(4)} \cap S_k^{(4)}| = 700$, for all $i, k \in \{1, 2, \ldots, 31\}$, $i \neq j$, $S_i^{(4)} = \{H_3^{g_j} \in \mathrm{ccl}_G(H_3) \mid H_1^{h_i} \cap H_3^{g_j} \cong D_8\}$.

It follows from (1) and (2) that the structure $\mathfrak{D}_2 = \mathfrak{D}(L(3,5), H_1, H_3; Z_5 : Z_4)$ is a block design $2-(31, 6, 100)$. The blocks of $\mathfrak{D}_2$ are conics in $\mathrm{PG}(2,5)$.

(3) and (4) imply that the structure $\mathfrak{D}_3 = \mathfrak{D}(L(3,5), H_1, H_3; D_{12})$ is a block design $2-(31, 10, 300)$. The blocks of $\mathfrak{D}_3$ are interior points of conics in $\mathrm{PG}(2,5)$.

Further, (5) and (6) show that the structure $\mathfrak{D}_4 = \mathfrak{D}(L(3,5), H_1, H_3; D_8)$ is a block design $2-(31, 15, 700)$. The blocks of $\mathfrak{D}_3$ are exterior points of conics in $\mathrm{PG}(2,5)$.

The full automorphism group of designs $\mathfrak{D}_2$, $\mathfrak{D}_3$, and $\mathfrak{D}_4$ has 372000 elements and is isomorphic to the group $L(3,5)$.

Let $H_4 \cong (Z_4 \times Z_4) : S_3$ be a maximal subgroup in G. The conjugacy class $\mathrm{ccl}_G(H_4)$ has 3875 elements. Using GAP (see [13]), one can check that $H_1^x \cap H_4^y$ is isomorphic to S_3, $Z_2 \times Z_4$, or $(Z_8 : Z_2) : Z_2$, for all $x, y \in G$. In the similar way as above, we obtain the following results:

(i) $\mathfrak{D}_5 = \mathfrak{D}(L(3,5), H_1, H_4; (Z_8 : Z_2) : Z_2))$ is a $2-(31, 3, 25)$ design. $\mathfrak{D}_5$ is isomorphic to the design constructed from the symmetric $2-(31, 6, 1)$ design $\mathfrak{D}_1$ in such a way that the blocks are all triples of noncollinear points.

(ii) The incidence structure $\mathfrak{D}_6 = \mathfrak{D}(L(3,5), H_1, H_4; Z_2 \times Z_4)$ is a block design $2-(31, 12, 550)$. The blocks of $\mathfrak{D}_6$ are unions of sides of triangles in $\mathrm{PG}(2,5)$.

(iii) $\mathfrak{D}_7 = \mathfrak{D}(L(3,5), H_1, H_4; S_3)$ is a block design $2-(31, 15, 875)$. The blocks of $\mathfrak{D}_7$ are unions of sides of triangles in $\mathrm{PG}(2,5)$, including the corners.

The full automorphism group of $\mathfrak{D}_5$, $\mathfrak{D}_6$, and $\mathfrak{D}_7$ is isomorphic to $L(3,5)$.

Six designs isomorphic to $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$, $\mathfrak{D}_5$, $\mathfrak{D}_6$, and $\mathfrak{D}_7$ can be obtain in the same way as described in this paper using the maximal subgroup H_2 instead of H_1. This is a consequence of the Remark 2.6 and the fact that there exists an automorphism of $L(3,5)$ which fixes $\mathrm{ccl}_G(H_3)$, $\mathrm{ccl}_G(H_4)$ and $\mathrm{ccl}_G(H_5)$, setwise and acts as a transposition which maps $\mathrm{ccl}_G(H_1)$ onto $\mathrm{ccl}_G(H_2)$.

We thank an anonymous referee of an earlier version of this paper for suggesting the construction of the designs $\mathfrak{D}_2$, $\mathfrak{D}_3$, $\mathfrak{D}_4$, $\mathfrak{D}_6$, and $\mathfrak{D}_7$ from triangles and conics in $\mathrm{PG}(2,5)$.

4. Strongly Regular Graphs Constructed from the Group $U(5,2)$

In this section, we describe structures constructed from a simple group isomorphic to the unitary group $U(5,2)$.

The intersection of two different elements $K_i^x, K_i^y \in \mathrm{ccl}_G(K_i)$, $i = 1,2,3,4$, denoted by $G_{i,i}^k$, is isomorphic to

(i) $G_{1,1}^1 \cong ((Ex_{27}^+ : Z_2) \cdot Z_4) : Z_3$ or $G_{1,1}^2 \cong ((E_8 \times Z_4) \cdot E_8) : E_9$,

(ii) $G_{2,2}^1 \cong (E_8 : Z_{12}) : Z_6$ or $G_{2,2}^2 \cong (((Ex_{27}^+ : Z_2) \cdot Z_4) : Z_3) : Z_3$,

(iii) $G_{3,3}^1 \cong A_5 \times Z_3$ or $G_{3,3}^2 \cong Ex_{128}^- : E_9$,

(iv) $G_{4,4}^1 \cong D_6$, $G_{4,4}^2 \cong S_3 \times S_3$, $G_{4,4}^3 \cong (A_4 : Z_2) : Z_3$, $G_{4,4}^4 \cong A_5 : Z_2$, or $G_{4,4}^5 \cong (Ex_{27}^+ : Z_2) \times S_3$.

Applying the method described in Section 2, we obtain the following results:

(i) the graph $\mathcal{G}_1 = \mathcal{G}(U(5,2), K_1; G_{1,1}^1)$ is a strongly regular graph with parameters $(165, 36, 3, 9)$,

(ii) the graph $\mathcal{G}_2 = \mathcal{G}(U(5,2), K_2; G_{2,2}^2)$ is a strongly regular graph with parameters $(176, 40, 12, 8)$,

(iii) the graph $\mathcal{G}_3 = \mathcal{G}(U(5,2), K_3; G_{3,3}^2)$ is a strongly regular graph with parameters $(297, 40, 7, 5)$,

(iv) the graph $\mathcal{G}_4 = \mathcal{G}(U(5,2), K_4; G_{4,4}^2, G_{4,4}^3, G_{4,4}^4)$ is a strongly regular graph with parameters $(1408, 567, 246, 216)$.

The full automorphism groups of the graphs $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are isomorphic to the group $U(5,2) : Z_2 \cong \mathrm{Aut}(U(5,2))$. $U(5,2)$ is a rank 3 group on 165, 176, and 297 points, and $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$ are rank 3 graphs of the group $U(5,2)$.

The full automorphism group of the graph $\mathcal{G}_4$ is of order 18393661440 and is isomorphic to the group $U(6,2) : Z_2 \cong Fi_{21} : Z_2$. Although $U(5,2)$ acts as a rank 7 group on 1408 points, the graph $\mathcal{G}_4$ can be constructed as a rank 3 graph from the Fischer group Fi_{21}.

5. Structures Constructed from the Group $S(6,2)$

In this section, we consider the structures constructed from a simple group isomorphic to the symplectic group $S(6,2)$.

The intersection of two different elements $M_i^x \in \mathrm{ccl}_G(M_i)$ and $M_j^y \in \mathrm{ccl}_G(M_j)$, denoted by $G_{i,j}^k$, is isomorphic to (we introduce only the intersection of elements of conjugacy classes that give rise to a strongly regular graph or a block design)

(i) $G_{1,3}^1 \cong E_{16} : S_5$ or $G_{1,3}^2 \cong A_6 : E_4$,

(ii) $G_{1,6}^1 \cong ((E_8 \cdot Z_{12}) : Z_6) : Z_2$ or $G_{1,6}^2 \cong S_4 \times D_8$,

(iii) $G_{1,7}^1 \cong E_{27} : (D_8 \times Z_2)$ or $G_{1,7}^2 \cong S_5 : Z_2$,

(iv) $G_{2,3}^1 \cong A_6 : E_4$ or $G_{2,3}^2 \cong (S_4 \times S_4) : Z_2$,

(v) $G_{2,5}^1 \cong E_8 : L(2,7)$ or $G_{2,5}^2 \cong E_{16} : S_4$,

(vi) $G_{2,6}^1 \cong E_{16} : S_4$ or $G_{2,6}^2 \cong S_4 \times D_8$,

(vii) $G_{2,7}^1 \cong S_5 \times S_3$ or $G_{2,7}^2 \cong E_9 : (D_8 \times Z_2)$,

(viii) $G_{3,3}^1 \cong Ex_{32}^+ : (D_{12} \times Z_2)$ or $G_{3,3}^2 \cong S_6$,

(ix) $G_{4,4}^1 \cong (Ex_{27}^+ : Z_2) \cdot Z_4$ or $G_{4,4}^2 \cong (Z_4 \times Z_4) : D_{12}$,

(x) $G_{5,5}^1 \cong L(2,7)$, $G_{5,5}^2 \cong E_{16} : D_{12}$, or $G_{5,5}^3 \cong Ex_{32}^+ : (A_4 : Z_2)$.

Applying the method described in Section 2, we obtain the following results:

(i) the incidence structure $\mathfrak{D}_8 = \mathfrak{D}(S(6,2), M_1, M_3; G^1_{1,3})$ is a block design with parameters $2-(28,12,11)$,

(ii) the incidence structure $\mathfrak{D}_9 = \mathfrak{D}(S(6,2), M_1, M_6; G^1_{1,6})$ is a block design with parameters $2-(28,4,5)$,

(iii) the incidence structure $\mathfrak{D}_{10} = \mathfrak{D}(S(6,2), M_1, M_7; G^1_{1,7})$ is a block design with parameters $2-(28,10,40)$,

(iv) the incidence structure $\mathfrak{D}_{11} = \mathfrak{D}(S(6,2), M_2, M_3; G^1_{2,3})$ is a block design with parameters $2-(36,16,12)$,

(v) the incidence structure $\mathfrak{D}_{12} = \mathfrak{D}(S(6,2), M_2, M_5; G^1_{2,5})$ is a block design with parameters $2-(36,8,6)$,

(vi) the incidence structure $\mathfrak{D}_{13} = \mathfrak{D}(S(6,2), M_2, M_6; G^1_{2,6})$ is a block design with parameters $2-(36,12,33)$,

(vii) the incidence structure $\mathfrak{D}_{14} = \mathfrak{D}(S(6,2), M_2, M_7; G^1_{2,7})$ is a block design with parameters $2-(36,6,8)$,

(viii) the incidence structure $\mathfrak{D}_{15} = \mathfrak{D}(S(6,2), M_3, M_3; G^1_{3,3}, M_3)$ is a block design with parameters $2-(63,31,15)$,

(ix) the graph $\mathcal{G}_5 = \mathcal{G}(S(6,2), M_3; G^1_{3,3})$ is a strongly regular graph with parameters $(63,30,13,15)$,

(x) the graph $\mathcal{G}_6 = \mathcal{G}(S(6,2), M_4; G^1_{4,4})$ is a strongly regular graph with parameters $(120,56,28,24)$,

(xi) the graph $\mathcal{G}_7 = \mathcal{G}(S(6,2), M_5; G^1_{5,5})$ is a strongly regular graph with parameters $(135,64,28,32)$.

The full automorphism groups of designs $\mathfrak{D}_8, \mathfrak{D}_9, \mathfrak{D}_{10}, \mathfrak{D}_{11}, \mathfrak{D}_{12}, \mathfrak{D}_{13}$, and $\mathfrak{D}_{14}$ and the full automorphism group of the graph $\mathcal{G}_5$ are isomorphic to $S(6,2)$.

Graphs $\mathcal{G}_6$ and $\mathcal{G}_7$ have $O_8^+(2) : Z_2 \cong \mathrm{Aut}O_8^+(2)$ as the full automorphism group. $O_8^+(2)$ acts as a rank 3 group on 120 and 135 points, and graphs $\mathcal{G}_6$ and $\mathcal{G}_7$ can be constructed as rank 3 graphs from the orthogonal group $O_8^+(2)$. On the other hand, the group $S(6,2)$ acts as a rank 3 group on 120 points and as a rank 4 group on 135 points. Therefore, the graph $\mathcal{G}_6$ can also be constructed as a rank 3 graph from the $S(6,2)$.

The design $\mathfrak{D}_8$ is a quasi-symmetric SDP design isomorphic to the design described in [15, 16]. Its block graph is isomorphic to the complement of the graph $\mathcal{G}_5$.

Furthermore, from the design $\mathfrak{D}_9$ on 28 points, one can construct two designs on 28 points, one isomorphic to the design $\mathfrak{D}_8$ and one isomorphic to the design $\mathfrak{D}_{10}$. The design on 28 points, whose blocks are unions of three disjoint blocks of the design $\mathfrak{D}_9$ such that stabilizer of that union under the action of the automorphism group $\mathrm{Aut}(\mathfrak{D}_9)$ is isomorphic to the maximal subgroup M_3, is isomorphic to the design $\mathfrak{D}_8$. On the other hand, unions of three blocks intersecting in exactly one point of the design $\mathfrak{D}_9$ such that stabilizer of that union under the action of the group $\mathrm{Aut}(\mathfrak{D}_9)$ is isomorphic to the maximal subgroup M_7 are blocks of a design isomorphic to the design $\mathfrak{D}_{10}$.

The design $\mathfrak{D}_{11}$ is also a quasi-symmetric design, isomorphic to the design described in [15, 16]. Its block graph is isomorphic to the graph $\mathcal{G}_5$.

The ovals (sets of 6 points, no three collinear) of the design $\mathfrak{D}_{12}$ form a block design with parameters $2-(36,6,8)$ isomorphic to the design $\mathfrak{D}_{14}$. Furthermore, the group $S(6,2)$

Table 1: Maximal subgroups of the group $L(3,5)$.

Maximal subgroup	Structure	Order	Index
H_1	$E_{25} : GL(2,5)$	12000	31
H_2	$E_{25} : GL(2,5)$	12000	31
H_3	S_5	120	3100
H_4	$(Z_4 \times Z_4) : S_3$	96	3875
H_5	F_{93}	93	4000

Table 2: Maximal subgroups of the group $U(5,2)$.

Maximal subgroup	Structure	Order	Index
K_1	$(E_{64} : Z_2) \cdot (E_9 : Z_3) \cdot SL(2,3)$	829444	165
K_2	$Z_3 \times U(4,2)$	46080	176
K_3	$(E_{16} : E_{16}) : (Z_3 \times A_5)$	77760	297
K_4	$E_{81} : S_5$	9720	1408
K_5	$(S_3 \times (E_9 : Z_3)) : SL(2,3)$	3888	3520
K_6	$L(2,11)$	660	20736

Table 3: Maximal subgroups of the group $S(6,2)$.

Maximal subgroup	Structure	Order	Index
M_1	$U(4,2) : Z_2$	51840	28
M_2	S_8	40320	36
M_3	$E_{32} : S_6$	23040	63
M_4	$U(3,3) : Z_2$	12096	120
M_5	$E_{64} : L(3,2)$	10752	135
M_6	$((E_{16} : Z_2) \times E_4) : (S_4 \times S_4)$	4608	315
M_7	$S_3 \times S_6$	4320	336
M_8	$L(2,8) : Z_3$	504	960

has a subgroup H isomorphic to the group $\mathrm{PG}(2,8)$. The group H acts on the set of all ovals of the design $\mathfrak{D}_{12}$ in four orbits of size 84 and one of these orbits forms the block set of a block design with parameters $2-(36,6,2)$ having $\mathrm{Aut}(\mathrm{PGL}(2,8))$ as the full automorphism group. The design $\mathfrak{D}_{12}$ is isomorphic to the $2-(36,8,6)$ design described in [17].

Unions of four disjoint blocks of the design $\mathfrak{D}_{14}$ such that stabilizer of that union under the action of the automorphism group $\mathrm{Aut}(\mathfrak{D}_{14})$ is isomorphic to the maximal subgroup M_6 are blocks of a design isomorphic to the complement of the design $\mathfrak{D}_{13}$.

The design $\mathfrak{D}_{15}$ is the point-hyperplane design in the projective geometry $\mathrm{PG}(6,2)$, and its full automorphism group is isomorphic to $\mathrm{PG}(6,2)$. $\mathfrak{D}_{15}$ possesses a symmetric incidence matrix with 1 everywhere on the diagonal, and therefore it gives rise to a strongly regular graph with parameters $(63,30,13,15)$ which is isomorphic to the graph $\mathcal{G}_5$.

The graph $\mathcal{G}_7$ can also be constructed from the design $\mathfrak{D}_{12}$. Any two blocks of $\mathfrak{D}_{12}$ intersect in $1,2$, or 4 points. The graph which has as its vertices the blocks of $\mathfrak{D}_{12}$, two vertices being adjacent if and only if the corresponding blocks intersect in one point, is isomorphic to $\mathcal{G}_7$.

Let $U \cong U(3,3)$ be a subgroup of the group $S(6,2)$ and let $\mathrm{ccl}_U(M_1)$ and $\mathrm{ccl}_U(M_6)$ be conjugacy classes of maximal subgroups M_1 and M_6 under the action of the group U. The intersection of M_1^g, $g \in U$, and M_6^h, $h \in U$, is isomorphic either to $G_{1,6}^1$ or $G_{1,6}^2$.

Table 4: Structures constructed from $L(3,5)$, $U(5,2)$, and $S(6,2)$.

Combinatorial structure	The full automorphism group
$2-(31,6,1)$ design	$L(3,5)$
$2-(31,6,100)$ design	$L(3,5)$
$2-(31,10,300)$ design	$L(3,5)$
$2-(31,15,700)$ design	$L(3,5)$
$2-(31,3,25)$ design	$L(3,5)$
$2-(31,12,550)$ design	$L(3,5)$
$2-(31,15,875)$ design	$L(3,5)$
$2-(28,12,11)$ design	$S(6,2)$
$2-(28,4,5)$ design	$S(6,2)$
$2-(28,10,40)$ design	$S(6,2)$
$2-(36,16,12)$ design	$S(6,2)$
$2-(36,8,6)$ design	$S(6,2)$
$2-(36,12,33)$ design	$S(6,2)$
$2-(36,6,8)$ design	$S(6,2)$
$2-(63,31,15)$ design	$\mathrm{PGL}(6,2)$
$2-(28,4,4)$ design	$U(3,3):Z_2$
$2-(28,4,1)$ design	$U(3,3):Z_2$
$\mathrm{SRG}(63,30,13,15)$	$S(6,2)$
$\mathrm{SRG}(120,56,28,24)$	$O_8^+(2):Z_2$
$\mathrm{SRG}(135,64,28,32)$	$O_8^+(2):Z_2$
$\mathrm{SRG}(165,36,3,9)$	$U(5,2):Z_2$
$\mathrm{SRG}(176,40,12,8)$	$U(5,2):Z_2$
$\mathrm{SRG}(297,40,7,5)$	$U(5,2):Z_2$
$\mathrm{SRG}(1408,567,246,216)$	$Fi_{21}:Z_2$

Let $\mathfrak{D}'$ be an incidence structure whose points are labeled by the elements of the class $\mathrm{ccl}_U(M_1)$ and whose blocks are labeled by the elements of the class $\mathrm{ccl}_U(M_6)$, point and block being incident if and only if the intersection of corresponding elements of conjugacy classes is isomorphic to the group $G^1_{1,6}$. The structure $\mathfrak{D}'$ is isomorphic to the block design with parameters $2-(28,4,4)$ having $\mathrm{Aut}U(3,3)\cong U(3,3):Z_2$ as the full automorphism group. Further, let $M=M_6^g,\ \ g\notin U$, be an element of the class $\mathrm{ccl}_{S(6,2)}(M_6)$. The incidence structure $\mathfrak{D}''$ whose points are labeled by the elements of the class $\mathrm{ccl}_U(M_1)$ and whose blocks are labeled by the elements of the class $\mathrm{ccl}_U(M)$, point and block being incident if and only if the intersection of corresponding elements of conjugacy classes is isomorphic to the group $G^1_{1,6}$, is isomorphic to the Hermitian unital with parameters $2-(28,4,1)$ having $\mathrm{Aut}U(3,3)$ as the full automorphism group. We conclude that the designs $\mathfrak{D}'$ and $\mathfrak{D}''$ are subdesigns of the design $\mathfrak{D}_9$. One can construct from the group $U(3,3)$ the design $\mathfrak{D}(U(3,3),(E_9:Z_3):Z_8,Z_4\cdot S_4;Z_3:Z_8)$ isomorphic to the design $\mathfrak{D}''$ (see [15]).

The Hölz design $H(q)$ of order q is a design with parameters $2-(q^3+1,q+1,q+2)$ which is a union of the Hermitian unital with parameters $2-(q^3+1,q+1,1)$ and a $2-(q^3+1,q+1,q+1)$ design whose blocks are arcs in the unital. To prove that design $\mathfrak{D}_9$ is isomorphic to the design $H(3)$, we constructed design $H(3)$, by the method described in [18], as the support design of the dual code of the code spanned by the incidence vectors of the design $\mathfrak{D}_8$. Constructed design is isomorphic to the design $\mathfrak{D}_9$.

From

(i) $|\mathrm{ccl}_{S(6,2)} M_1| = |\mathrm{ccl}_U M_1|$,

(ii) $|\mathrm{ccl}_{S(6,2)} M_2| = |\mathrm{ccl}_U M_2|$,

(iii) $|\mathrm{ccl}_{S(6,2)} M_3| = |\mathrm{ccl}_U M_3|$,

(iv) $|\mathrm{ccl}_{S(6,2)} M_7| = |\mathrm{ccl}_U \mathrm{M}_7|$,

one can conclude that, in order to construct designs $\mathfrak{D}_8$, $\mathfrak{D}_{10}$, $\mathfrak{D}_{11}$, $\mathfrak{D}_{14}$ and $\mathfrak{D}_{15}$, corresponding maximal subgroups need not to be conjugate by the elements of the whole group $S(6,2)$. The conjugation by the elements of the subgroup U is sufficient to obtain the desired structures.

In Table 4 we give a list of the constructed designs and strongly regular graphs and their full automorphism groups. Table 1, Table 2, and Table 3 give a list of maximal subgroups of the groups $L(3,5)$, $U(5,2)$, and $S(6,2)$, respectively.

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