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Research Article

Bitranslations and Symmetric Nets

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It is known that every class-regular symmetric (μ, m)-net is tactical. Also it is known that all (μ, m)-nets with $m = 2$ or $\mu = 1$ are tactical. In the work of Al-Kenani and Mavron (2003), it is proved that every symmetric net with $m = 3$ is tactical if and only if it is class regular. In this paper, we construct $(2, 4)$-net and show that it is class regular and therefore tactical. New necessary and sufficient conditions are given for a symmetric net to admit a nonidentity bitranslation.

1. Introduction

A $t - (v, k, \lambda)$ design Π is an incidence structure with v points, k points on a block, and any subset of t points is contained in exactly λ blocks, where $v > k$, $\lambda > 0$. The number of blocks is b and the number of blocks on a point is r.

The design Π is *resolvable* if its blocks can be partitioned into r *parallel classes*, such that each parallel class partitions the point set of Π. Blocks in the same parallel class are *parallel*. Clearly each parallel class has $m = v/k$ blocks. Π is *affine resolvable*, or simply *affine*, if it can be resolved so that any two nonparallel blocks meet in μ points, where $\mu = k/m = k^2/v$ is constant. Affine 1-designs are also called *nets*. The dual design of a design Π is denoted by $\Pi^\star$. If Π and $\Pi^\star$ are both affine, we call Π a *symmetric net*. We use the terminology of Jungnickel [1] (see also [2]). In this case, if $r > 1$, then $v = b = \mu m^2$ and $k = r = \mu m$. That is, Π is an affine $1 - (\mu m^2, \mu m, \mu m)$ design whose dual $\Pi^\star$ is also affine with the same parameters. For short we call such a symmetric net a (μ, m)-*net*.

If Π is a symmetric net, we shall refer to the parallel classes of Π as *block classes* of Π and to the parallel classes of $\Pi^\star$ as *point classes* of Π.

A *bitranslation* of Π is an automorphism fixing every point and block class which is fixed-point-free or is the identity. It is well known that the bitranslations form a group of order of a factor of m. The bitranslation group has order m if and only if it acts regularly on each point and block class. In which case we say that Π is *class regular*.

In this paper we find necessary and sufficient conditions for a permutation in S_m to induce a bitranslation, by considering regular subsets of S_m.

Let Π be a symmetric (μ, m)-net and let A, B be block classes. If $\theta : A \to B$ is a bijection, then the point subset $S = \bigcup a \cap \theta(a)$, where a ranges over all elements of A, is called an (A, B)-*transversal* of Π. If S is a union of point classes of Π, then S is said to be a *regular transversal* of Π, and θ is called an (A, B)-*syntax* of Π. We will denote the set of all (A, B)-syntaxes by $\Sigma(A, B)$.

Π is defined to be *tactical* if and only if $|\Sigma(A, B)| = m$ for all pairs of distinct block classes A and B of Π. Equivalently, the intersection of any two nonparallel blocks is contained in a (unique) transversal. See [3] for more details.

Label the m blocks in each block class of Π by $\{1, 2, \dots, m\}$ and similarly for point classes of Π. Then a bijection between point or block classes of Π may be regarded as an element of the symmetric group S_m. If A, B are any two distinct block classes of Π, then we may regard $\Sigma(A, B)$ as a semiregular subset of S_m; that is, if $\theta_1, \theta_2 \in \Sigma(A, B)$ and $\theta_1(a) = \theta_2(a)$ for some $a \in A$, then $\theta_1 = \theta_2$ (see [3]).

Let Y be a given block class of Π which we will call the *base block class*, and let its blocks be labelled $\{Y_1, Y_2, \dots, Y_m\}$ arbitrarily. We call Y_i the ith block of Y. Label any point class P of Π with integers $\{1, 2, \dots, m\}$ such that its ith point p_i is on Y_i for $i = 1, 2, \dots, m$.

Now choose a fixed point class $x = \{x_1, x_2, \dots, x_m\}$, called the *base point class*, and label any block class A of Π with $\{1, 2, \dots, m\}$ such that the ith block A_i is on x_i, $i = 1, 2, \dots, m$. We can therefore refer to the ith block of a block class, and dually.

We call this labelling the *standard labelling*, relative to the given base block class Y and point class x.

Result 1.1 (see [3]). If Π is a tactical (μ, m)-net with standard labelling for its block and point classes, then the identity bijection $1 \in \Sigma(A, B)$, for any two block classes A and B of Π.

2. Bitranslation Groups

Let Π be a symmetric (μ, m)-net.

Let $Y = \{Y_1, Y_2, \dots, Y_m\}$ be the base block class of Π and $x = \{x_1, x_2, \dots, x_m\}$ the base point class of Π in the standard labelling as above. If Φ is a bitranslation of Π, then Φ induces the permutation $\sigma \in S_m$ defined by: $\Phi(Y_i) = Y_{\sigma(i)}$, $i = 1, 2, \dots, m$.

Then by definition of standard labelling it follows that $\Phi : K_i \to K_{\sigma(i)}$ for any point or block class K of Π.

Notation. If X is a subset of a group G, then $C_G(X)$ denotes the centralizer subgroup $\{g \in G \mid gx = xg$, for all $x \in G\}$ of X in G.

Theorem 2.1. *Let Π be a tactical symmetric (μ, m)-net with standard labelling.*

Let $\sigma \in S_m$ and define a mapping $\Phi : \Pi \to \Pi$ by $\Phi(u_i) = u_{\sigma(i)}$ and $\Phi(A_i) = A_{\sigma(i)}$ for all point classes u and block classes A ($i = 1, 2, \dots, m$).

Then Φ is a bitranslation of Π if and only if $\sigma \in H$, where H is the subgroup

$$\bigcap_{(A,B)} C_{S_m}(\Sigma(A, B)). \tag{2.1}$$

Here (A, B) runs over all pairs of distinct block classes of Π.

Proof. Let A, B be distinct block classes of Π. Then $S = \bigcup_{i=1}^{m} A_i \cap B_{\theta(i)}$ is a union of point classes for all $\theta \in \Sigma(A, B)$, by definition of syntaxes.

Assume first that Φ is a bitranslation.

Since Φ is a bitranslation, it fixes every point and block classes. Hence Φ fixes S since S is a union of point classes.

Therefore, $S = \Phi(S) = \bigcup_{i=1}^{m} A_{\sigma(i)} \cap B_{\sigma\theta(i)} = \bigcup_{i=1}^{m} A_i \cap B_{\sigma\theta\sigma^{-1}(i)}$.

It follows that $\sigma\theta\sigma^{-1} \in \Sigma(A, B)$ for all $\theta \in \Sigma(A, B)$ and all A, B. Hence $\sigma \in H$.

Conversely, suppose $\sigma \in H$. Define Φ as in the statement of the theorem. Clearly Φ is bijective and fixes every point and block classes. Let p_j be any point and A_i any block of Π. Suppose $p_j \in A_i$. To show Φ is a bitranslation, it is enough to show that $p_{\sigma(j)} \in A_{\sigma(i)}$ (i.e., that Φ is an automorphism).

By definition of standard labelling, $p_j \in Y_j$, where Y is the base block class. Therefore, $p_j \in Y_j \cap A_i$. Since Π is tactical, then there is a unique $\theta \in \Sigma(Y, A)$ such that $i = \sigma(j)$.

Note that θ depends only on p_j and A_i.

Since p_j and $p_{\sigma(j)}$ are parallel, they are in the same transversal determined by θ: that is, $p_{\sigma(j)} \in Y_{\sigma(j)} \cap A_{\theta\sigma(j)}$. So $p_{\sigma(j)} \in A_{\theta\sigma(j)} = A_{\sigma\theta(j)}$, since $\theta \in H \leq C_{S_m}(\Sigma(Y, A))$. But $\theta(j) = i$, therefore $p_{\sigma(j)} \in A_{\sigma(i)}$, as required. □

With the notation and hypothesis of the theorem, we prove the following corollaries.

Corollary 2.2. (a) *H is isomorphic to the bitranslation group of Π.*

(b) *Π is class regular if and only if $|H| = m$.*

Proof. (a) The mapping $\sigma \to \Phi$ of the theorem is easily verified to be an isomorphism from H onto the bitranslation group.

(b) This follows easily from the definition of class regular. □

Corollary 2.3. *If all syntax sets of Π are the same subgroup G of S_m, then the bitranslation group of Π is isomorphic to $C_{S_M}(G) = H$ and $G \cap H = Z(G)$.*

Proof. It is clear that $H = C_{S_m}(G)$. The rest follows from Lemma 3.2. □

3. Regular Subsets

Let $n \geq 2$ be an integer and Ω a set of size n. Let S_Ω be the symmetric group on Ω.

A subset T of S_Ω is a *semiregular subset* of S_Ω if for any $\alpha, \beta \in \Omega$, there exists at most one element $t \in T$ such that $\alpha t = \beta$.

If there exists always exactly one such $t \in T$, then T is a *regular subset* of S_Ω.

Suppose T is a regular subset of S_Ω.

Clearly $|T| = n$. Let $C = C_{S_\Omega}(T) = \{x \in S_\Omega \mid xt = tx \text{ for all } t \in T\}$.

Let $G = \langle T \rangle$, the subgroup generated by T in S_Ω. Then it is easy to see that

(a) G is transitive on Ω;

(b) $C = C_{S_\Omega}(G)$.

Using this notation, we prove the following results.

Lemma 3.1. *If T is regular, then C is semiregular on Ω and $|C|$ divides n.*

Proof. Let $x \in C$ and suppose $\alpha x = \alpha$ for some $\alpha \in \Omega$.

Let β be any element of Ω and let $t \in T$ be such that $\alpha t = \beta$. Then $\beta x = \alpha t x = \alpha x t = \alpha t = \beta$. Therefore, $x = 1$. The rest of the proof is straightforward.

It is clear that $T = G(= \langle T \rangle)$ if and only if T is a subgroup of S_Ω. □

Lemma 3.2. (a) *If $T \neq G$, then $|C| < n$.*

(b) *If $T = G$, then $|C| = n$. Moreover, $C \cong G$ and $C \cap G = Z(G)$.*

Proof. (a) $T \neq G$. Let $\alpha \in \Omega$. Since $G \neq T$, then $|G| > |T| \geq n$.

Since $|G : G_\alpha| \leq |\Omega| = n$, then $|G_\alpha| \geq |G|/n > 1$.

$\therefore$ If Δ is the set of fixed points of G_α, then $\alpha \in \Delta$ and $\Delta \neq \Omega$.

Hence $|\Delta| < n$.

Let $c \in C$ and $\beta = \alpha c$. For any $g \in G_\alpha$, we have $\beta g = \alpha c g = \alpha g c = \alpha c = \beta$.

Hence $\beta \in \Delta$. So Δ contains the C-orbit αC of α.

Therefore, $|\alpha C| \leq |\Delta| < n$. We also have $|\alpha C| = |C|$, since C is semiregular on Ω. Hence $|C| < n$.

(b) $T = G$. Choose any $\alpha \in G$. For each $t \in T$, define $g_t \in S_\Omega$ by

$$(\alpha u) g_t = \alpha t^{-1} u \quad \forall u \in T. \tag{3.1}$$

Note that $\{\alpha u \mid u \in T\} = \Omega$, since T is regular on Ω.

Let $H = \{g_t \mid t \in T\}$. If $t_1, t_2 \in T$ and $g_{t_1} = g_{t_2}$, then $\alpha t_1^{-1} = \alpha t_2^{-1}$ and hence $t_1 = t_2$ (since T is regular). Therefore, $|H| = |T| = n$.

If $t_1, t_2, u \in T$, then $(\alpha u) g_{t_1} g_{t_2} = (\alpha t_1^{-1} u) g_{t_2} = \alpha t_2^{-1} t_1^{-1} u = \alpha g_{t_1 t_2}$.

Hence $g_{t_1} g_{t_2} = g_{t_1 t_2}$, which means that the mapping $t \to g_t$ defines an isomorphism $T \to H$ (H is just the left regular representation of T). Therefore, $|H| = |T| = n$.

If $t_1, t_2 \in T$, then $(\alpha u) g_{t_1} t_2 = \alpha t_1^{-1} u t_2 = (\alpha u t_2) g_{t_1} = (\alpha u) t_2 g_{t_1}$. So $g_{t_1} t_2 = t_2 g_{t_1}$ for all $t_2 \in T$ and $g_{t_1} \in H$. Hence $H \subseteq C$.

Since $|H| = n$ and, by Lemma 3.1, $|C|$ divides n, then $|C| = n$ and $C = H$.

Finally, $C \cap G = C_{S_\Omega}(G) \cap G = Z(G)$. □

Lemma 3.3. *If $g, h \in S_\Omega$, then gTh is also a regular set.*

Proof. Let $\alpha, \beta \in \Omega$. There is a unique $t \in T$ such that $(\alpha g)t = \beta h^{-1}$; that is, $\alpha(gth) = \beta$.

Since $gth \in gTh$, the result follows immediately. □

Definition 3.4. T and gTh are *congruent* regular sets. If $g = h^{-1}$, they are said to be *similar*.

Lemma 3.5. *If T, T' are regular sets in S_Ω and $|T \cap T'| \geq n - 1$, then $T = T'$.*

Proof. The result is obvious if $|T \cap T'| = n$, since $|T| = |T'| = n$. So suppose $|T \cap T'| = n - 1$.

Let $T = \{t_1, t_2, \ldots, t_{n-1}, t\}$ and $T' = \{t_1, t_2, \ldots, t_{n-1}, t'\}$. Then $T \cap T' = \{t_1, t_2, \ldots, t_{n-1}\}$ and $t \neq t'$.

Let $\alpha \in \Omega$. Then $\Omega = \alpha T$ and $\{\alpha t\} = \Omega \setminus \alpha(T \cap T') = \{\alpha t'\}$, since T is a regular set. Hence $\alpha t = \alpha t'$ and so $t = t'$, since T is regular. Therefore, $T = T'$. □

The above lemma shows that distinct regular sets must differ in at least 2 elements.

4. Syntax Sets as Regular Subsets

The general theory of regular sets developed in the previous section is now applied syntax sets.

From the introduction to this paper, we know that syntax sets of a tactical symmetric (μ, m)-net are semiregular subsets of S_m.

Lemma 4.1. *Let* Π *be a tactical symmetric* (μ, m)*-net and* Σ *any of its syntax sets. If* Σ *is not a subgroup of* S_m*, then*

(a) $|C_{S_m}(\Sigma)|$ *divides* m *and is less than* m*;*

(b) *the bitranslation group of* Π *has order at most* $m - 1$*.*

Proof. (a) Follows from Lemmas 3.1 and 3.2.

(b) It is clear from Theorem 2.1 that the bitranslation group G of Π has order $|H|$. Since H is a subgroup of $C_{S_m}(\Sigma)$, then $|H|$ divides m and $|H| < m$ by part (a). □

Now, consider the special case $n = 4$. Let $\Omega = \{1, 2, 3, 4\}$.

Theorem 4.2. *If* $n = 4$ *and* T *is a regular set in* S_4*, then* T *is congruent to a regular subgroup of* S_4*.*

The regular subgroups of S_4 *are the three cyclic groups generated by 4-cycles and the Klein 4-group* $\{1, (12)(34), (13)(24), (14)(23)\}$.

Proof. From Lemma 3.3 we may assume that $1 \in T$. The order of any of the 3 nonidentity elements of T is therefore either 2 or 4.

Suppose T has an element of order 4. Without loss of generality, we may assume that the 4-cycle $(1234) = \omega \in T$. If t, u are the remaining nonidentity elements of T, then, say, $1t = 3$, $1u = 4$.

Therefore, $3t \neq 3$ or 4; so $3t = 1$ or 2.

If $3t = 1$, then $t = (13)(24)$, $u = (1432)$, and so $T = \langle \omega \rangle$.

Suppose $3t = 2$. Then $2t \neq 2$ or 3. Therefore, $2t = 1$ or 4.

If $2t = 1$, then $4t \neq 1, 2, 3, 4$, using the fact that t is a permutation and T is a regular set. This is impossible.

Similarly, $2t = 4$ would imply $4t \neq 1, 2, 3$, or 4, which again is impossible.

Therefore, the case $3t = 2$ is impossible and so $3t = 1$ as above.

If no element of T has order 4, then all nonidentity elements of T have order 2. Then from the regularity of T it follows that T must be the Klein 4-group. □

We continue with the notation and hypothesis of Theorem 2.1.

(1) Suppose all syntax sets of Π are the same subgroup G of S_Ω. Then by Lemma 3.2, $H = G$ and the bitranslation group of Π is isomorphic to $C_{S_\Omega}(G) \cong G$. Furthermore, $C_{S_\Omega}(G) \cap G = Z(G)$.

(2) Consider the special case $m = 4$. By Theorem 4.2, we know that any syntax set of Π is congruent to a subgroup of S_4 of order 4. This must be either the Klein 4-group or one of the 3 cyclic subgroups of order 4. Since Π is tactical, all its syntax sets contain the identity. Therefore, we can say that any syntax set of Π is conjugate to a subgroup of order 4 in S_4.

Suppose all syntax sets of Π are the same subgroup G. Then by Corollary 2.3, we have $H = G$ and the bitranslation group of Π is isomorphic to $C_{S_4}(G)$.

From (1), $G \cong C_{S_4}(G)$ and $C_{S_4} \cap G = G$, since G is abelian. Hence $C_{S_4}(G) = G$. It follows that the bitranslation group of Π has order 4 and hence Π is class regular.

Below is an example of a tactical symmetric $(2,4)$-net in which every syntax set $\Sigma(A,B)$, $A \neq B$, is the Klein 4-group: $\{1, (12)(34), (13)(24), (14)(23)\}$. (The full automorphism group has order 5376. The author is grateful to V. D. Tonchev for this information.)

The incidence matrix M of this symmetric $(2,4)$-net is as follows:

$$M = \begin{pmatrix}
1&1&1&1&0&0&1&0&0&0&0&0&1&0&0&0&0&0&0&1&1&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&1&0&0&1&0&0&0&0&0&0&1&0&0&0&0&1&0&0&0&1&0&1&0&1\\
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0&0&0&0&1&0&1&0&0&0&0&1&0&0&1&0&1&0&0&0&1&0&0&0&0&0&0&0&1&0&0&1
\end{pmatrix}. \tag{4.1}$$

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