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Research Article

Fixed-Point Results for Generalized Contractions on Ordered Gauge Spaces with Applications

Cristian Chifu and Gabriela Petruşel

Faculty of Business, Babeş-Bolyai University, Horia Street no. 7, 400174 Cluj-Napoca, Romania

Correspondence should be addressed to Cristian Chifu, cochifu@tbs.ubbcluj.ro

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The purpose of this paper is to present some fixed-point results for single-valued φ-contractions on ordered and complete gauge space. Our theorems generalize and extend some recent results in the literature. As an application, existence results for some integral equations on the positive real axis are given.

1. Introduction

Throughout this paper $\mathbb{E}$ will denote a nonempty set E endowed with a separating gauge structure $\mathfrak{D} = \{d_\alpha\}_{\alpha\in\Lambda}$, where Λ is a directed set (see [1] for definitions). Let $\mathbb{N} := \{0,1,2,\ldots\}$ and $\mathbb{N}^* := \mathbb{N} \setminus \{0\}$. We also denote by $\mathbb{R}$ the set of all real numbers and by $\mathbb{R}_+ := [0,+\infty)$.

A sequence (x_n) of elements in E is said to be Cauchy if for every $\varepsilon > 0$ and $\alpha \in \Lambda$, there is an N with $d_\alpha(x_n, x_{n+p}) \le \varepsilon$ for all $n \ge N$ and $p \in \mathbb{N}^*$. The sequence (x_n) is called convergent if there exists an $x_0 \in X$ such that for every $\varepsilon > 0$ and $\alpha \in \Lambda$, there is an $N \in \mathbb{N}^*$ with $d_\alpha(x_0, x_n) \le \varepsilon$, for all $n \ge N$.

A gauge space $\mathbb{E}$ is called complete if any Cauchy sequence is convergent. A subset of X is said to be closed if it contains the limit of any convergent sequence of its elements. See also Dugundji [1] for other definitions and details.

If $f : E \to E$ is an operator, then $x \in E$ is called fixed point for f if and only if $x = f(x)$. The set $F_f := \{x \in E \mid x = f(x)\}$ denotes the fixed-point set of f.

On the other hand, Ran and Reurings [2] proved the following Banach-Caccioppoli type principle in ordered metric spaces.

Theorem 1.1 (Ran and Reurings [2]). *Let X be a partially ordered set such that every pair $x, y \in X$ has a lower and an upper bound. Let d be a metric on X such that the metric space (X, d) is complete.*

Let $f : X \to X$ be a continuous and monotone (i.e., either decreasing or increasing) operator. Suppose that the following two assertions hold:

(1) *there exists $a \in]0,1[$ such that $d(f(x), f(y)) \leq a \cdot d(x,y)$, for each $x, y \in X$ with $x \geq y$;*

(2) *there exists $x_0 \in X$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$.*

Then f has an unique fixed point $x^* \in X$, that is, $f(x^*) = x^*$, and for each $x \in X$ the sequence $(f^n(x))_{n\in\mathbb{N}}$ of successive approximations of f starting from x converges to $x^* \in X$.

Since then, several authors considered the problem of existence (and uniqueness) of a fixed point for contraction-type operators on partially ordered sets.

In 2005, Nieto and Rodrguez-López proved a modified variant of Theorem 1.1, by removing the continuity of f. The case of decreasing operators is treated in Nieto and Rodrguez-López [3], where some interesting applications to ordinary differential equations with periodic boundary conditions are also given. Nieto, Pouso, and Rodrguez-López, in a very recent paper, improve some results given by Petruşel and Rus in [4] in the setting of abstract L-spaces in the sense of Fréchet, see, for example, [5, Theorems 3.3 and 3.5]. Another fixed-point result of this type was given by O'Regan and Petruşel in [6] for the case of φ-contractions in ordered complete metric spaces.

The aim of this paper is to present some fixed-point theorems for φ-contractions on ordered and complete gauge space. As an application, existence results for some integral equations on the positive real axis are given. Our theorems generalize the above-mentioned theorems as well as some other ones in the recent literature (see; Ran and Reurings [2], Nieto and Rodrguez-López [3, 7], Nieto et al. [5], Petruşel and Rus [4], Agarwal et al. [8], O'Regan and Petruşel [6], etc.).

2. Preliminaries

Let X be a nonempty set and $f : X \to X$ be an operator. Then, $f^0 := 1_X$, $f^1 := f, \ldots, f^{n+1} = f \circ f^n$, $n \in \mathbb{N}$ denote the iterate operators of f. Let X be a nonempty set and let $s(X) := \{(x_n)_{n\in N} \mid x_n \in X,\ n \in N\}$. Let $c(X) \subset s(X)$ a subset of $s(X)$ and $\mathrm{Lim} : c(X) \to X$ an operator. By definition the triple $(X, c(X), \mathrm{Lim})$ is called an L-space (Fréchet [9]) if the following conditions are satisfied.

(i) If $x_n = x$, for all $n \in N$, then $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$.

(ii) If $(x_n)_{n\in N} \in c(X)$ and $\mathrm{Lim}(x_n)_{n\in N} = x$, then for all subsequences, $(x_{n_i})_{i\in N}$, of $(x_n)_{n\in N}$ we have that $(x_{n_i})_{i\in N} \in c(X)$ and $\mathrm{Lim}(x_{n_i})_{i\in N} = x$.

By definition, an element of $c(X)$ is a convergent sequence, $x := \mathrm{Lim}(x_n)_{n\in N}$ is the limit of this sequence and we also write $x_n \to x$ as $n \to +\infty$.

In what follow we denote an L-space by $(X, \to)$.

In this setting, if $U \subset X \times X$, then an operator $f : X \to X$ is called orbitally U-continuous (see [5]) if $[x \in X$ and $f^{n(i)}(x) \to a \in X$, as $i \to +\infty$ and $(f^{n(i)}(x), a) \in U$ for any $i \in \mathbb{N}]$ imply $[f^{n(i)+1}(x) \to f(a)$, as $i \to +\infty]$. In particular, if $U = X \times X$, then f is called orbitally continuous.

Let $(X, \leq)$ be a partially ordered set, that is, X is a nonempty set and $\leq$ is a reflexive, transitive, and antisymmetric relation on X. Denote

$$X_{\leq} := \{(x,y) \in X \times X \mid x \leq y \text{ or } y \leq x\}. \tag{2.1}$$

Also, if $x, y \in X$, with $x \leq y$ then by $[x, y]_{\leq}$ we will denote the ordered segment joining x and y, that is, $[x, y]_{\leq} := \{z \in X \mid x \leq z \leq y\}$. In the same setting, consider $f : X \to X$. Then, $(\mathrm{LF})_f := \{x \in X \mid x \leq f(x)\}$ is the lower fixed-point set of f, while $(\mathrm{UF})_f := \{x \in X \mid x \geq f(x)\}$ is the upper fixed-point set of f. Also, if $f : X \to X$ and $g : Y \to Y$, then the cartesian product of f and g is denoted by $f \times g$, and it is defined in the following way: $f \times g : X \times Y \to X \times Y$, $(f \times g)(x, y) := (f(x), g(y))$.

Definition 2.1. Let X be a nonempty set. By definition $(X, \to, \leq)$ is an ordered L-space if and only if

(i) $(X, \to)$ is an L-space;

(ii) $(X, \leq)$ is a partially ordered set;

(iii) $(x_n)_{n\in\mathbb{N}} \to x$, $(y_n)_{n\in\mathbb{N}} \to y$ and $x_n \leq y_n$, for each $n \in \mathbb{N} \Rightarrow x \leq y$.

If $\mathbb{E} := (E, \mathfrak{D})$ is a gauge space, then the convergence structure is given by the family of gauges $\mathfrak{D} = \{d_\alpha\}_{\alpha\in\Lambda}$. Hence, $(E, \mathfrak{D}, \leq)$ is an ordered L-space, and it will be called an ordered gauge space, see also [10, 11].

Recall that $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a comparison function if it is increasing and $\varphi^k(t) \to 0$, as $k \to +\infty$. As a consequence, we also have $\varphi(t) < t$, for each $t > 0$, $\varphi(0) = 0$ and φ is right continuous at 0. For example, $\varphi(t) = at$ (where $a \in [0, 1[\,$), $\varphi(t) = t/(1 + t)$ and $\varphi(t) = \ln(1 + t)$, $t \in \mathbb{R}_+$ are comparison functions.

Recall now the following important abstract concept.

Definition 2.2 (Rus [12]). Let $(X, \to)$ be an L-space. An operator $f : X \to X$ is, by definition, a Picard operator if

(i) $F_f = \{x^*\}$;

(ii) $(f^n(x))_{n\in\mathbb{N}} \to x^*$ as $n \to \infty$, for all $x \in X$.

Several classical results in fixed-point theory can be easily transcribed in terms of the Picard operators, see [4, 13, 14]. In Rus [12] the basic theory of Picard operators is presented.

3. Fixed-Point Results

Our first main result is the following existence, uniqueness, and approximation fixed-point theorem.

Theorem 3.1. *Let $(E, \mathfrak{D}, \leq)$ be an ordered complete gauge space and $f : E \to E$ be an operator. Suppose that*

(i) *for each $x, y \in E$ with $(x, y) \notin X_{\leq}$ there exists $c(x, y) \in E$ such that $(x, c(x, y)) \in X_{\leq}$ and $(y, c(x, y)) \in X_{\leq}$;*

(ii) *$X_{\leq} \in I(f \times f)$;*

(iii) *if $(x, y) \in X_{\leq}$ and $(y, z) \in X_{\leq}$, then $(x, z) \in X_{\leq}$;*

(iv) *there exists $x_0 \in X_{\leq}$ such that $(x_0, f(x_0)) \in X_{\leq}$;*

(v) f *is orbitally continuous;*

(vi) *there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *such that, for each* $\alpha \in \Lambda$ *one has*

$$d_\alpha(f(x), f(y)) \leq \varphi(d_\alpha(x,y)), \quad \text{for each } (x,y) \in X_\leq. \tag{3.1}$$

Then, f *is a Picard operator.*

Proof. Let $x_0 \in E$ be such that $(x_0, f(x_0)) \in X_\leq$. Suppose first that $x_0 \neq f(x_0)$. Then, from (ii) we obtain

$$\left(f(x_0), f^2(x_0)\right), \left(f^2(x_0), f^3(x_0)\right), \ldots, \left(f^n(x_0), f^{n+1}(x_0)\right), \ldots, \in X_\leq. \tag{3.2}$$

From (vi), by induction, we get, for each $\alpha \in \Lambda$, that

$$d_\alpha\left(f^n(x_0), f^{n+1}(x_0)\right) \leq \varphi^n(d_\alpha(x_0, f(x_0))), \quad \text{for each } n \in \mathbb{N}. \tag{3.3}$$

Since $\varphi^n(d_\alpha(x_0, f(x_0)) \to 0$ as $n \to +\infty$, for an arbitrary $\varepsilon > 0$ we can choose $N \in \mathbb{N}^*$ such that $d_\alpha(f^n(x_0), f^{n+1}(x_0)) < \varepsilon - \varphi(\varepsilon)$, for each $n \geq N$. Since $(f^n(x_0), f^{n+1}(x_0)) \in X_\leq$ for all $n \in \mathbb{N}$, we have for all $n \geq N$ that

$$\begin{aligned} d_\alpha\left(f^n(x_0), f^{n+2}(x_0)\right) &\leq d_\alpha\left(f^n(x_0), f^{n+1}(x_0)\right) + d_\alpha\left(f^{n+1}(x_0), f^{n+2}(x_0)\right) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi\left(d_\alpha\left(f^n(x_0), f^{n+1}(x_0)\right) \leq \varepsilon. \end{aligned} \tag{3.4}$$

Now since $(f^n(x_0), f^{n+2}(x_0)) \in X_\leq$ (see (iii)) we have for any $n \geq N$ that

$$\begin{aligned} d_\alpha\left(f^n(x_0), f^{n+3}(x_0)\right) &\leq d_\alpha\left(f^n(x_0), f^{n+1}(x_0)\right) + d\left(f^{n+1}(x_0), f^{n+3}(x_0)\right) \\ &< \varepsilon - \varphi(\varepsilon) + \varphi\left(d_\alpha\left(f^n(x_0), f^{n+2}(x_0)\right) \leq \varepsilon. \end{aligned} \tag{3.5}$$

By induction, for each $\alpha \in \Lambda$, we have

$$d_\alpha\left(f^n(x_0), f^{n+k}(x_0)\right) < \varepsilon, \quad \text{for any } k \in \mathbb{N}^*, \ n \geq N. \tag{3.6}$$

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{E}$. From the completeness of the gauge space we have $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$.

Let $x \in E$ be arbitrarily chosen. Then;

(1) If $(x, x_0) \in X_\leq$ then $(f^n(x), f^n(x_0)) \in X_\leq$ and thus, for each $\alpha \in \Lambda$, we have $d_\alpha(f^n(x), f^n(x_0)) \leq \varphi^n(d_\alpha(x, x_0))$, for each $n \in \mathbb{N}$. Letting $n \to +\infty$ we obtain that $(f^n(x))_{n\in\mathbb{N}} \to x^*$.

(2) If $(x, x_0) \notin X_{\leq}$ then, by (i), there exists $c(x, x_0) \in E$ such that $(x, c(x, x_0)) \in X_{\leq}$ and $(x_0, c(x, x_0)) \in X_{\leq}$. From the second relation, as before, we get, for each $\alpha \in \Lambda$, that $d_\alpha(f^n(x_0), f^n(c(x, x_0))) \leq \varphi^n(d_\alpha(x_0, c(x, x_0)))$, for each $n \in \mathbb{N}$ and hence $(f^n(c(x, x_0)))_{n\in\mathbb{N}} \to x^*$, as $n \to +\infty$. Then, using the first relation we infer that, for each $\alpha \in \Lambda$, we have $d_\alpha(f^n(x), f^n(c(x, x_0))) \leq \varphi^n(d_\alpha(x, c(x, x_0)))$, for each $n \in \mathbb{N}$. Letting again $n \to +\infty$, we conclude $(f^n(x))_{n\in\mathbb{N}} \to x^*$.

By the orbital continuity of f we get that $x^* \in F_f$. Thus $x^* = f(x^*)$. If we have $f(y) = y$ for some $y \in E$, then from above, we must have $f^n(y) \to x^*$, so $y = x^*$.

If $f(x_0) = x_0$, then x_0 plays the role of x^*. □

Remark 3.2. Equivalent representation of condition (iv) are as follows.

(iv)′ There exists $x_0 \in E$ such that $x_0 \leq f(x_0)$ or $x_0 \geq f(x_0)$

(iv)″ $(\mathrm{LF})_f \cup (\mathrm{UF})_f \neq \emptyset$.

Remark 3.3. Condition (ii) can be replaced by each of the following assertions:

(ii)′ $f : (E, \leq) \to (E, \leq)$ is increasing,

(ii)″ $f : (E, \leq) \to (E, \leq)$ is decreasing.

However, it is easy to see that assertion (ii) in Theorem 3.1. is more general.

As a consequence of Theorem 3.1, we have the following result very useful for applications.

Theorem 3.4. *Let $(E, \mathfrak{D}, \leq)$ be an ordered complete gauge space and $f : E \to E$ be an operator. One supposes that*

(i) *for each $x, y \in E$ with $(x, y) \notin X_{\leq}$ there exists $c(x, y) \in E$ such that $(x, c(x, y)) \in X_{\leq}$ and $(y, c(x, y)) \in X_{\leq}$;*

(ii) *$f : (E, \leq) \to (E, \leq)$ is increasing;*

(iii) *there exists $x_0 \in E$ such that $x_0 \leq f(x_0)$;*

(iv)

(a) *f is orbitally continuous or*

(b) *if an increasing sequence $(x_n)_{n\in\mathbb{N}}$ converges to x in E, then $x_n \leq x$ for all $n \in \mathbb{N}$;*

(v) *there exists a comparison function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ such that*

$$d_\alpha(f(x), f(y)) \leq \varphi(d_\alpha(x, y)), \quad \textit{for each } (x, y) \in X_{\leq},\ \alpha \in \Lambda; \tag{3.7}$$

(vi) *if $(x, y) \in X_{\leq}$ and $(y, z) \in X_{\leq}$, then $(x, z) \in X_{\leq}$.*

Then f is a Picard operator.

Proof. Since $f : (E, \leq) \to (E, \leq)$ is increasing and $x_0 \leq f(x_0)$ we immediately have $x_0 \leq f(x_0) \leq f^2(x_0) \leq \cdots f^n(x_0) \leq \cdots$. Hence from (v) we obtain $d_\alpha(f^n(x_0), f^{n+1}(x_0)) \leq \varphi^n(d_\alpha(x_0, f(x_0))$, for each $n \in \mathbb{N}$. By a similar approach as in the proof of Theorem 3.1 we obtain

$$d_\alpha\left(f^n(x_0), f^{n+k}(x_0)\right) < \varepsilon, \quad \text{for any } k \in \mathbb{N}^*,\ n \geq N, \tag{3.8}$$

Hence $(f^n(x_0))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{E}$. From the completeness of the gauge space we have that $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$ as $n \to +\infty$.

By the orbital continuity of the operator f we get that $x^* \in F_f$. If (iv)(b) takes place, then, since $(f^n(x_0))_{n\in\mathbb{N}} \to x^*$, given any $\epsilon > 0$ there exists $N_\epsilon \in \mathbb{N}^*$ such that for each $n \geq N_\epsilon$ we have $d_\alpha(f^n(x_0), x^*) < \epsilon$. On the other hand, for each $n \geq N_\epsilon$, since $f^n(x_0) \leq x^*$, we have, for each $\alpha \in \Lambda$ that

$$\begin{aligned} d_\alpha(x^*, f(x^*)) &\leq d_\alpha\left(x^*, f^{n+1}(x_0)\right) + d_\alpha(f(f^n(x_0)), f(x^*)) \\ &\leq d_\alpha\left(x^*, f^{n+1}(x_0)\right) + \varphi(d_\alpha(f^n(x_0), x^*)) < 2\epsilon. \end{aligned} \tag{3.9}$$

Thus $x^* \in F_f$.

The uniqueness of the fixed point follows by contradiction. Suppose there exists $y^* \in F_f$, with $x^* \neq y^*$. There are two possible cases.

(a) If $(x^*, y^*) \in X_\leq$, then we have $0 < d_\alpha(y^*, x^*) = d_\alpha(f^n(y^*), f^n(x^*)) \leq \varphi^n(d_\alpha(y^*, x^*)) \to 0$ as $n \to +\infty$, which is a contradiction. Hence $x^* = y^*$.

(b) If $(x^*, y^*) \notin X_\leq$ then there exists $c^* \in E$ such that $(x^*, c^*) \in X_\leq$ and $(y^*, c^*) \in X_\leq$. The monotonicity condition implies that $f^n(x^*)$ and $f^n(c^*)$ are comparable as well as $f^n(c^*)$ and $f^n(y^*)$. Hence $0 < d_\alpha(y^*, x^*) = d_\alpha(f^n(y^*), f^n(x^*)) \leq d_\alpha(f^n(y^*), f^n(c^*)) + d_\alpha(f^n(c^*), f^n(x^*)) \leq \varphi^n(d_\alpha(y^*, c^*)) + \varphi^n(d_\alpha(c^*, x^*)) \to 0$ as $n \to +\infty$, which is again a contradiction. Thus $x^* = y^*$. □

4. Applications

We will apply the above result to nonlinear integral equations on the real axis

$$x(t) = \int_0^t K(t, s, x(s))ds + g(t), \qquad t \in \mathbb{R}_+. \tag{4.1}$$

Theorem 4.1. *Consider* (4.1). *Suppose that*

(i) $K : \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \mathbb{R}_+ \to \mathbb{R}^n$ *are continuous;*

(ii) $K(t, s, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is increasing for each* $t, s \in \mathbb{R}_+$*;*

(iii) *there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, *with* $\varphi(\lambda t) \leq \lambda\varphi(t)$ *for each* $t \in \mathbb{R}_+$ *and any* $\lambda \geq 1$, *such that*

$$|K(t, s, u) - K(t, s, v)| \leq \varphi(|u - v|), \quad \textit{for each } t, s \in \mathbb{R}_+,\ u, v \in \mathbb{R}^n,\ u \leq v; \tag{4.2}$$

(iv) *there exists* $x_0 \in C(\mathbb{R}_+, \mathbb{R}^n)$ *such that*

$$x_0(t) \le \int_0^t K(t,s,x_0(s))ds + g(t), \quad \text{for any } t \in \mathbb{R}_+. \tag{4.3}$$

Then the integral equation (4.1) *has a unique solution* x^* *in* $C([0,+\infty), \mathbb{R}^n)$.

Proof. Let $E := C([0,+\infty), \mathbb{R}^n)$ and the family of pseudonorms

$$\|x\|_n := \max_{t\in[0,n]} |x(t)|e^{-\tau t}, \quad \text{where } \tau > 0. \tag{4.4}$$

Define now $d_n(x,y) := \|x-y\|_n$ for $x, y \in E$.

Then $\mathfrak{D} := (d_n)_{n\in\mathbb{N}^*}$ is family of gauges on E. Consider on E the partial order defined by

$$x \le y \text{ if and only if } x(t) \le y(t) \quad \text{for any } t \in \mathbb{R}_+. \tag{4.5}$$

Then $(E, \mathfrak{D}, \le)$ is an ordered and complete gauge space. Moreover, for any increasing sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to some $x^* \in E$ we have $x_n(t) \le x^*(t)$, for any $t \in [0,+\infty)$. Also, for every $x, y \in E$ there exists $c(x,y) \in E$ which is comparable to x and y.

Define $A : C([0,+\infty), \mathbb{R}^n) \to C([0,+\infty), \mathbb{R}^n)$, by the formula

$$Ax(t) := \int_0^t K(t,s,x(s))ds + g(t), \quad t \in \mathbb{R}_+. \tag{4.6}$$

First observe that from (ii) A is increasing. Also, for each $x, y \in E$ with $x \le y$ and for $t \in [0,n]$, we have

$$\begin{aligned} \left|Ax(t) - Ay(t)\right| &\le \int_0^t \left|K(t,s,x(s)) - K(t,s,y(s))\right| ds \le \int_0^t \varphi\left(\left|x(s)-y(s)\right|\right) ds \\ &= \int_0^t \varphi\left(\left|x(s)-y(s)\right| e^{-\tau s} e^{\tau s}\right) ds \le \int_0^t e^{\tau s} \varphi\left(\left|x(s)-y(s)\right| e^{-\tau s}\right) ds \\ &\le \varphi\left(d_n(x,y)\right) \int_0^t e^{\tau s} ds \le \frac{1}{\tau} \varphi\left(d_n(x,y)\right) e^{\tau t}. \end{aligned} \tag{4.7}$$

Hence, for $\tau \ge 1$ we obtain

$$d_n(Ax, Ay) \le \varphi\left(d_n(x,y)\right), \quad \text{for each } x, y \in X, x \le y. \tag{4.8}$$

From (iv) we have that $x_0 \le Ax_0$. The conclusion follows now from Theorem 3.4. □

Consider now the following equation:

$$x(t) = \int_{-t}^{t} K(t, s, x(s))ds + g(t), \qquad t \in \mathbb{R}. \tag{4.9}$$

Theorem 4.2. *Consider* (4.9). *Suppose that*

(i) $K : \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ *and* $g : \mathbb{R} \to \mathbb{R}^n$ *are continuous;*

(ii) $K(t, s, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is increasing for each* $t, s \in \mathbb{R}$;

(iii) *there exists a comparison function* $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$, *with* $\varphi(\lambda t) \leq \lambda\varphi(t)$ *for each* $t \in \mathbb{R}_+$ *and any* $\lambda \geq 1$, *such that*

$$|K(t, s, u) - K(t, s, v)| \leq \varphi(|u - v|), \quad \text{for each } t, s \in \mathbb{R},\ u, v \in \mathbb{R}^n,\ u \leq v; \tag{4.10}$$

(iv) *there exists* $x_0 \in C(\mathbb{R}, \mathbb{R}^n)$ *such that*

$$x_0(t) \leq \int_{-t}^{t} K(t, s, x_0(s))ds + g(t), \quad \text{for any } t \in \mathbb{R}. \tag{4.11}$$

Then the integral equation (4.9) *has a unique solution* x^* *in* $C(\mathbb{R}, \mathbb{R}^n)$.

Proof. We consider the gauge space $E := (C(\mathbb{R}, \mathbb{R}^n), \mathfrak{D} := (d_n)_{n\in\mathbb{N}})$ where

$$d_n(x, y) = \max_{-n\leq t\leq n}\left(\left|x(t) - y(t)\right| \cdot e^{-\tau|t|}\right), \quad \tau > 0, \tag{4.12}$$

and the operator $B : E \to E$ defined by

$$Bx(t) = \int_{-t}^{t} K(t, s, x(s))ds + g(t). \tag{4.13}$$

As before, consider on E the partial order defined by

$$x \leq y \quad \text{iff } x(t) \leq y(t) \quad \text{for any } t \in \mathbb{R}. \tag{4.14}$$

Then $(E, \mathfrak{D}, \leq)$ is an ordered and complete gauge space. Moreover, for any increasing sequence $(x_n)_{n\in\mathbb{N}}$ in E converging to a certain $x^* \in E$ we have $x_n(t) \leq x^*(t)$, for any $t \in \mathbb{R}$. Also, for every $x, y \in E$ there exists $c(x, y) \in E$ which is comparable to x and y. Notice that (ii) implies that B is increasing.

From condition (iii), for $x, y \in E$ with $x \leq y$, we have

$$\begin{aligned} \left|Bx(t) - By(t)\right| &\leq \int_{-t}^{t} \varphi\left(\left|x(s) - y(s)\right| e^{-\tau|s|} e^{\tau|s|}\right) ds \\ &\leq \int_{-t}^{t} e^{\tau|s|} \varphi\left(\left|x(s) - y(s)\right| e^{-\tau|s|}\right) ds \leq \varphi(d_n(x, y)) \left|\int_{-t}^{t} e^{\tau|s|} ds\right| \\ &\leq \varphi(d_n(x, y)) \int_{-|t|}^{|t|} e^{\tau|s|} ds \leq \frac{2}{\tau} \varphi(d_n(x, y)) e^{\tau|t|}, \quad t \in [-n; n]. \end{aligned} \tag{4.15}$$

Thus, for any $\tau \geq 2$, we obtain

$$d_n(B(x), B(y)) \leq \varphi(d_n(x, y)), \quad \forall\, x, y \in E,\ x \leq y,\ \text{for } n \in \mathbb{N}. \tag{4.16}$$

As before, from (iv) we have that $x_0 \leq Bx_0$. The conclusion follows again by Theorem 3.4. $\square$

Remark 4.3. It is worth mentioning that it could be of interest to extend the above technique for other metrical fixed-point theorems, see [15, 16], and so forth.

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