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*Research Article*

# A Viscosity of Cesàro Mean Approximation Methods for a Mixed Equilibrium, Variational Inequalities, and Fixed Point Problems

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We introduce a new iterative method for finding a common element of the set of solutions for mixed equilibrium problem, the set of solutions of the variational inequality for a $\beta$-inverse-strongly monotone mapping, and the set of fixed points of a family of finitely nonexpansive mappings in a real Hilbert space by using the viscosity and Cesàro mean approximation method. We prove that the sequence converges strongly to a common element of the above three sets under some mind conditions. Our results improve and extend the corresponding results of Kumam and Katchang (2009), Peng and Yao (2009), Shimizu and Takahashi (1997), and some authors.

## 1. Introduction

Throughout this paper, we assume that $H$ is a real Hilbert space with inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively and let $C$ be a nonempty closed convex subset of $H$. A mapping $T : C \to C$ is called *nonexpansive* if $\|Tx - Ty\| \leq \|x - y\|$, for all $x, y \in C$. We use $F(T)$ to denote the set of *fixed points* of $T$, that is, $F(T) = \{x \in C : Tx = x\}$. It is assumed throughout the paper that $T$ is a nonexpansive mapping such that $F(T) \neq \emptyset$. Recall that a self-mapping $f : C \to C$ is a *contraction* on $C$ if there exists a constant $\alpha \in [0,1)$ and $x, y \in C$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$.

Let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper extended real-valued function and $\phi$ be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. Ceng and Yao [1] considered the following *mixed equilibrium problem* for finding $x \in C$ such that

$$\phi(x, y) + \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\text{MEP}(\phi, \varphi)$. We see that $x$ is a solution of problem (1.1) implies that $x \in \text{dom}\,\varphi = \{x \in C \mid \varphi(x) < +\infty\}$. If $\varphi = 0$, then the mixed equilibrium problem (1.1) becomes the following *equilibrium problem* is to find $x \in C$ such that

$$\phi(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $\text{EP}(\phi)$. The mixed equilibrium problems include fixed point problems, variational inequality problems, optimization problems, Nash equilibrium problems, and the equilibrium problem as special cases. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.2). Some methods have been proposed to solve the equilibrium problem (see [2–14]).

Let $B : C \to H$ be a mapping. The *variational inequality problem*, denoted by $\text{VI}(C, B)$, is to find $x \in C$ such that

$$\langle Bx, y - x \rangle \geq 0, \tag{1.3}$$

for all $y \in C$. The variational inequality problem has been extensively studied in the literature. See, for example, [15, 16] and the references therein. A mapping $B$ of $C$ into $H$ is called *monotone* if

$$\langle Bx - By, x - y \rangle \geq 0, \tag{1.4}$$

for all $x, y \in C$. $B$ is called *$\beta$-inverse-strongly monotone* if there exists a positive real number $\beta > 0$ such that for all $x, y \in C$

$$\langle Bx - By, x - y \rangle \geq \beta \|Bx - By\|^2. \tag{1.5}$$

Let $A$ be a strongly positive linear bounded operator on $H$: that is, there is a constant $\overline{\gamma} > 0$ with property

$$\langle Ax, x \rangle \geq \overline{\gamma} \|x\|^2, \quad \forall x \in H. \tag{1.6}$$

A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space $H$:

$$\min_{x \in F(T)} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.7}$$

where $A$ is strongly positive linear bounded operator and $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). Moreover, it is shown in [17] that the sequence $\{x_n\}$ defined by the scheme

$$x_{n+1} = \epsilon_n \gamma f(x_n) + (1 - \epsilon_n A) T x_n, \tag{1.8}$$

converges strongly to $z = P_{F(T)}(I - A + \gamma f)(z)$.

In 1997, Shimizu and Takahashi [18] originally studied the convergence of an iteration process $\{x_n\}$ for a family of nonexpansive mappings in the framework of a real Hilbert space. They restate the sequence $\{x_n\}$ as follows:

$$x_{n+1} = \alpha_n x + (1-\alpha_n)\frac{1}{n+1}\sum_{j=0}^{n} T^j x_n, \quad \text{for } n = 0, 1, 2, \dots, \tag{1.9}$$

where $x_0$ and $x$ are all elements of $C$ and $\alpha_n$ is an appropriate in $[0,1]$. They proved that $\{x_n\}$ converges strongly to an element of fixed point of $T$ which is the nearest to $x$.

In 2007, Plubtieng and Punpaeng [19] proposed the following iterative algorithm:

$$\begin{gathered} \phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in H, \\ x_{n+1} = \epsilon_n \gamma f(x_n) + (I - \epsilon_n A)T u_n. \end{gathered} \tag{1.10}$$

They proved that if the sequence $\{\epsilon_n\}$ and $\{r_n\}$ of parameters satisfy appropriate condition, then the sequences $\{x_n\}$ and $\{u_n\}$ both converge to the unique solution $z$ of the variational inequality

$$\langle (A - \gamma f)z, x - z\rangle \geq 0, \quad \forall x \in F(T) \cap \mathrm{EP}(\phi), \tag{1.11}$$

which is the optimality condition for the minimization problem

$$\min_{x \in F(T)\cap \mathrm{EP}(\phi)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{1.12}$$

where $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

In 2008, Peng and Yao [20] introduced an iterative algorithm based on extragradient method which solves the problem of finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a family of finitely nonexpansive mappings and the set of the variational inequality for a monotone, Lipschitz continuous mapping in a real Hilbert space. The sequences generated by $v \in C$,

$$\begin{gathered} x_1 = x \in C, \\ \phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \gamma_n B u_n), \\ x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \lambda_n B y_n), \end{gathered} \tag{1.13}$$

for all $n \geq 1$, where $W_n$ is $W$-mapping. They proved the strong convergence theorems under some mind conditions.

In this paper, motivated by the above results and the iterative schemes considered in [9, 18–20], we introduce a new iterative process below based on viscosity and Cesàro mean approximation method for finding a common element of the set of fixed points of a family of finitely nonexpansive mappings, the set of solutions of the variational inequality problem for a $\beta$-inverse-strongly monotone mapping and the set of solutions of a mixed equilibrium problem in a real Hilbert space. Then, we prove strong convergence theorems which are connected with [5, 21–24]. We extend and improve the corresponding results of Kumam and Katchang [9], Peng and Yao [20], Shimizu and Takahashi [18] and some authors.

## 2. Preliminaries

Let $H$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$ and let $C$ be a nonempty closed convex subset of $H$. Then

$$\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle, \tag{2.1}$$

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{2.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. For every point $x \in H$, there exists a unique *nearest point* in $C$, denoted by $P_C x$, such that

$$\|x - P_C x\| \le \|x-y\|, \quad \forall y \in C. \tag{2.3}$$

$P_C$ is called the *metric projection* of $H$ onto $C$. It is well known that $P_C$ is a nonexpansive mapping of $H$ onto $C$ and satisfies

$$\langle x-y, P_C x - P_C y\rangle \ge \|P_C x - P_C y\|^2, \tag{2.4}$$

for every $x, y \in H$. Moreover, $P_C x$ is characterized by the following properties: $P_C x \in C$ and

$$\langle x - P_C x, y - P_C x\rangle \le 0, \tag{2.5}$$

$$\|x-y\|^2 \ge \|x - P_C x\|^2 + \|y - P_C x\|^2, \tag{2.6}$$

for all $x \in H, y \in C$. Let $B$ be a monotone mapping of $C$ into $H$. In the context of the variational inequality problem the characterization of projection (2.5) implies the following:

$$u \in \mathrm{VI}(C,B) \Longleftrightarrow u = P_C(u - \lambda Bu), \quad \lambda > 0. \tag{2.7}$$

It is also known that $H$ satisfies the Opial condition [25], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty}\|x_n - x\| < \liminf_{n\to\infty}\|x_n - y\|, \tag{2.8}$$

holds for every $y \in H$ with $x \ne y$.

A set-valued mapping $U : H \to 2^H$ is called *monotone* if for all $x, y \in H$, $f \in Ux$ and $g \in Uy$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $U : H \to 2^H$ is *maximal* if the graph of $G(U)$ of $U$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping $U$ is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(U)$ implies $f \in Ux$. Let $B$ be a monotone mapping of $C$ into $H$ and let $N_C\overline{y}$ be the *normal cone* to $C$ at $\overline{y} \in C$, that is, $N_C\overline{y} = \{w \in H : \langle u - \overline{y}, w\rangle \leq 0, \forall u \in C\}$ and define

$$U\overline{y} = \begin{cases} B\overline{y} + N_C\overline{y}, & \overline{y} \in C, \\ \emptyset, & \overline{y} \notin C. \end{cases} \tag{2.9}$$

Then $U$ is the *maximal monotone* and $0 \in U\overline{y}$ if and only if $\overline{y} \in \mathrm{VI}(C, B)$; see [26].

For solving the mixed equilibrium problem, let us give the following assumptions for a bifunction $\phi : C \times C \to \mathbb{R}$ and a proper extended real-valued function $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions:

(A1) $\phi(x, x) = 0$ for all $x \in C$;

(A2) $\phi$ is monotone, that is, $\phi(x, y) + \phi(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0}\phi(tz + (1 - t)x, y) \leq \phi(x, y)$;

(A4) for each $x \in C$, $y \mapsto \phi(x, y)$ is convex and lower semicontinuous;

(A5) for each $y \in C$, $x \mapsto \phi(x, y)$ is weakly upper semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\phi(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z); \tag{2.10}$$

(B2) $C$ is a bounded set.

We need the following lemmas for proving our main results.

**Lemma 2.1** (Peng and Yao [20]). *Let $C$ be a nonempty closed convex subset of $H$. Let $\phi : C\times C \to \mathbb{R}$ be a bifunction satisfies (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either (B1) or (B2) holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{z \in C : \phi(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z), \quad \forall y \in C\right\}, \tag{2.11}$$

*for all $x \in H$. Then, the following hold.*

(1) *For each $x \in H, T_r(x) \neq \emptyset$;*

(2) *$T_r$ is single-valued;*

(3) *$T_r$ is firmly nonexpansive, that is, for any $x, y \in H$, $\|T_rx - T_ry\|^2 \leq \langle T_rx - T_ry, x - y\rangle$;*

(4) *$F(T_r) = MEP(\phi, \varphi)$;*

(5) *$MEP(\phi, \varphi)$ is closed and convex.*

**Lemma 2.2** (Xu [27]). *Assume $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\alpha_n)a_n + \delta_n, \quad n \geq 0, \tag{2.12}$$

*where $\{\alpha_n\}$ is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that*

(1) $\sum_{n=1}^{\infty} \alpha_n = \infty$,

(2) $\limsup_{n\to\infty}(\delta_n/\alpha_n) \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

*Then* $\lim_{n\to\infty} a_n = 0$.

**Lemma 2.3** (Osilike and Igbokwe [28]). *Let $(C, \langle\cdot,\cdot\rangle)$ be an inner product space. Then for all $x, y, z \in C$ and $\alpha, \beta, \gamma \in [0,1]$ with $\alpha+\beta+\gamma = 1$, we have*

$$\|\alpha x + \beta y + \gamma z\|^2 = \alpha\|x\|^2 + \beta\|y\|^2 + \gamma\|z\|^2 - \alpha\beta\|x-y\|^2 - \alpha\gamma\|x-z\|^2 - \beta\gamma\|y-z\|^2. \tag{2.13}$$

**Lemma 2.4** (Suzuki [29]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space $X$ and let $\{\beta_n\}$ be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \geq 0$ and $\limsup_{n\to\infty}(\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \leq 0$. Then, $\lim_{n\to\infty}\|y_n - x_n\| = 0$.*

**Lemma 2.5** (Marino and Xu [17]). *Assume $A$ is a strongly positive linear bounded operator on a Hilbert space $H$ with coefficient $\overline{\gamma} > 0$ and $0 < \rho \leq \|A\|^{-1}$. Then $\|I - \rho A\| \leq 1 - \rho\overline{\gamma}$.*

**Lemma 2.6** (Bruck [30]). *Let $C$ be a nonempty bounded closed convex subset of a uniformly convex Banach space $E$ and $T : C \to C$ a nonexpansive mapping. For each $x \in C$ and the Cesàro means $T_n x = (1/(n+1))\sum_{i=0}^{n} T^i x$, then $\limsup_{n\to\infty}\|T_n x - T(T_n x)\| = 0$.*

## 3. Main Results

In this section, we show a strong convergence theorem for finding a common element of the set of fixed points of a family of finitely nonexpansive mappings, the set of solutions of mixed equilibrium problem and the set of solutions of a variational inequality problem for a $\beta$-inverse-strongly monotone mapping in a real Hilbert space by using the viscosity of Cesàro mean approximation method.

**Theorem 3.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert space $H$. Let $\phi$ be a bifunction of $C \times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5) and let $\varphi : C \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuos and convex function. Let $T^i : C \to C$ be a nonexpansive mappings for all $i = 1, 2, 3, \ldots, n$, such that $\Theta := \bigcap_{i=1}^{n} F(T^i) \cap VI(C,B) \cap MEP(\phi,\varphi) \neq \emptyset$. Let $f$ be a contraction of $C$*

*into itself with coefficient $\alpha \in (0,1)$ and let $B$ be a $\beta$-inverse-strongly monotone mapping of $C$ into $H$. Let $A$ be a strongly positive bounded linear self-adjoint on $H$ with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Assume that either $B_1$ or $B_2$ holds. Let $\{x_n\}$, $\{y_n\}$ and $\{u_n\}$ be sequences generated by $x_0 \in C$, $u_n \in C$ and*

$$\phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C,$$
$$y_n = \delta_n u_n + (1 - \delta_n) P_C(u_n - \lambda_n B u_n), \tag{3.1}$$
$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1 - \beta_n) I - \alpha_n A\right)\frac{1}{n+1}\sum_{i=0}^{n} T^i y_n, \quad \forall n \geq 0,$$

*where $\{\alpha_n\}$, $\{\beta_n\}$ and $\delta_n \subset (0,1)$, $\{\lambda_n\} \subset (0, 2\beta)$ and $\{r_n\} \subset (0,\infty)$ satisfy the following conditions:*

(i) $\sum_{n=0}^{\infty} \alpha_n = \infty$ *and* $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\lim_{n\to\infty}\delta_n = 0$,

(iii) $0 < \liminf_{n\to\infty}\beta_n \leq \limsup_{n\to\infty}\beta_n < 1$,

(iv) $\{\lambda_n\} \subset [a, b]$, $\exists a, b \in (0, 2\beta)$ *and* $\lim_{n\to\infty}|\lambda_{n+1} - \lambda_n| = 0$,

(v) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

*Then, $\{x_n\}$ converges strongly to $z \in \Theta$, where $z = P_\Theta(I - A + \gamma f)(z)$, which is the unique solution of the variational inequality*

$$\langle (A - \gamma f) z, x - z\rangle \geq 0, \quad \forall x \in \Theta. \tag{3.2}$$

*Proof.* Now, we have $\|(1 - \beta_n) I - \alpha_n A\| \leq 1 - \beta_n - \alpha_n \overline{\gamma}$ (see [9, page 479]). Since $\lambda_n \in (0, 2\beta)$ and $B$ is a $\beta$-inverse-strongly monotone mapping. For any $x, y \in C$, we have

$$\begin{aligned}
\left\|(I - \lambda_n B)x - (I - \lambda_n B)y\right\|^2 &= \left\|(x - y) - \lambda_n(Bx - By)\right\|^2 \\
&= \|x - y\|^2 - 2\lambda_n\langle x - y, Bx - By\rangle + \lambda_n^2\|Bx - By\|^2 \\
&\leq \|x - y\|^2 - 2\lambda_n\beta\|Bx - By\|^2 + \lambda_n^2\|Bx - By\|^2 \\
&= \|x - y\|^2 + \lambda_n(\lambda_n - 2\beta)\|Bx - By\|^2 \\
&\leq \|x - y\|^2.
\end{aligned} \tag{3.3}$$

It follows that $\|(I - \lambda_n B)x - (I - \lambda_n B)y\| \leq \|x - y\|$, hence $I - \lambda_n B$ is nonexpansive.

Let $x^* \in \Theta, T_{r_n}$ be a sequence of mapping defined as in Lemma 2.1 and $u_n = T_{r_n}x_n$, for all $n \geq 0$, we have

$$\|u_n - x^*\| = \|T_{r_n}x_n - T_{r_n}x^*\| \leq \|x_n - x^*\|. \tag{3.4}$$

By the fact that $P_C$ and $I - \lambda_n B$ are nonexpansive and $x^* = P_C(x^* - \lambda_n Bx^*)$, we get

$$\begin{aligned}
\|y_n - x^*\| &= \|\delta_n u_n + (1-\delta_n)P_C(u_n - \lambda_n Bu_n) - x^*\| \\
&\leq \delta_n\|u_n - x^*\| + (1-\delta_n)\|P_C(u_n - \lambda_n Bu_n) - P_C(x^* - \lambda_n Bx^*)\| \\
&\leq \delta_n\|u_n - x^*\| + (1-\delta_n)\|(I - \lambda_n B)u_n - (I - \lambda_n B)x^*\| \\
&\leq \delta_n\|u_n - x^*\| + (1-\delta_n)\|u_n - x^*\| \\
&= \|u_n - x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned} \tag{3.5}$$

Let $T_n = (1/(n+1))\sum_{i=0}^{n} T^i$; it follows that

$$\begin{aligned}
\|T_n x - T_n y\| &= \left\| \frac{1}{n+1}\sum_{i=0}^{n} T^i x - \frac{1}{n+1}\sum_{i=0}^{n} T^i y \right\| \\
&\leq \frac{1}{n+1}\sum_{i=0}^{n}\|T^i x - T^i y\| \\
&\leq \frac{1}{n+1}\sum_{i=0}^{n}\|x - y\| \\
&= \frac{n+1}{n+1}\|\mathrm{x} - y\| \\
&= \|x - y\|,
\end{aligned} \tag{3.6}$$

which implies that $T_n$ is nonexpansive. Since $x^* \in \Theta$, we have $T_n x^* = (1/(n+1))\sum_{i=0}^{n} T^i x^* = (1/(n+1))\sum_{i=0}^{n} x^* = x^*$, for all $x, y \in C$. By (3.5) and (3.6), we have

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(T_n y_n - x^*)\| \\
&\leq \alpha_n\|\gamma f(x_n) - Ax^*\| + \beta_n\|x_n - x^*\| + (1 - \beta_n - \alpha_n\overline{\gamma})\|y_n - x^*\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \alpha_n\|\gamma f(x_n) - Ax^*\| + \beta_n\|x_n - x^*\| + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|x_n - x^*\| \\
&\leq \alpha_n\gamma\|f(x_n) - f(x^*)\| + \alpha_n\|\gamma f(x^*) - Ax^*\| + \left(1 - \alpha_n\overline{\gamma}\right)\|x_n - x^*\| \\
&\leq \alpha_n\gamma\alpha\|x_n - x^*\| + \alpha_n\|\gamma f(x^*) - Ax^*\| + \left(1 - \alpha_n\overline{\gamma}\right)\|x_n - x^*\| \\
&= \left(1 - \alpha_n\left(\overline{\gamma} - \gamma\alpha\right)\right)\|x_n - x^*\| + \alpha_n\left(\overline{\gamma} - \gamma\alpha\right)\frac{\|\gamma f(x^*) - Ax^*\|}{\left(\overline{\gamma} - \gamma\alpha\right)} \\
&\leq \max\left\{\|x_0 - x^*\|, \frac{\|\gamma f(x^*) - Ax^*\|}{\left(\overline{\gamma} - \gamma\alpha\right)}\right\}.
\end{aligned}
\tag{3.7}
$$

Hence $\{x_n\}$ is bounded and also $\{u_n\}$, $\{y_n\}$ and $\{T_n y_n\}$ are bounded.

Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. Observing that $u_n = T_{r_n}x_n \in \operatorname{dom}\varphi$ and $u_{n+1} = T_{r_{n+1}}x_{n+1} \in \operatorname{dom}\varphi$, we get

$$
\phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \qquad \forall y \in C, \tag{3.8}
$$

$$
\phi(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y - u_{n+1}, u_{n+1} - x_{n+1}\rangle \geq 0, \qquad \forall y \in C. \tag{3.9}
$$

Take $y = u_{n+1}$ in (3.8) and $y = u_n$ in (3.9), by using condition (A2); it follows that

$$
\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}}\right\rangle \geq 0. \tag{3.10}
$$

Thus $\langle u_{n+1} - u_n, u_n - u_{n+1} + x_{n+1} - x_n + (1 - r_n/r_{n+1})(u_{n+1} - x_{n+1})\rangle \geq 0$. Without loss of generality, let us assume that there exists a nonnegative real number $c$ such that $r_n > c$, for all $n \geq 1$. Then, we have

$$
\|u_{n+1} - u_n\|^2 \leq \|u_{n+1} - u_n\|\left\{\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right|\|u_{n+1} - x_{n+1}\|\right\} \tag{3.11}
$$

and hence

$$
\begin{aligned}
\|u_{n+1} - u_n\| &\leq \|x_{n+1} - x_n\| + \frac{1}{r_{n+1}}|r_{n+1} - r_n|\|u_{n+1} - x_{n+1}\| \\
&\leq \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M_1,
\end{aligned}
\tag{3.12}
$$

where $M_1 = \sup\{\|u_n - x_n\| : n \in \mathbb{N}\}$. On the other hand, let $v_n = P_C(u_n - \lambda_n B u_n)$; it follows from the definition of $\{y_n\}$ that

$$\begin{aligned}
\|y_{n+1} - y_n\| &= \|\{\delta_{n+1}u_{n+1} + (1-\delta_{n+1})P_C(u_{n+1} - \lambda_{n+1}Bu_{n+1})\} \\
&\quad - \{\delta_n u_n + (1-\delta_n)P_C(u_n - \lambda_n B u_n)\}\| \\
&= \|\delta_{n+1}(u_{n+1} - u_n) + (\delta_{n+1} - \delta_n)u_n + (1-\delta_{n+1})P_C(u_{n+1} - \lambda_{n+1}Bu_{n+1}) \\
&\quad - (1-\delta_{n+1})P_C(u_n - \lambda_n B u_n) + (1-\delta_{n+1})P_C(u_n - \lambda_n B u_n) \\
&\quad - (1-\delta_n)P_C(u_n - \lambda_n B u_n)\| \\
&= \|\delta_{n+1}(u_{n+1} - u_n) + (\delta_{n+1} - \delta_n)u_n \\
&\quad + (1-\delta_{n+1})\{P_C(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - P_C(u_n - \lambda_n B u_n)\} \\
&\quad + (\delta_n - \delta_{n+1})P_C(u_n - \lambda_n B u_n)\| \\
&\le \delta_{n+1}\|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) \\
&\quad + (1-\delta_{n+1})\|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_n B u_n)\| \\
&= \delta_{n+1}\|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) \\
&\quad + (1-\delta_{n+1})\|(u_{n+1} - \lambda_{n+1}Bu_{n+1}) - (u_n - \lambda_{n+1} B u_n) \\
&\qquad\qquad + (u_n - \lambda_{n+1}Bu_n) - (u_n - \lambda_n B u_n)\| \\
&\le \delta_{n+1}\|u_{n+1} - u_n\| + |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) \\
&\quad + (1-\delta_{n+1})\{\|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|Bu_n\|\} \\
&= |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) + \|u_{n+1} - u_n\| + (1-\delta_{n+1})|\lambda_{n+1} - \lambda_n|\|Bu_n\| \\
&\le |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) + \|u_{n+1} - u_n\| + |\lambda_{n+1} - \lambda_n|\|Bu_n\| \\
&\le |\delta_{n+1} - \delta_n|(\|u_n\| + \|v_n\|) + \|x_{n+1} - x_n\| + \frac{1}{c}|r_{n+1} - r_n|M_1 \\
&\quad + |\lambda_{n+1} - \lambda_n|\|Bu_n\|.
\end{aligned} \tag{3.13}$$

We compute that

$$\begin{aligned}
\|T_{n+1}y_{n+1} - T_n y_n\| &\le \|T_{n+1}y_{n+1} - T_{n+1}y_n\| + \|T_{n+1}y_n - T_n y_n\| \\
&\le \|y_{n+1} - y_n\| + \left\|\frac{1}{n+2}\sum_{i=0}^{n+1} T^i y_n - \frac{1}{n+1}\sum_{i=0}^{n} T^i y_n\right\| \\
&= \|y_{n+1} - y_n\| + \left\|\frac{1}{n+2}\sum_{i=0}^{n} T^i y_n + \frac{1}{n+2}T^{n+1}y_n - \frac{1}{n+1}\sum_{i=0}^{n} T^i y_n\right\|
\end{aligned}$$

$$
\begin{aligned}
&= \|y_{n+1} - y_n\| + \left\| -\frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} T^i y_n + \frac{1}{n+2} T^{n+1} y_n \right\| \\
&\le \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} \|T^i y_n\| + \frac{1}{n+2} \|T^{n+1} y_n\| \\
&\le \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} \left( \|T^i y_n - T^i x^*\| + \|x^*\| \right) \\
&\quad + \frac{1}{n+2} \left( \|T^{n+1} y_n - T^{n+1} x^*\| + \|x^*\| \right) \\
&\le \|y_{n+1} - y_n\| + \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} \left( \|y_n - x^*\| + \|x^*\| \right) \\
&\quad + \frac{1}{n+2} \left( \|y_n - x^*\| + \|x^*\| \right) \\
&\le \|y_{n+1} - y_n\| + \frac{n+1}{(n+1)(n+2)} \left( \|y_n - x^*\| + \|x^*\| \right) \\
&\quad + \frac{1}{n+2} \|y_n - x^*\| + \frac{1}{n+2} \|x^*\| \\
&= \|y_{n+1} - y_n\| + \frac{2}{n+2} \|y_n - x^*\| + \frac{2}{n+2} \|x^*\| \\
&\le |\delta_{n+1} - \delta_n| (\|u_n\| + \|v_n\|) + \|x_{n+1} - x_n\| + \frac{1}{c} |r_{n+1} - r_n| M_1 \\
&\quad + |\lambda_{n+1} - \lambda_n| \|Bu_n\| + \frac{2}{n+2} \|y_n - x^*\| + \frac{2}{n+2} \|x^*\|.
\end{aligned}
\tag{3.14}
$$

Let $x_{n+1} = (1 - \beta_n) z_n + \beta_n x_n$; it follows that

$$
\begin{aligned}
z_n &= \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\
&= \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n A) T_n y_n}{1 - \beta_n},
\end{aligned}
\tag{3.15}
$$

and hence

$$
\begin{aligned}
\|z_{n+1} - z_n\| = \Bigg\| &\frac{\alpha_{n+1} \gamma f(x_{n+1}) + ((1 - \beta_{n+1}) I - \alpha_{n+1} A) T_{n+1} y_{n+1}}{1 - \beta_{n+1}} \\
&- \frac{\alpha_n \gamma f(x_n) + ((1 - \beta_n) I - \alpha_n A) T_n y_n}{1 - \beta_n} \Bigg\|
\end{aligned}
$$

$$
\begin{aligned}
&= \left\| \frac{\alpha_{n+1}\gamma f(x_{n+1})}{1-\beta_{n+1}} + \frac{(1-\beta_{n+1})T_{n+1}y_{n+1}}{1-\beta_{n+1}} \right.\\
&\quad - \frac{\alpha_{n+1}AT_{n+1}y_{n+1}}{1-\beta_{n+1}} - \frac{\alpha_n\gamma f(x_n)}{1-\beta_n}\\
&\quad \left. - \frac{(1-\beta_n)T_ny_n}{1-\beta_n} + \frac{\alpha_nAT_ny_n}{1-\beta_n} \right\|\\
&= \left\| \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}\right) + \frac{\alpha_n}{1-\beta_n}\left(AT_ny_n - \gamma f(x_n)\right) + T_{n+1}y_{n+1} - T_ny_n \right\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|AT_ny_n - \gamma f(x_n)\| + \|T_{n+1}y_{n+1} - T_ny_n\|\\
&\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|AT_ny_n - \gamma f(x_n)\|\\
&\quad + |\delta_{n+1}-\delta_n|(\|u_n\| + \|v_n\|) + \|x_{n+1}-x_n\| + \frac{1}{c}|r_{n+1}-r_n|M_1\\
&\quad + |\lambda_{n+1}-\lambda_n|\|Bu_n\| + \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\|.
\end{aligned}
\tag{3.16}
$$

Therefore,

$$
\begin{aligned}
\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\| \le{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - AT_{n+1}y_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|AT_ny_n - \gamma f(x_n)\|\\
&+ |\delta_{n+1}-\delta_n|(\|u_n\| + \|v_n\|) + \frac{1}{c}\|r_{n+1}-r_n\|M_1\\
&+ |\lambda_{n+1}-\lambda_n|\|Bu_n\| + \frac{2}{n+2}\|y_n - x^*\| + \frac{2}{n+2}\|x^*\|.
\end{aligned}
\tag{3.17}
$$

It follows from $n \to \infty$ and the conditions (i)–(v), that

$$
\limsup_{n\to\infty}(\|z_{n+1}-z_n\| - \|x_{n+1}-x_n\|) \le 0.
\tag{3.18}
$$

From Lemma 2.4 and (3.18), we obtain $\lim_{n\to\infty}\|z_n - x_n\| = 0$ and also

$$
\lim_{n\to\infty}\|x_{n+1}-x_n\| = \lim_{n\to\infty}(1-\beta_n)\|z_n - x_n\| = 0.
\tag{3.19}
$$

Next, we show that $\|x_n - u_n\| \to 0$ as $n \to \infty$. For $x^* \in \Theta$, we obtain

$$\begin{aligned}
\|u_n - x^*\|^2 &= \|T_{r_n}x_n - T_{r_n}x^*\|^2 \\
&\le \langle T_{r_n}x_n - T_{r_n}x^*, x_n - x^* \rangle \\
&= \langle u_n - x^*, x_n - x^* \rangle \\
&= \frac{1}{2}\left(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x^* - x_n + x^*\|^2\right) \\
&= \frac{1}{2}\left(\|u_n - x^*\|^2 + \|x_n - x^*\|^2 - \|u_n - x_n\|^2\right),
\end{aligned} \tag{3.20}$$

and hence

$$\|u_n - x^*\|^2 \le \|x_n - x^*\|^2 - \|u_n - x_n\|^2. \tag{3.21}$$

Since $\|y_n - x^*\| \le \|u_n - x^*\|$ and from Lemma 2.3 and (3.21), we obtain

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &= \left\|\alpha_n\gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_n y_n - x^*\right\|^2 \\
&= \left\|\alpha_n(\gamma f(x_n) - Ax^*) + \beta_n(x_n - x^*) + ((1-\beta_n)I - \alpha_n A)(T_n y_n - x^*)\right\|^2 \\
&\le \alpha_n\left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\overline{\gamma})\left\|y_n - x^*\right\|^2 \\
&\le \alpha_n\left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\overline{\gamma})\|u_n - x^*\|^2 \\
&\le \alpha_n\left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n\|x_n - x^*\|^2 + (1 - \beta_n - \alpha_n\overline{\gamma})\left(\|x_n - x^*\|^2 - \|x_n - u_n\|^2\right) \\
&\le \alpha_n\left\|\gamma f(x_n) - Ax^*\right\|^2 + \|x_n - x^*\|^2 - (1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - u_n\|^2.
\end{aligned} \tag{3.22}$$

Then, we have

$$\begin{aligned}
(1 - \beta_n - \alpha_n\overline{\gamma})\|x_n - u_n\|^2 &\le \alpha_n\left\|\gamma f(x_n) - Ax^*\right\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&= \alpha_n\left\|\gamma f(x_n) - Ax^*\right\|^2 + \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|).
\end{aligned} \tag{3.23}$$

By $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, (i) and (iv), imply that

$$\lim_{n\to\infty}\|u_n - x_n\| = 0. \tag{3.24}$$

Since $\liminf_{n\to\infty} r_n > 0$, we obtain

$$\lim_{n\to\infty}\left\|\frac{x_n - u_n}{r_n}\right\| = \lim_{n\to\infty}\frac{1}{r_n}\|x_n - u_n\| = 0. \tag{3.25}$$

Next, we show that $\lim_{n\to\infty}\|T_n y_n - x_n\| = 0$. Indeed, observe that

$$
\begin{aligned}
\|x_n - T_n y_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) + \beta_n x_n + ((1-\beta_n)I - \alpha_n A)T_n y_n - T_n y_n\| \\
&= \|x_n - x_{n+1}\| + \|\alpha_n \gamma f(x_n) - \alpha_n A T_n y_n + \alpha_n A T_n y_n + \beta_n x_n - \beta_n T_n y_n + \beta_n T_n y_n \\
&\qquad + ((1-\beta_n)I - \alpha_n A)T_n y_n - T_n y_n\| \\
&\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - A T_n y_n\| + \beta_n \|x_n - T_n y_n\|
\end{aligned}
\tag{3.26}
$$

and then

$$
\|x_n - T_n y_n\| \le \frac{1}{1-\beta_n}\|x_n - x_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n) - A T_n y_n\|. \tag{3.27}
$$

Since $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$, (i) and (iv), we get $\lim_{n\to\infty}\|x_n - T_n y_n\| = 0$.

Next, we show that $\lim_{n\to\infty}\|y_n - v_n\| = 0$, where $v_n = P_C(u_n - \lambda_n B u_n)$. From Lemma 2.3 and (3.3), we obtain

$$
\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n\|x_n - x^*\|^2 + ((1-\beta_n)I - \alpha_n A)\left\|T_n y_n - x^*\right\|^2 \\
&\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\overline{\gamma})\left\|y_n - x^*\right\|^2 \\
&= \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n\|x_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\overline{\gamma})\|\delta_n(u_n - x^*) + (1-\delta_n)\{P_C(u_n - \lambda_n B u_n) - P_C(x^* - \lambda_n B x^*)\}\|^2 \\
&\le \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\overline{\gamma})\delta_n\|u_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\overline{\gamma})(1-\delta_n)\|(u_n - \lambda_n B u_n) - (x^* - \lambda_n B x^*)\|^2 \\
&= \alpha_n \|\gamma f(x_n) - Ax^*\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\overline{\gamma})\delta_n\|u_n - x^*\|^2 \\
&\quad + (1-\beta_n - \alpha_n\overline{\gamma})(1-\delta_n)\|(u_n - x^*) - \lambda_n(B u_n - B x^*)\|^2 \\
&\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n\|x_n - x^*\|^2 + (1-\beta_n - \alpha_n\overline{\gamma})\delta_n\|u_n - x^*\|^2
\end{aligned}
$$

$$
\begin{aligned}
&+(1-\beta_n-\alpha_n\overline{\gamma})(1-\delta_n)\left\{\|x_n-x^*\|^2+\lambda_n(\lambda_n-2\beta)\|Bu_n-Bx^*\|^2\right\}\\
&\leq \alpha_n\left\|\gamma f(x_n)-Ax^*\right\|^2+(1-\alpha_n\overline{\gamma})\|x_n-x^*\|^2\\
&\quad+(1-\beta_n-\alpha_n\overline{\gamma})(1-\delta_n)\lambda_n(\lambda_n-2\beta)\|Bu_n-Bx^*\|^2\\
&\leq \alpha_n\left\|\gamma f(x_n)-Ax^*\right\|^2+\|x_n-x^*\|^2\\
&\quad+(1-\beta_n-\alpha_n\overline{\gamma})(1-\delta_n)a(b-2\beta)\|Bu_n-Bx^*\|^2.
\end{aligned}
\tag{3.28}
$$

It follows that

$$
\begin{aligned}
0&\leq (1-\beta_n-\alpha_n\overline{\gamma})(1-\delta_n)a(2\beta-b)\|Bu_n-Bx^*\|^2\\
&\leq \alpha_n\left\|\gamma f(x_n)-Ax^*\right\|^2+\|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\leq \alpha_n\left\|\gamma f(x_n)-Ax^*\right\|^2+\|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_{n+1}-x^*\|).
\end{aligned}
\tag{3.29}
$$

Since $\alpha_n \to 0$ and $\|x_{n+1}-x_n\| \to 0$, as $n \to \infty$, we obtain $\|Bu_n-Bx^*\| \to 0$ as $n \to \infty$. Using (2.1), we have

$$
\begin{aligned}
\|v_n-x^*\|^2&=\|P_C(u_n-\lambda_nBu_n)-P_C(x^*-\lambda_nBx^*)\|^2\\
&\leq \langle (u_n-\lambda_nBu_n)-(x^*-\lambda_nBx^*),v_n-x^*\rangle\\
&=\frac{1}{2}\left\{\|(u_n-\lambda_nBu_n)-(x^*-\lambda_nBx^*)\|^2+\|v_n-x^*\|^2\right\}\\
&\quad-\frac{1}{2}\left\{\|(u_n-\lambda_nBu_n)-(x^*-\lambda_nBx^*)-(v_n-x^*)\|^2\right\}\\
&=\frac{1}{2}\left\{\|u_n-x^*\|^2+\|v_n-x^*\|^2-\|(u_n-v_n)-\lambda_n(Bu_n-Bx^*)\|^2\right\}\\
&=\frac{1}{2}\Big\{\|u_n-x^*\|^2+\|v_n-x^*\|^2\\
&\qquad-\left(\|u_n-v_n\|^2+\lambda_n^2\|Bu_n-Bx^*\|^2-2\lambda_n\langle u_n-v_n,Bu_n-Bx^*\rangle\right)\Big\}\\
&\leq\frac{1}{2}\Big\{\|u_n-x^*\|^2+\|v_n-x^*\|^2-\|u_n-v_n\|^2-\lambda_n^2\|Bu_n-Bx^*\|^2\\
&\qquad+2\lambda_n\langle u_n-v_n,Bu_n-Bx^*\rangle\Big\},
\end{aligned}
\tag{3.30}
$$

so, we obtain

$$
\|v_n-x^*\|^2\leq\|u_n-x^*\|^2-\|u_n-v_n\|^2-\lambda_n^2\|Bu_n-Bx^*\|^2+2\lambda_n\langle u_n-v_n,Bu_n-Bx^*\rangle, \tag{3.31}
$$

and hence

$$\begin{aligned}
\|x_{n+1} - x^*\|^2 &\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\left\|y_n - x^*\right\|^2 \\
&= \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|\delta_n u_n + (1-\delta_n)v_n - x^*\|^2 \\
&\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\delta_n \|u_n - x^*\|^2 \\
&\quad + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)(1-\delta_n)\|v_n - x^*\|^2 \\
&\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\delta_n \|u_n - x^*\|^2 \\
&\quad + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|v_n - x^*\|^2 \\
&\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\delta_n \|u_n - x^*\|^2 \\
&\quad + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\Big\{\|u_n - x^*\|^2 - \|u_n - v_n\|^2 - \lambda_n^2\|Bu_n - Bx^*\|^2 \\
&\qquad\qquad + 2\lambda_n\langle u_n - v_n, Bu_n - Bx^*\rangle\Big\} \\
&\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \beta_n \|x_n - x^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\delta_n \|u_n - x^*\|^2 \\
&\quad + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|x_n - x^*\|^2 - \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|u_n - v_n\|^2 \\
&\quad - \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\lambda_n^2\|Bu_n - Bx^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)2\lambda_n\langle u_n - v_n, Bu_n - Bx^*\rangle \\
&\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \|x_n - x^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\delta_n \|u_n - x^*\|^2 \\
&\quad - \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|u_n - v_n\|^2 \\
&\quad - \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\lambda_n^2\|Bu_n - Bx^*\|^2 + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)2\lambda_n\|u_n - v_n\|\|Bu_n - Bx^*\|,
\end{aligned} \tag{3.32}$$

which implies that

$$\begin{aligned}
\left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\|u_n - v_n\|^2 &\le \alpha_n \left\|\gamma f(x_n) - Ax^*\right\|^2 + \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 \\
&\quad + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\delta_n\|u_n - x^*\|^2 \\
&\quad - \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)\lambda_n^2\|Bu_n - Bx^*\|^2 \\
&\quad + \left(1 - \beta_n - \alpha_n\overline{\gamma}\right)2\lambda_n\|u_n - v_n\|\|Bu_n - Bx^*\|
\end{aligned}$$

$$\leq \alpha_n \left\| \gamma f(x_n) - Ax^* \right\|^2 + \|x_n - x_{n+1}\| (\|x_n - x^*\| + \|x_{n+1} - x^*\|)$$
$$+ (1 - \beta_n - \alpha_n \overline{\gamma}) \delta_n \|u_n - x^*\|^2 - (1 - \beta_n - \alpha_n \overline{\gamma}) \lambda_n^2 \|Bu_n - Bx^*\|^2$$
$$+ (1 - \beta_n - \alpha_n \overline{\gamma}) 2\lambda_n \|u_n - v_n\| \|Bu_n - Bx^*\|. \tag{3.33}$$

Since $\|Bu_n - Bx^*\| \to 0$, $\|x_{n+1} - x_n\| \to 0$ as $n \to 0$ and the condition (i)–(iii), we have $\|u_n - v_n\| \to 0$ as $n \to 0$. At the same time, we note that

$$\|y_n - v_n\| = \|\delta_n (u_n - v_n)\| = \delta_n \|u_n - v_n\|, \tag{3.34}$$

since $\delta_n \to 0$, we have $\|y_n - v_n\| \to 0$ as $n \to \infty$. Consequently, we observe that

$$\|T_n y_n - y_n\| \leq \|T_n y_n - x_n\| + \|x_n - u_n\| + \|u_n - v_n\| + \|v_n - y_n\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{3.35}$$

It is easy to see that $P_\Theta(I - A + \gamma f)(z)$ is a contradiction of $H$ into itself. Hence $H$ is complete, there exists a unique fixed point $z \in H$, such that $z = P_\Theta(I - A + \gamma f)(z)$.

Next, we show that

$$\limsup_{n \to \infty} \langle (A - \gamma f) z, z - x_n \rangle \leq 0. \tag{3.36}$$

Indeed, we can choose a subsequence $\{y_{n_i}\}$ of $\{y_n\}$, such that

$$\lim_{i \to \infty} \langle (A - \gamma f) z, z - y_{n_i} \rangle = \limsup_{n \to \infty} \langle (A - \gamma f) z, z - y_n \rangle. \tag{3.37}$$

Since $\{y_{n_i}\}$ is bounded, there exists a subsequence $\{y_{n_{i_j}}\}$ of $\{y_{n_i}\}$ which converge weakly to $\overline{y} \in C$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \overline{y}$. From $\|T_n y_n - y_n\| \to 0$, we obtain $T_n y_{n_i} \rightharpoonup \overline{y}$.

Let us show $\overline{y} \in \mathrm{MEP}(\phi, \varphi)$. Since $u_n = T_{r_n} x_n \in \operatorname{dom} \varphi$, we obtain

$$\phi(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C. \tag{3.38}$$

From (A2), we also have

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \geq \phi(y, u_n), \quad \forall y \in C \tag{3.39}$$

and hence

$$\varphi(y) - \varphi(u_n) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \geq \phi(y, u_{n_i}), \quad \forall y \in C. \tag{3.40}$$

From $\|u_n - x_n\| \to 0, \|x_n - T_n y_n\| \to 0$ and $\|T_n y_n - y_n\| \to 0$, we get $u_{n_i} \to \overline{y}$. Since $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$, thus that from (A4) and the weakly lower semicontinuity of $\varphi$ that $\phi(y,\overline{y}) + \varphi(\overline{y}) - \varphi(y) \leq 0$, for all $y \in C$. For $t$ with $0 < t \leq 1$ and $x \in C$, let $x_t = tx + (1-t)\overline{y}$. Since $x \in C$ and $\overline{y} \in C$, we have $x_t \in C$ and hence $\phi(x_t,\overline{y}) + \varphi(\overline{y}) - \varphi(x_t) \leq 0$. So, from (A1), (A4) and the convexity of $\varphi$, we have

$$\begin{aligned} 0 &= \phi(x_t,x_t) + \varphi(x_t) - \varphi(x_t) \\ &\leq t\phi(x_t,x) + (1-t)\phi(x_t,\overline{y}) + t\varphi(x) + (1-t)\varphi(\overline{y}) - \varphi(x_t) \\ &= t\big(\phi(x_t,x) + \varphi(x) - \varphi(x_t)\big) + (1-t)\big(\phi(x_t,\overline{y}) + \varphi(\overline{y}) - \varphi(x_t)\big) \\ &\leq t\big(\phi(x_t,x) + \varphi(x) - \varphi(x_t)\big). \end{aligned} \tag{3.41}$$

Dividing by $t$, we get $\phi(x_t,x) + \varphi(x) - \varphi(x_t) \geq 0$. From (A3) and the weakly lower semicontinuity of $\varphi$, we have $\phi(\overline{y},y) + \varphi(y) - \varphi(\overline{y}) \geq 0$, for all $y \in C$ and hence $\overline{y} \in \mathrm{MEP}(\phi,\varphi)$.

Next, we show that $\overline{y} \in F(T_n) = (1/(n+1))\sum_{i=0}^{n} F(T^i)$. Assume that $\overline{y} \notin (1/(n+1))\sum_{i=0}^{n} F(T^i)$, since $y_{n_i} \rightharpoonup \overline{y}$ and $T_n\overline{y} \neq \overline{y}$. From Opial's condition, we have

$$\begin{aligned} \liminf_{i\to\infty}\|y_{n_i} - \overline{y}\| &< \liminf_{i\to\infty}\|y_{n_i} - T_n\overline{y}\| \\ &\leq \liminf_{i\to\infty}\big(\|y_{n_i} - T_n y_{n_i}\| + \|T_n y_{n_i} - T_n\overline{y}\|\big) \\ &\leq \liminf_{i\to\infty}\|y_{n_i} - \overline{y}\|, \end{aligned} \tag{3.42}$$

which is a contradiction. Thus, we obtain $\overline{y} \in F(T_n) = (1/(n+1))\sum_{i=0}^{n} F(T^i)$.

Now, let us show that $v \in \mathrm{VI}(C,B)$. Let $U : H \to 2^H$ be a set-valued mapping is defined by

$$Uv = \begin{cases} Bv + N_C v, & v \in C, \\ \emptyset, & v \notin C, \end{cases} \tag{3.43}$$

where $N_C v$ is the normal cone to $C$ at $v \in C$. We have $U$ is maximal monotone and $0 \in Uv$ if and only if $v \in \mathrm{VI}(C,B)$. Let $(v,w) \in G(U)$, hence $w - Bv \in N_C v$ and $v_n \in C$, we have $\langle v - v_n, w - Bv\rangle \geq 0$. On the other hand, from $v_n = P_C(u_n - \lambda_n B u_n)$, we have

$$\langle v - v_n, v_n - (u_n - \lambda_n B u_n)\rangle \geq 0, \tag{3.44}$$

that is

$$\left\langle v - v_n, \frac{v_n - u_n}{\lambda_n} + Bu_n \right\rangle \geq 0. \tag{3.45}$$

Therefor, we have

$$\begin{aligned}
\langle v - v_{n_i}, w \rangle &\geq \langle v - v_{n_i}, Bv \rangle \\
&\geq \langle v - v_{n_i}, Bv \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} + Bu_{n_i} \right\rangle \\
&= \left\langle v - v_{n_i}, Bv - \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} - Bu_{n_i} \right\rangle \\
&= \langle v - v_{n_i}, Bv - Bv_{n_i} \rangle + \langle v - v_{n_i}, Bv_{n_i} - Bu_{n_i} \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \langle v - v_{n_i}, Bv_{n_i} - Bu_{n_i} \rangle - \left\langle v - v_{n_i}, \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\rangle \\
&\geq \|v - v_{n_i}\| \|Bv_{n_i} - Bu_{n_i}\| - \|v - v_{n_i}\| \left\| \frac{v_{n_i} - u_{n_i}}{\lambda_{n_i}} \right\|.
\end{aligned} \tag{3.46}$$

Noting that $\|v_{n_i} - u_{n_i}\| \to 0$ as $i \to \infty$ and $B$ is $\beta$-inverse-strongly monotone, hence from (3.46), we obtain $\langle v - \overline{y}, w \rangle \geq 0$ as $i \to \infty$. Since $U$ is maximal monotone, we have $\overline{y} \in U^{-1}0$, and hence $\overline{y} \in \mathrm{VI}(C, B)$. Therefore, $\overline{y} \in \Theta := \bigcap_{i=1}^{n} F(T^i) \cap \mathrm{VI}(C, B) \cap \mathrm{MEP}(\phi, \varphi)$.

Since $z = P_\Theta(I - A + \gamma f)(z)$, we have

$$\begin{aligned}
\limsup_{n \to \infty} \langle (\gamma f - A)z, x_n - z \rangle &= \limsup_{n \to \infty} \langle (\gamma f - A)z, T_n y_n - z \rangle \\
&= \limsup_{i \to \infty} \langle (\gamma f - A)z, T_n y_{n_i} - z \rangle \\
&= \langle (\gamma f - A)z, \overline{y} - z \rangle \leq 0.
\end{aligned} \tag{3.47}$$

Finally, we show that $\{x_n\}$ converge strongly to $z$, we obtain that

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \left\| \alpha_n \gamma f(x_n) + \beta_n x_n + ((1 - \beta_n)I - \alpha_n A) T_n y_n - z \right\|^2 \\
&= \left\| \alpha_n (\gamma f(x_n) - Az) + \beta_n (x_n - z) + ((1 - \beta_n)I - \alpha_n A)(T_n y_n - z) \right\|^2 \\
&= \alpha_n^2 \left\| \gamma f(x_n) - Az \right\|^2 + \left\| \beta_n (x_n - z) + ((1 - \beta_n)I - \alpha_n A)(T_n y_n - z) \right\|^2
\end{aligned}$$

$$
\begin{aligned}
&+2\left\langle\beta_n(x_n-z)+\left((1-\beta_n)I-\alpha_nA\right)(T_ny_n-z),\alpha_n(\gamma f(x_n)-Az)\right\rangle\\
&\le\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2+\left\{\beta_n\|x_n-z\|+(1-\beta_n-\alpha_n\overline{\gamma})\|y_n-z\|\right\}^2\\
&\quad+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(x_n)-Az\right\rangle+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\left\langle T_ny_n-z,\gamma f(x_n)-Az\right\rangle\\
&\le\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2+\left\{\beta_n\|x_n-z\|+(1-\beta_n-\alpha_n\overline{\gamma})\|x_n-z\|\right\}^2\\
&\quad+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(x_n)-\gamma f(z)\right\rangle+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(z)-Az\right\rangle\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\left\langle T_ny_n-z,\gamma f(x_n)-\gamma f(z)\right\rangle\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\left\langle T_ny_n-z,\gamma f(z)-Az\right\rangle\\
&\le\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2+(1-\alpha_n\overline{\gamma})^2\|x_n-z\|^2+2\alpha_n\beta_n\gamma\|x_n-z\|\|f(x_n)-f(z)\|\\
&\quad+2\alpha_n\beta_n\left\langle x_n-z,\gamma f(z)-Az\right\rangle+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\gamma\|T_ny_n-z\|\|f(x_n)-f(z)\|\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\left\langle T_ny_n-z,\gamma f(z)-Az\right\rangle\\
&\le\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2+(1-\alpha_n\overline{\gamma})^2\|x_n-z\|^2+2\alpha_n\beta_n\gamma\alpha\|x_n-z\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\gamma\alpha\|x_n-z\|^2\\
&\quad+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\left\langle T_ny_n-z,\gamma f(z)-Az\right\rangle\\
&=\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2+\left(1-2\alpha_n\overline{\gamma}+\alpha_n^2\overline{\gamma}^2+2\alpha_n\gamma\alpha-2\alpha_n^2\overline{\gamma}\gamma\alpha\right)\|x_n-z\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\left\langle T_ny_n-z,\gamma f(z)-Az\right\rangle\\
&\le\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\|x_n-z\|^2+\alpha_n^2\left\|\gamma f(x_n)-Az\right\|^2\\
&\quad+2\alpha_n\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2\alpha_n(1-\beta_n-\alpha_n\overline{\gamma})\left\langle T_ny_n-z,\gamma f(z)-Az\right\rangle\\
&\le\left\{1-\alpha_n\left(2\overline{\gamma}-\alpha_n\overline{\gamma}^2-2\gamma\alpha+2\alpha_n\overline{\gamma}\gamma\alpha\right)\right\}\|x_n-z\|^2+\alpha_n\sigma_n,
\end{aligned}
\tag{3.48}
$$

where $\sigma_n=\alpha_n\|\gamma f(x_n)-Az\|^2+2\beta_n\langle x_n-z,\gamma f(z)-Az\rangle+2(1-\beta_n-\alpha_n\overline{\gamma})\langle T_ny_n-z,\gamma f(z)-Az\rangle$. By (3.47), (i) and (iii), we get $\limsup_{n\to\infty}\sigma_n\le 0$. Applying Lemma 2.2 to (3.48) we conclude that $x_n\to z$. This completes the proof. □

Using Theorem 3.1, we obtain the following corollaries.

**Corollary 3.2.** *Let $C$ be a nonempty closed convex subset of a real Hilbert Space $H$. Let $\phi$ be a bifunction of $C\times C$ into real numbers $\mathbb{R}$ satisfying (A1)–(A5). Let $f$ be a contraction of $C$ into itself* *with coefficient* $\alpha \in (0,1)$ *and let* $T : C \to C$ *be a nonexpansive mapping such that* $\Theta := F(T) \cap VI(C,B) \cap EP(\phi) \neq \emptyset$. *Let* $B$ *be a* $\beta$*-inverse-strongly monotone mapping of* $C$ *into* $H$. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$ *and*

$$\begin{aligned}
\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \lambda_n B u_n), \\
x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n) T y_n, \quad \forall n \geq 0,
\end{aligned} \tag{3.49}$$

*where* $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\lambda_n\} \subset (0,2\beta)$, *for all* $n \geq 0$ *satisfy the condition (i), (iii)–(v). Then,* $\{x_n\}$ *converges strongly to* $z \in \Theta$ *where* $z = P_\Theta z$.

*Proof.* Taking $T^i = T$ for $i = 0, 1, \ldots, n$, $\delta_n = 0$, $A = I$ and $\varphi \equiv 0$ in Theorem 3.1, we can conclude the desired conclusion easily. □

**Corollary 3.3.** *Let* $C$ *be a nonempty closed convex subset of a real Hilbert Space* $H$. *Let* $\phi$ *be a bifunction of* $C \times C$ *into real numbers* $\mathbb{R}$ *satisfying (A1)–(A5) and let* $T : C \to C$ *be a nonexpansive mapping such that* $\Theta := F(T) \cap VI(C,B) \cap EP(\phi) \neq \emptyset$. *Let* $B$ *be a* $\beta$*-inverse-strongly monotone mapping of* $C$ *into* $H$. *Let* $\{x_n\}$, $\{y_n\}$ *and* $\{u_n\}$ *be sequences generated by* $x_0 \in C$, $u_n \in C$ *and*

$$\begin{aligned}
\phi(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \lambda_n B u_n), \\
x_{n+1} = \alpha_n v + \beta_n x_n + (1 - \beta_n - \alpha_n) T y_n, \quad \forall n \geq 0,
\end{aligned} \tag{3.50}$$

*where* $\{\alpha_n\}, \{\beta_n\} \subset (0,1), \{\lambda_n\} \subset (0,2\beta)$, *for all* $n \geq 0$ *satisfy the condition (i), (iii)–(v). Then,* $\{x_n\}$ *converges strongly to* $z \in \Theta$, *where* $z = P_\Theta z$.

*Proof.* Taking $\gamma = 1$ and $f(x) = v$, for all $x \in C$ in Corollary 3.2, we can conclude the desired conclusion easily. □

## 4. Applications to Optimization Problem

Let $H$ be a real Hilbert space, $C$ a nonempty closed convex subset of $H$, $A$ a strongly positive linear bounded operator on $H$ with a constant $\overline{\gamma} > 0$, and $T : C \to C$ be a nonexpansive mapping. In this section we will utilize the results present in section main results to study the following *optimization problem:*

$$\min_{x \in F(T)} \frac{1}{2}\langle Ax, x\rangle - h(x), \tag{4.1}$$

where $F(T)$ is the set of fixed points of $T$ in $C$ and $h$ is a potential function for $\gamma f$ (i.e., $h'(x) = \gamma f(x)$ for $x \in H$). We have the following theorem.

**Theorem 4.1.** *Let $C$ be a nonempty closed convex subset of a real Hilbert Space $H$. Let $T : C \to C$ be a nonexpansive mappings and let $f$ be a contraction of $C$ into itself with coefficient $\alpha \in (0,1)$. Let $A$ be a strongly positive linear bounded operator on $H$ with coefficient $\overline{\gamma} > 0$ and $0 < \gamma < \overline{\gamma}/\alpha$. Let $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ satisfying condition (i) and (iii) in Theorem 3.1. If $F(T)$ is a nonempty compact subset of $C$, then for each $n \geq 0$ there is a unique $x_n \in C$ such that*

$$x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + \left((1-\beta_n)I - \alpha_n A\right)T x_n, \quad \forall n \geq 0, \tag{4.2}$$

*and the sequence $\{x_n\}$ converges strongly to some point $z \in F(T)$, which solves the following optimization problem* (4.1)*.*

*Proof.* Taking $\phi \equiv 0$, $B \equiv 0$ in Corollary 3.2, we get $P_C = I$ and we also have $y_n = x_n$. Hence the sequence $\{x_n\}$ converges strongly to some point $z \in F(T)$ which is the unique solution of the following variational inequality:

$$\left\langle (A - \gamma f)z, x - z \right\rangle \geq 0, \quad x \in F(T). \tag{4.3}$$

Since $T$ is nonexpansive, then $F(T)$ is convex. Again by the assumption that $F(T)$ is compact, therefore, it is a compact and convex subset of $C$, and

$$\frac{1}{2}\langle Ax, x\rangle - h(x) : C \longrightarrow \mathcal{R} \tag{4.4}$$

is a continuous mapping. By virtue of the well-know Weierstrass theorem, there exists a point $z^* \in F(T)$ which is a minimal point of optimization problem (4.1). As is know to all, (4.3) is the optimality necessary condition (see Xu [31]) for the optimization problem (4.1). Then, we also have

$$\left\langle (A - \gamma f)z^*, x - z^* \right\rangle \geq 0, \quad x \in F(T). \tag{4.5}$$

Since $z$ is the unique solution of (4.3), therefor, $z = z^*$. This complete the proof of Theorem 4.1. □

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## References

[1] L.-C. Ceng and J.-C. Yao, “A hybrid iterative scheme for mixed equilibrium problems and fixed point problems,” *Journal of Computational and Applied Mathematics*, vol. 214, no. 1, pp. 186–201, 2008.

[2] R. S. Burachik, J. O. Lopes, and G. J. P. Da Silva, “An inexact interior point proximal method for the variational inequality problem,” *Computational & Applied Mathematics*, vol. 28, no. 1, pp. 15–36, 2009.

[3] E. Blum and W. Oettli, "From optimization and variational inequalities to equilibrium problems," *The Mathematics Student*, vol. 63, no. 1–4, pp. 123–145, 1994.

[4] S. D. Flåm and A. S. Antipin, "Equilibrium programming using proximal-like algorithms," *Mathematical Programming*, vol. 78, no. 1, pp. 29–41, 1997.

[5] P. Kumam, "Strong convergence theorems by an extragradient method for solving variational inequalities and equilibrium problems in a Hilbert space," *Turkish Journal of Mathematics*, vol. 33, no. 1, pp. 85–98, 2009.

[6] P. Kumam, "A hybrid approximation method for equilibrium and fixed point problems for a monotone mapping and a nonexpansive mapping," *Nonlinear Analysis: Hybrid Systems*, vol. 2, no. 4, pp. 1245–1255, 2008.

[7] P. Kumam, "A new hybrid iterative method for solution of equilibrium problems and fixed point problems for an inverse strongly monotone operator and a nonexpansive mapping," *Journal of Applied Mathematics and Computing*, vol. 29, no. 1-2, pp. 263–280, 2009.

[8] P. Kumam and C. Jaiboon, "A new hybrid iterative method for mixed equilibrium problems and variational inequality problem for relaxed cocoercive mappings with application to optimization problems," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 510–530, 2009.

[9] P. Kumam and P. Katchang, "A viscosity of extragradient approximation method for finding equilibrium problems, variational inequalities and fixed point problems for nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 475–486, 2009.

[10] A. Moudafi and M. Théra, "Proximal and dynamical approaches to equilibrium problems," in *Ill-Posed Variational Problems and Regularization Techniques*, vol. 477 of *Lecture Notes in Econom. and Math. Systems*, pp. 187–201, Springer, Berlin, Germany, 1999.

[11] Z. Wang and Y. Su, "Strong convergence theorems of common elements for equilibrium problems and fixed point problems in Banach paces," *Journal of Applied Mathematics & Informatics*, vol. 28, no. 3-4, pp. 783–796, 2010.

[12] R. Wangkeeree and R. Wangkeeree, "Strong convergence of the iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems of an infinite family of nonexpansive mappings," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 719–733, 2009.

[13] Y. Yao, M. A. Noor, S. Zainab, and Y.-C. Liou, "Mixed equilibrium problems and optimization problems," *Journal of Mathematical Analysis and Applications*, vol. 354, no. 1, pp. 319–329, 2009.

[14] Y. Yao, Y. J. Cho, and R. Chen, "An iterative algorithm for solving fixed point problems, variational inequality problems and mixed equilibrium problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 71, no. 7-8, pp. 3363–3373, 2009.

[15] J.-C. Yao and O. Chadli, "Pseudomonotone complementarity problems and variational inequalities," in *Handbook of Generalized Convexity and Generalized Monotonicity*, vol. 76 of *Nonconvex Optim. Appl.*, pp. 501–558, Springer, New York, NY, USA, 2005.

[16] L. C. Zeng, S. Schaible, and J. C. Yao, "Iterative algorithm for generalized set-valued strongly nonlinear mixed variational-like inequalities," *Journal of Optimization Theory and Applications*, vol. 124, no. 3, pp. 725–738, 2005.

[17] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.

[18] T. Shimizu and W. Takahashi, "Strong convergence to common fixed points of families of nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 211, no. 1, pp. 71–83, 1997.

[19] S. Plubtieng and R. Punpaeng, "A general iterative method for equilibrium problems and fixed point problems in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 336, no. 1, pp. 455–469, 2007.

[20] J.-W. Peng and J.-C. Yao, "Strong convergence theorems of iterative scheme based on the extragradient method for mixed equilibrium problems and fixed point problems," *Mathematical and Computer Modelling*, vol. 49, no. 9-10, pp. 1816–1828, 2009.

[21] Y. Su, M. Shang, and X. Qin, "An iterative method of solution for equilibrium and optimization problems," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 69, no. 8, pp. 2709–2719, 2008.

[22] R. Wangkeeree, "An extragradient approximation method for equilibrium problems and fixed point problems of a countable family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 134148, 17 pages, 2008.

[23] Y.-C. Liou and Y. Yao, "Iterative algorithms for nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2008, Article ID 384629, 10 pages, 2008.

[24] Y. Yao, Y.-C. Liou, and J.-C. Yao, "Convergence theorem for equilibrium problems and fixed point

problems of infinite family of nonexpansive mappings," *Fixed Point Theory and Applications*, vol. 2007, Article ID 64363, 12 pages, 2007.

[25] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, pp. 591–597, 1967.

[26] R. T. Rockafellar, "On the maximality of sums of nonlinear monotone operators," *Transactions of the American Mathematical Society*, vol. 149, pp. 75–88, 1970.

[27] H.-K. Xu, "Viscosity approximation methods for nonexpansive mappings," *Journal of Mathematical Analysis and Applications*, vol. 298, no. 1, pp. 279–291, 2004.

[28] M. O. Osilike and D. I. Igbokwe, "Weak and strong convergence theorems for fixed points of pseudocontractions and solutions of monotone type operator equations," *Computers & Mathematics with Applications*, vol. 40, no. 4-5, pp. 559–567, 2000.

[29] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

[30] R. E. Bruck, "On the convex approximation property and the asymptotic behavior of nonlinear contractions in Banach spaces," *Israel Journal of Mathematics*, vol. 38, no. 4, pp. 304–314, 1981.

[31] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.