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Research Article

Existence of Positive Solutions for Nonlocal Fourth-Order Boundary Value Problem with Variable Parameter

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By using the Krasnoselskii's fixed point theorem and operator spectral theorem, the existence of positive solutions for the nonlocal fourth-order boundary value problem with variable parameter $u^{(4)}(t) + B(t)u''(t) = \lambda f(t, u(t), u''(t))$, $0 < t < 1$, $u(0) = u(1) = \int_0^1 p(s)u(s)ds$, $u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds$ is considered, where $p, q \in L^1[0,1]$, $\lambda > 0$ is a parameter, and $B \in C[0,1]$, $f \in C([0,1] \times [0,\infty) \times (-\infty, 0], [0,\infty))$.

1. Introduction

The existence of positive solutions for nonlinear fourth-order multipoint boundary value problems has been studied by many authors using nonlinear alternatives of Leray-Schauder, the fixed point theory, and the method of upper and lower solutions (see, e.g., [1–15] and references therein). The multipoint boundary value problem is in fact a special case of the boundary value problem with integral boundary conditions.

Recently, Bai [16] studied the existence of positive solutions of nonlocal fourth-order boundary value problem

$$\begin{gathered} u^{(4)}(t) + \beta u''(t) = \lambda f\left(t, u(t), u''(t)\right), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds. \end{gathered} \tag{1.1}$$

under the assumption:

(A1) $\lambda > 0$ and $0 < \beta < \pi^2$,

(A2) $f \in C([0,1]\times[0,\infty)\times(-\infty,0],[0,\infty))$, $p, q \in L^1[0,1]$, $p(s) \geq 0$, $q(s) \geq 0$, $\int_0^1 p(s)ds < 1$, $\int_0^1 q(s)\sin\sqrt{\beta}s ds + \int_0^1 q(s)\sin\sqrt{\beta}(1-s)ds < \sin\sqrt{\beta}$.

In this paper, we study the above generalizing form with variable parameters BVP

$$u^{(4)}(t) + B(t)u''(t) = \lambda f(t, u(t), u''(t)), \quad 0 < t < 1,$$
$$u(0) = u(1) = \int_0^1 p(s)u(s)ds, \tag{1.2}$$
$$u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds,$$

where $B \in C[0,1]$, $\lambda > 0$ is a parameter.

Obviously, BVP(1.1) can be regarded as the special case of BVP(1.2) with $B(t) = \beta$. Since the parameters $B(t)$ is variable, we cannot expect to transform directly BVP(1.2) into an integral equation as in [16]. We will apply the cone fixed point theory, combining with the operator spectra theorem to establish the existence of positive solutions of BVP(1.2). Our results generalize the main result in [16].

Let $\beta = \inf_{t\in[0,1]} B(t)$, and we assume that the following conditions hold throughout the paper:

(H1) $B \in C[0,1]$ and $0 < \beta < \pi^2$,

(H2) $f \in C([0,1] \times [0,\infty) \times (-\infty,0],[0,\infty))$, $p, q \in L^1[0,1]$, $p(s) \geq 0$, $q(s) \geq 0$ and $\int_0^1 p(s)ds < 1$, $\int_0^1 q(s)\sin\sqrt{\beta}s ds + \int_0^1 q(s)\sin\sqrt{\beta}(1-s)ds < \sin\sqrt{\beta}$.

2. The Preliminary Lemmas

Set $\lambda_1 = 0$, $-\pi^2 < \lambda_2 = -\beta < 0$ and

$$\delta_1 = 1 - \int_0^1 p(s)ds, \qquad \delta_2 = \sin\sqrt{\beta} - \int_0^1 q(s)\sin\sqrt{\beta}s ds - \int_0^1 q(s)\sin\sqrt{\beta}(1-s)ds. \tag{2.1}$$

By (H1), (H2), we get $\delta_i \neq 0$, $i = 1, 2$. Denote by $K_1(t,s)$ the Green's function of the problem

$$-u''(t) + \lambda_1 u(t) = 0, \quad 0 < t < 1,$$
$$u(0) = u(1) = \int_0^1 p(s)u(s)ds \tag{2.2}$$

and $K_2(t,s)$ the Green's function of the problem

$$
\begin{gathered}
-u''(t) + \lambda_2 u(t) = 0, \quad 0 < t < 1, \\
u(0) = u(1) = \int_0^1 q(s)u(s)ds.
\end{gathered}
\tag{2.3}
$$

Then, carefully calculation yield

$$
\begin{gathered}
K_1(t,s) = G_1(t,s) + \rho_1 \int_0^1 G_1(s,x)p(x)dx, \\
K_2(t,s) = G_2(t,s) + \rho_2(t) \int_0^1 G_2(s,x)q(x)dx, \\
G_1(t,s) = \begin{cases} t(1-s), & 0 \le t \le s \le 1, \\ s(1-t), & 0 \le s \le t \le 1, \end{cases} \\
G_2(t,s) = \begin{cases} \dfrac{\sin\sqrt{\beta}t \sin\sqrt{\beta}(1-s)}{\sqrt{\beta}\sin\sqrt{\beta}}, & 0 \le t \le s \le 1, \\[2ex] \dfrac{\sin\sqrt{\beta}s \sin\sqrt{\beta}(1-t)}{\sqrt{\beta}\sin\sqrt{\beta}}, & 0 \le s \le t \le 1, \end{cases} \\
\rho_1 = \frac{1}{\delta_1}, \qquad \rho_2(t) = \frac{\sin\sqrt{\beta}t + \sin\sqrt{\beta}(1-t)}{\delta_2}.
\end{gathered}
\tag{2.4}
$$

Lemma 2.1 (see [16]). *Suppose that (A1), (A2) hold. Then, for any $h \in C[0,1]$, u solves the problem*

$$
\begin{gathered}
u^{(4)}(t) + \beta u''(t) = h(t), \quad 0 < t < 1, \\
u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\
u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds,
\end{gathered}
\tag{2.5}
$$

if and only if $u(t) = \int_0^1 \int_0^1 K_1(t,s)K_2(s,\tau)h(\tau)d\tau ds$.

Let $Y = C[0,1]$, $Y_+ = \{u \in Y : u(t) \ge 0, t \in [0,1]\}$, and $\|u\|_0 = \max_{0\le t\le 1}|u(t)|$, for $u \in Y$. $X = \{u \in C^2[0,1] : u(0) = u(1) = \int_0^1 p(s)u(s)ds,\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds\}$, $\|u\|_1 = \|u''\|_0$, $\|u\|_2 = \|u\|_0 + \|u\|_1$, for $u \in X$.

It is easy to show that $\|u\|_1$, $\|u\|_2$ are norms on X.

Lemma 2.2 (see [16]). *$\|\cdot\|_1 \le \|\cdot\|_2 \le (1+\delta_1)\|\cdot\|_1$ and $(X, \|\cdot\|_2)$ is a Banach space.*

Lemma 2.3 (see [5]). *Assume that (A1), (A2) hold. Then,*

(i) *$K_i(t,s) \ge 0$, for $t, s \in [0,1]$, $i = 1,2$; $K_i(t,s) > 0$, for $t, s \in (0,1)$, $i = 1,2$,*

(ii) *$G_i(t,s) \ge b_i G_i(t,t) G_i(s,s)$, $G_i(t,s) \le C_i G_i(s,s)$ for $t, s \in [0,1]$, $i = 1,2$,*

where $C_1 = 1$, $b_1 = 1$; $C_2 = 1/\sin\sqrt{\beta}$, $b_2 = \sqrt{\beta}\sin\sqrt{\beta}$.

Denote

$$d_i = \min_{1/4 \le t \le 3/4} b_i G_i(t,t) \quad (i = 1,2),$$
$$\xi = \frac{\min_{1/4 \le t \le 3/4} \rho_2(t)}{\max_{1/4 \le t \le 3/4} \rho_2(t)}, \tag{2.6}$$
$$D_i = \max_{t \in [0,1]} \int_0^1 K_i(t,s)\,ds \quad (i = 1,2).$$

Computations yield the following results.

Lemma 2.4 (see [3]). $D_i^1 = \max_{t \in [0,1]} \int_0^1 G_i(t,s)\,ds > 0$ $(i = 1,2)$

(i) *when $\lambda_i > 0$, $D_i^1 = (1/\lambda_i)(1 - 1/\cos(\omega_i/2))$,*

(ii) *when $\lambda_i = 0$, $D_i^1 = 1/8$,*

(iii) *when $-\pi^2 < \lambda_i < 0$, $D_i^1 = (1/\lambda_i)(1 - 1/\cos(\omega_i/2))$.*

Lemma 2.5 (see [16]). *Suppose that (A1), (A2) hold and $\rho_2(t)$, d_i, ξ are given as above. Then,*

(i) $\max_{t \in [0,1]} \rho_2(t) = \rho_2(1/2)$,

(ii) $0 < d_i < 1$, $0 < \xi < 1$.

By Lemmas 2.4 and 2.5, $D_2 = \max_{t \in [0,1]} \int_0^1 K_2(1/2, s)\,ds$.
Take $\theta = \min\{d_1, d_2\xi/C_2\}$, by Lemma 2.5, $0 < \theta < 1$.
Define

$$(Th)(t) = \int_0^1 \int_0^1 K_1(t,s) K_2(s,\tau) h(\tau)\,d\tau\,ds, \quad t \in [0,1],$$
$$(Ah)(t) = (Th)''(t) = -\int_0^1 K_2(t,\tau) h(\tau)\,d\tau, \quad t \in [0,1]. \tag{2.7}$$

Lemma 2.6. *$T : Y \to (X, \|\cdot\|_2)$ is completely continuous, and $\|T\| \le D_2$.*

Proof. It is similar to Lemma 6 of [3], so we omit it. □

Lemma 2.7 (see [17]). *Let E be a Banach space, $P \subseteq E$ a cone, and Ω_1, Ω_2 be two bounded open sets of E with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$. Suppose that $A : P \cap (\overline{\Omega}_2 \setminus \Omega_1) \to P$ is a completely continuous operator such that either*

(i) $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_2$, *or*

(ii) $\|Ax\| \geq \|x\|$, $x \in P \cap \partial\Omega_1$ *and* $\|Ax\| \leq \|x\|$, $x \in P \cap \partial\Omega_2$

holds. Then, A has a fixed point in $P \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

3. The Main Results

Suppose that K_1, K_2, G_2, ρ_2, C_2, θ, and D_2, are defined as in Section 2, we introduce some notations as follows:

$$A = \int_0^1 \int_0^1 K_1(s,s) K_2(s,\tau) d\tau \, ds, \qquad B = \int_0^1 \left[G_2(s,s) + \rho_2\left(\frac{1}{2}\right) \int_0^1 G_2(s,x) q(x) dx \right] ds,$$

$$K = \sup_{t \in [0,1]} [B(t) - \beta], \qquad L = D_2 K, \qquad \eta_0 = \frac{1-L}{A + C_2 B}, \qquad \eta_1 = \frac{1}{\theta \int_{1/4}^{3/4} K_2(1/2,\tau) d\tau},$$

$$\overline{f}_0 = \limsup_{|u|+|v| \to 0} \max_{t \in [0,1]} \frac{f(t,u,v)}{|u|+|v|}, \qquad \underline{f}_0 = \liminf_{|u|+|v| \to 0} \min_{t \in [1/4,3/4]} \frac{f(t,u,v)}{|u|+|v|},$$

$$\overline{f}_\infty = \limsup_{|u|+|v| \to +\infty} \max_{t \in [0,1]} \frac{f(t,u,v)}{|u|+|v|}, \qquad \underline{f}_\infty = \liminf_{|u|+|v| \to +\infty} \min_{t \in [1/4,3/4]} \frac{f(t,u,v)}{|u|+|v|}. \tag{3.1}$$

Theorem 3.1. *Assume that (H1), (H2) hold and $L = D_2 K < 1$. Then BVP*(1.2) *has at least one positive solution if one of the following cases holds:*

(i) $\overline{f}_0 < (1/\lambda)\eta_0$, $\underline{f}_\infty > (1/\lambda)\eta_1$,

(ii) $\underline{f}_0 > (1/\lambda)\eta_1$, $\overline{f}_\infty < (1/\lambda)\eta_0$.

Proof. For any $h \in Y$, consider the following BVP:

$$\begin{aligned} u^{(4)}(t) + B(t) u''(t) &= h(t), \quad 0 < t < 1, \\ u(0) = u(1) &= \int_0^1 p(s) u(s) ds, \\ u''(0) = u''(1) &= \int_0^1 q(s) u''(s) ds. \end{aligned} \tag{3.2}$$

It is easy to see that the above question is equivalent to the following question:

$$\begin{gathered} u^{(4)}(t) + \beta u''(t) = -\big(B(t) - \beta\big)u''(t) + h(t), \quad 0 < t < 1, \\ u(0) = u(1) = \int_0^1 p(s)u(s)ds, \\ u''(0) = u''(1) = \int_0^1 q(s)u''(s)ds. \end{gathered} \tag{3.3}$$

For any $v \in X$, let $Gv = -(B(t) - \beta)v''$. Obviously, the operator $G : X \to Y$ is linear. By Lemma 2.2, for all $v \in X$, $t \in [0,1]$, $|(Gv)(t)| \le (B(t) - \beta)\|v\|_1 \le K\|v\|_1 \le K\|v\|_2$. Hence $\|Gv\|_0 \le K\|v\|_2$, and so $\|G\| \le K$. On the other hand, $u \in C^2[0,1] \cap C^4(0,1)$ is a solution of (3.3) if and only if $u \in X$ satisfies $u = T(Gu + h)$, that is,

$$u \in X, \quad (I - TG)u = Th. \tag{3.4}$$

Owing to $G : X \to Y$ and $T : Y \to X$, the operator $I - TG$ maps X into X. From $\|T\| \le D_2$ (by Lemma 2.6) together with $\|G\| \le K$ and condition $L < 1$, applying operator spectral theorem, we have that the $(I - TG)^{-1}$ exists and is bounded. Let $H = (I - TG)^{-1}T$, then (3.4) is equivalent to $u = Hh$. By the Neumann expansion formula, H can be expressed by

$$H = \big(I + TG + \cdots + (TG)^n + \cdots\big)T = T + (TG)T + \cdots + (TG)^n T + \cdots. \tag{3.5}$$

The complete continuity of T with the continuity of $(I - TG)^{-1}$ yields that the operator $H : Y \to X$ is completely continuous. For all $h \in Y_+$, let $u = Th$, then $u \in X \cap Y_+$, and $u'' < 0$. So, we have $(Gu)(t) = -(B(t) - \beta)u''(t) \ge 0$, $t \in [0,1]$. Hence,

$$\forall h \in Y_+, \quad (GTh)(t) \ge 0, \quad t \in [0,1], \tag{3.6}$$

and so $(TG)(Th)(t) = T(GTh)(t) \ge 0$, $t \in [0,1]$.

Assume that for all $h \in Y_+$, $(TG)^k(Th)(t) \ge 0$, $t \in [0,1]$, let $h_1 = GTh$, by (3.6) we have $h_1 \in Y_+$, and so $(TG)^{k+1}(Th)(t) = (TG)^k(TGTh)(t) = (TG)^k(Th_1)(t) \ge 0$, $t \in [0,1]$. Thus by induction, it follows that $(TG)^n(Th)(t) \ge 0$, for all $n \ge 1$, $h \in Y_+$, $t \in [0,1]$. By (3.5), for all $h \in Y_+$, we have

$$\begin{aligned} (Hh)(t) &= (Th)(t) + (TG)(Th)(t) + \cdots + (TG)^n(Th)(t) + \cdots \ge (Th)(t), \quad t \in [0,1], \\ (Hh)''(t) &= (Ah)(t) + (AG)(Th)(t) + \cdots + \Big(AG(TG)^{n-1}\Big)(Th)(t) + \cdots \\ &\le (Ah)(t) = (Th)''(t) \le 0, \quad t \in [0,1], \end{aligned} \tag{3.7}$$

and so $H : Y_+ \to Y_+ \cap X$.

On the other hand, for all $h \in Y_+$, we have

$$
\begin{aligned}
(Hh)(t) &\le (Th)(t) + |TG|(Th)(t) + \cdots + |TG|^n (Th)(t) + \cdots \\
&\le (1 + L + \cdots + L_n + \cdots)(Th)(t) \\
&= \frac{1}{1-L}(Th)(t) \quad t \in [0,1],
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
\left|(Hh)''(t)\right| &\le |(Ah)(t)| + |(AG)(Th)(t)| + \cdots + \left|\left(AG(TG)^{n-1}\right)(Th)(t)\right| + \cdots \\
&\le |(Ah)(t)| + L|(Ah)(t)| + \cdots + L^n|(Ah)(t)| + \cdots \\
&= (1 + L + \cdots + L_n + \cdots)|(Ah)(t)| \\
&= \frac{1}{1-L}\left|(Th)''(t)\right| \quad t \in [0,1],
\end{aligned}
\tag{3.9}
$$

$$
\begin{aligned}
&\|Hh\|_0 \ge \|Th\|_0, \qquad \|Hh\|_0 \le \frac{1}{1-L}\|Th\|_0, \\
&\|Hh\|_1 \ge \|Th\|_1, \qquad \|Hh\|_1 \le \frac{1}{1-L}\|Th\|_1.
\end{aligned}
\tag{3.10}
$$

For any $u \in Y_+$, define $Fu = \lambda f(t,u,u'')$. By (H1) and (H2), we have that $F : Y_+ \to Y_+$ is continuous. It is easy to see that $u \in C^2[0,1] \cap C^4(0,1)$ being a positive solution of BVP(1.2) is equivalent to $u \in Y_+$ being a nonzero solution equation as follows:

$$
u = HFu. \tag{3.11}
$$

Let $Q = HF$. Obviously, $Q : Y_+ \to Y_+$ is completely continuous. We next show that the operator Q has a nonzero fixed point in Y_+. Let

$$
P = \left\{ u \in X : u \ge 0, u'' \le 0, \min_{1/4 \le t \le 3/4} u(t) \ge (1-L)d_1\|u\|_0, \ \max_{1/4 \le t \le 3/4} u''(t) \le -(1-L)\frac{d_2\xi}{C_2}\left\|u''\right\|_0 \right\}.
\tag{3.12}
$$

It is easy to know that P is a cone in X, $P \subset Y_+$. Now, we show $QP \subset P$.

For $h \in Y_+$, by (2.7), there is $Th \ge 0$, $(Th)'' \le 0$. Hence, by (3.7), $Qu \ge 0$, $(Qu)'' \le 0$, $u \in P$. By proof of Lemma 2.5 in [16],

$$
\min_{1/4 \le t \le 3/4} (Th)(t) \ge d_1\|Th\|_0, \qquad \max_{1/4 \le t \le 3/4} (Th)''(t) \le -\frac{d_2\xi}{C_2}\left\|(Th)''\right\|_0.
\tag{3.13}
$$

By (3.7) and (3.10),

$$\min_{1/4\le t\le 3/4}(Qu)(t) \ge \min_{1/4\le t\le 3/4}(TFu)(t) \ge d_1\|TFu\|_0 \ge (1-L)d_1\|Qu\|_0,$$
$$\max_{1/4\le t\le 3/4}(Qu)''(t) \le \max_{1/4\le t\le 3/4}(TFu)''(t) \le -\frac{d_2\xi}{C_2}\left\|(TFu)''\right\|_0 \le -(1-L)\frac{d_2\xi}{C_2}\left\|(Qu)''\right\|_0. \tag{3.14}$$

Thus $QP \subset P$.

(i) Since $\overline{f}_0 < (1/\lambda)\eta_0$, by the definition of $\overline{f}_0$, there exists $r_1 > 0$ such that

$$\max_{0\le t\le 1, |u(t)|+|u''(t)|\le r_1} f\left(t, u(t), u''(t)\right) \le \frac{r_1}{\lambda}\eta_0. \tag{3.15}$$

Let $\Omega_{r_1} = \{u \in P : \|u\|_2 < r_1\}$, one has

$$f\left(t, u(t), u''(t)\right) \le \frac{r_1}{\lambda}\eta_0, \quad u \in \partial\Omega_{r_1},\ t \in [0,1]. \tag{3.16}$$

So, by (3.10), we get

$$\begin{aligned}
\|Qu\|_0 = \|HFu\|_0 &\le \frac{1}{1-L}\|TFu\|_0 \\
&= \frac{\lambda}{1-L}\left\|\int_0^1\int_0^1 K_1(t,s)K_2(s,\tau)f\left(\tau,u(\tau),u''(\tau)\right)d\tau\, ds\right\|_0 \\
&\le \frac{r_1\eta_0}{1-L}\int_0^1\int_0^1 K_1(s,s)K_2(s,\tau)d\tau\, ds \le \frac{A\eta_0 r_1}{1-L}, \\
\|Qu\|_1 = \|HFu\|_1 &\le \frac{1}{1-L}\|TFu\|_1 \\
&\le \lambda C_2\frac{1}{1-L}\int_0^1\left[G_2(\tau,\tau) + \rho_2\left(\frac{1}{2}\right)\int_0^1 G_2(\tau,x)q(x)dx\right]f\left(\tau,u(\tau),u''(\tau)\right)d\tau \\
&\le \frac{C_2 B\eta_0 r_1}{1-L}.
\end{aligned} \tag{3.17}$$

Hence, for $u \in \partial\Omega_{r_1}$,

$$\|Qu\|_2 = \|HFu\|_2 \le \frac{1}{1-L}\|TFu\|_2 \le \frac{(A+BC_2)\eta_0 r_1}{1-L} = r_1 = \|u\|_2. \tag{3.18}$$

On the other hand, since $\underline{f}_\infty > (1/\lambda)\eta_1$, there exists $r_2' > r_1 > 0$ such that

$$\min_{1/4\le t\le 3/4,\theta(|u(t)|+|u''(t)|)\ge r_2'} \frac{f(t,u(t),u''(t))}{|u(t)|+|u''(t)|} \ge \frac{1}{\lambda}\eta_1. \tag{3.19}$$

Choose $r_2 > (1/\theta)r_2'$, let $\Omega_{r_2} = \{u \in P : \|u\|_2 < r_2\}$. For $u \in \partial\Omega_{r_2}$, $t \in [1/4, 3/4]$, there is $r_2' \le \theta r_2 \le |u(t)| + |u''(t)| \le r_2$. Thus,

$$f(t,u(t),u''(t)) \ge \frac{\theta r_2}{\lambda}\eta_1, \quad u \in \partial\Omega_{r_2},\ t \in \left[\frac{1}{4}, \frac{3}{4}\right].$$

$$\begin{aligned}\left|(TFu)''\left(\frac{1}{2}\right)\right| &= \lambda\int_0^1 K_2\left(\frac{1}{2},\tau\right) f(\tau,u(\tau),u''(\tau))\,d\tau \\ &\ge \lambda\int_{1/4}^{3/4} K_2\left(\frac{1}{2},\tau\right) f(\tau,u(\tau),u''(\tau))\,d\tau \ge \eta_1\theta r_2\int_{1/4}^{3/4} K_2\left(\frac{1}{2},\tau\right)d\tau = r_2.\end{aligned} \tag{3.20}$$

Hence, for $u \in \Omega_{r_2}$,

$$\|Qu\|_2 \ge \|TFu\|_2 \ge \left|(TFu)''\left(\frac{1}{2}\right)\right| \ge r_2 = \|u\|_2. \tag{3.21}$$

By the use of the Krasnoselskii's fixed point theorem, we know there exists $u_0 \in \overline{\Omega}_2 \setminus \Omega_1$ such that $Qu_0 = u_0$, namely, u_0 is a solution of (1.2) and satisfied $u_0 \ge 0$, $u_0'' \le 0$, $r_1 \le \|u_0\|_2 \le r_2$.

(ii) The proof is similar to (i), so we omit it. □

Corollary 3.2. *Assume that (H1), (H2) hold, and $L < 1$. Then that* (1.2) *has at least two positive solution, if f satisfy*

(i) $\overline{f}_0 < (1/\lambda)\eta_0$, $\overline{f}_\infty < (1/\lambda)\eta_0$,

(ii) *There exists $R_0 > 0$ such that $f(t,u,v) \ge (\theta R_0/\lambda)\eta_1$, for $t \in [1/4, 3/4]$, $|u| + |v| \ge \theta R_0$.*

Proof. By the proof of Theorem 3.1, we know that (1) from the condition $\overline{f}_0 < (1/\lambda)\eta_0$, there exists $\Omega_{r_1} = \{u \in P : \|u\|_2 < r_1\}$, such that $\|Qu\|_2 \le \|u\|_2$, $u \in \partial\Omega_{r_1}$, (2) from the condition $\overline{f}_\infty < (1/\lambda)\eta_0$, there exists $\Omega_{r_2} = \{u \in P : \|u\|_2 < r_2\}$, $r_2 > r_1$, such that $\|Qu\|_2 \le \|u\|_2$, $u \in \partial\Omega_{r_2}$, (3) from the condition (ii), there exists $\Omega_{r_3} = \{u \in P : \|u\|_2 < r_3\}$, $r_2 > r_3 > r_1$, such that $\|Qu\|_2 \ge \|u\|_2$, $u \in \partial\Omega_{r_3}$. By the use of Krasnoselskii's fixed point theorem, it is easy to know that (1.2) has at least two positive solutions. □

Corollary 3.3. *Assume (H1), (H2) hold, and $L < 1$. Then problem* (1.2) *has at least two positive solution, if f satisfy*

(i) $\underline{f}_0 > (1/\lambda)\eta_1$, $\underline{f}_\infty > (1/\lambda)\eta_1$,

(ii) *There exists $R_0 > 0$ such that $f(t,u,v) \le (\theta R_0/\lambda)\eta_0$, for $t \in [0,1]$, $|u| + |v| \le R_0$.*

Proof. The proof is similar to Corollary 3.2, so we omit it. □

Example 3.4. Consider the following boundary value problem

$$u^{(4)}(t) + \left(\frac{\pi^2}{4} + t\right)u''(t) = \pi^2\left[18\left(u(t) - u''(t)\right) - 17.9\sin\left(u(t) - u''(t)\right)\right], \quad 0 < t < 1,$$
$$u(0) = u(1) = \int_0^1 su(s)ds, \tag{3.22}$$
$$u''(0) = u''(1) = 0.$$

In this problem, we know that $B(t) = \pi^2/4 + t$, $p(t) = t, q(t) = 0$, $\lambda = \pi^2$, then we can get $C_1 = 1$, $C_2 = 1$, $\rho_1 = 1$, $\rho_2 = \sqrt{2}$, $\beta = \pi^2/4$, $K = 1$, $D_2 = 4(\sqrt{2} - 1)/\pi^2$. Further more, we obtain $A = (48 - 13\pi^2)/\pi^3$, $B = 2/\pi^2$, then $\eta_0 = (1 - L)\pi^3/(48 - 11\pi)$, $\eta_1 = 4\pi^2/\sqrt{2}\cos(\pi/8) - 1$, so

$$\overline{f}_0 = 0.1 < \frac{1}{\pi^2}\eta_0 \approx 0.19, \qquad \underline{f}_\infty = 18 > \frac{1}{\pi^2}\eta_1 \approx 13.3. \tag{3.23}$$

Thus, $B(t)$, $p(t)$, $q(t)$, and f satisfy the conditions of Theorem 3.1, and there exists at least a positive solution of the above problem.

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