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Research Article

On Properties of Solutions for Two Functional Equations Arising in Dynamic Programming

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We introduce and study two new functional equations, which contain a lot of known functional equations as special cases, arising in dynamic programming of multistage decision processes. By applying a new fixed point theorem, we obtain the existence, uniqueness, iterative approximation, and error estimate of solutions for these functional equations. Under certain conditions, we also study properties of solutions for one of the functional equations. The results presented in this paper extend, improve, and unify the results according to Bellman, Bellman and Roosta, Bhakta and Choudhury, Bhakta and Mitra, Liu, Liu and Ume, and others. Two examples are given to demonstrate the advantage of our results over existing results in the literature.

1. Introduction and Preliminaries

The existence, uniqueness, and successive approximations of solutions for the following functional equations arising in dynamic programming:

$$f(x) = \max_{y\in D}\{p(x,y) + q(x,y)f(a(x,y))\}, \quad \forall x \in S,$$

$$f(x) = \max_{y\in D}\{p(x,y) + f(a(x,y))\}, \quad \forall x \in S,$$

$$f(x) = \min_{y\in D}\max\{p(x,y), f(a(x,y))\}, \quad \forall x \in S,$$

$$f(x)=\min_{y\in D}\max\{p(x,y),q(x,y)f(a(x,y))\},\quad \forall x\in S,$$

$$f(x)=\sup_{y\in D}\left\{p(x,y)+\sum_{i=1}^{m}q_i(x,y)f(a_i(x,y))\right\},\quad \forall x\in S, \tag{1.1}$$

were first introduced and discussed by Bellman [1, 2]. Afterwards, further analyses on the properties of solutions for the functional equations (1.1) and (1.2) and others have been studied by several authors in [3–7] and [8–11] by using various fixed point theorems and monotone iterative technique, where (1.2) are as follows:

$$f(x)=\inf_{y\in D}H(x,y,f),\quad \forall x\in S,$$

$$f(x)=\operatorname*{opt}_{y\in D}\left\{p(x,y)+\sum_{i=1}^{m}q_i(x,y)\operatorname{opt}\{v_i(x,y),f(a_i(x,y))\}\right\},\quad \forall x\in S \tag{1.2}$$

$$f(x)=\operatorname*{opt}_{y\in D}\{t[u(x,y)+f(a(x,y))]+(1-t)\operatorname{opt}\{v(x,y),f(a(x,y))\}\},\quad \forall x\in S.$$

The aim of this paper is to investigate properties of solutions for the following more general functional equations arising in dynamic programming of multistage decision processes:

$$f(x)=\operatorname*{opt}_{y\in D}\{p(x,y)+H(x,y,f)\},\quad \forall x\in S, \tag{1.3}$$

$$\begin{aligned}f(x)=\operatorname*{opt}_{y\in D}\Bigg\{r(x,y)+\sum_{i=1}^{m}&\operatorname{opt}\{p_i(x,y)+q_i(x,y)f(a_i(x,y)),\\ &u_i(x,y)+v_i(x,y)f(b_i(x,y))\}\Bigg\},\quad \forall x\in S,\end{aligned} \tag{1.4}$$

where X and Y are real Banach spaces, $S\subseteq X$ is the state space, $D\subseteq Y$ is the decision space, opt denotes the sup or inf, x and y stand for the state and decision vectors, respectively, $a_1,a_2,\ldots,a_m,\ b_1,b_2,\ldots,b_m$ represent the transformations of the processes, and $f(x)$ denotes the optimal return function with initial state x. The rest of the paper is organized as follows. In Section 2, we state the definitions, notions, and a lemma and establish a new fixed point theorem, which will be used in the rest of the paper. The main results are presented in Section 3. By applying the new fixed point theorem, we establish the existence, uniqueness, iterative approximation, and error estimate of solutions for the functional equation (1.3) and (1.4). Under certain conditions, we also study other properties of solutions for the functional equations (1.4). The results present in this paper extend, improve, and unify the corresponding results according to Bellman [1], Bellman and Roosta [5], Bhakta and Choudhury [6], Bhakta and Mitra [7], Liu [8], Liu and Ume [11], and others. Two examples are given to demonstrate the advantage of our results over existing results in the literature.

Throughout this paper, we assume that $R = (-\infty, +\infty)$, $R^+ = [0, +\infty)$, and $R^- = (-\infty, 0]$. For any $t \in R$, $[t]$ denotes the largest integer not exceeding t. Define

$$\begin{aligned}
\Phi_1 &= \{\varphi : \varphi : R^+ \longrightarrow R^+ \text{ is upper semicontinuous from the right on } R^+\}, \\
\Phi_2 &= \{\varphi : \varphi : R^+ \longrightarrow R^+ \text{ and } \varphi(t) < t \text{ for } t > 0\}, \\
\Phi_3 &= \{\varphi : \varphi : R^+ \longrightarrow R^+ \text{ is nondecreasing}\}, \\
\Phi_4 &= \left\{ (\varphi, \psi) : \varphi, \psi \in \Phi_3,\ \psi(t) > 0,\ \sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty \text{ for } t > 0 \right\}.
\end{aligned} \tag{1.5}$$

2. A Fixed Point Theorem

Let $\{d_k\}_{k\geq 1}$ be a countable family of pseudometrics on a nonvoid set X such that for any two different points $x, y \in X$, $d_k(x, y) > 0$ for some $k \geq 1$. For any $x, y \in X$, let

$$d(x, y) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(x, y)}{1 + d_k(x, y)}, \tag{2.1}$$

then d is a metric on X. A sequence $\{x_n\}_{n\geq 1}$ in X is said to converge to a point $x \in X$ if $d_k(x_n, x) \to 0$ as $n \to \infty$ for any $k \geq 1$ and to be a Cauchy sequence if $d_k(x_n, x_m) \to 0$ as $n, m \to \infty$ for any $k \geq 1$.

Theorem 2.1. *Let (X, d) be a complete metric space, and let d be defined by* (2.1). *If $f : X \to X$ satisfies the following inequality:*

$$d_k(fx, fy) \leq \varphi(d_k(x, y)), \quad \forall x, y \in X,\ k \geq 1, \tag{2.2}$$

where φ is some element in $\Phi_1 \cap \Phi_2$, then

(i) *f has a unique fixed point $w \in X$ and $\lim_{n\to\infty} f^n x = w$ for any $x \in X$,*

(ii) *if, in addition, $\varphi \in \Phi_3$, then*

$$d_k(f^n x, w) \leq \varphi^n(d_k(x, w)), \quad \forall x \in X,\ n \geq 1,\ k \geq 1. \tag{2.3}$$

Proof. Given $x \in X$ and $k \geq 1$, define $c_n = d_k(f^n x, f^{n-1} x)$ for each $n \geq 1$. In view of (2.2), we know that

$$c_{n+1} = d_k\left(f^{n+1} x, f^n x\right) \leq \varphi\left(d_k\left(f^n x, f^{n-1} x\right)\right) = \varphi(c_n), \quad \forall n \geq 1. \tag{2.4}$$

Since $\varphi \in \Phi_1 \cap \Phi_2$, by (2.4) we easily conclude that $\{c_n\}_{n\geq 1}$ is nonincreasing. It follows that $\{c_n\}_{n\geq 1}$ has a limit $c \geq 0$. We claim that $c = 0$. Otherwise, $c > 0$. On account of (2.4) and $\varphi \in \Phi_1 \cap \Phi_2$, we deduce that

$$c \leq \limsup_{n\to\infty} \varphi(c_n) \leq \varphi(c) < c, \tag{2.5}$$

which is impossible. That is, $c = 0$. We now show that $\{f^n x\}_{n\geq 1}$ is a Cauchy sequence. Suppose that $\{f^n x\}_{n\geq 1}$ is not a Cauchy sequence, then there exist $\varepsilon > 0$, $k \geq 1$, and two sequences of positive integers $\{m(i)\}_{i\geq 1}$ and $\{n(i)\}_{i\geq 1}$ with $m(i) > n(i)$ and

$$a_i = d_k\left(f^{m(i)}x, f^{n(i)}x\right) \geq \varepsilon, \quad d_k\left(f^{m(i)-1}x, f^{n(i)}x\right) < \varepsilon, \quad \forall i \geq 1, \tag{2.6}$$

which yields that

$$\varepsilon \leq a_i \leq d_k\left(f^{m(i)}x, f^{m(i)-1}x\right) + d_k\left(f^{m(i)-1}x, f^{n(i)}x\right) \leq c_{m(i)} + \varepsilon, \quad \forall i \geq 1. \tag{2.7}$$

As $i \to \infty$ in (2.7), we derive that $\lim_{i\to\infty} a_i = \varepsilon$. Note that (2.2) and (2.7) mean that

$$\begin{aligned} a_i &\leq d_k\left(f^{m(i)}x, f^{m(i)+1}x\right) + d_k\left(f^{m(i)+1}x, f^{n(i)+1}x\right) + d_k\left(f^{n(i)+1}x, f^{n(i)}x\right) \\ &\leq c_{m(i)+1} + \varphi(a_i) + c_{n(i)+1}, \end{aligned} \tag{2.8}$$

for any $i \geq 1$. Letting $i \to \infty$ in (2.8), we see that

$$\varepsilon \leq \varphi(\varepsilon) < \varepsilon. \tag{2.9}$$

This is a contradiction. By completeness of (X, d), there exists a point $w \in X$, such that $\lim_{n\to\infty} f^n x = w$. Using (2.1), (2.2), and $\varphi \in \Phi_1 \cap \Phi_2$, we obtain that for each $x, y \in X$

$$\begin{aligned} d(fx, fy) &= \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(fx, fy)}{1 + d_k(fx, fy)} \leq \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{\varphi(d_k(x, y))}{1 + \varphi(d_k(x, y))} \\ &\leq \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(x, y)}{1 + d_k(x, y)} = d(x, y), \end{aligned} \tag{2.10}$$

which yields that

$$d(w, fw) \leq d(w, f^n x) + d(f^n x, fw) \leq d(w, f^n x) + d\left(f^{n-1}x, w\right) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{2.11}$$

that is, w is a fixed point of f. If f has a fixed point v different from w, then there exists $k \geq 1$ such that $d_k(w, v) > 0$. By (2.2), we have

$$d_k(w, v) = d_k(fw, fv) \leq \varphi(d_k(w, v)) < d_k(w, v), \tag{2.12}$$

which is a contradiction. Consequently, w is a unique fixed point of f.

Suppose that $\varphi \in \Phi_3$. By (2.2), we get that for any $x \in X$, $n \geq 1$, and $k \geq 1$

$$d_k(f^n x, w) = d_k(f^n x, f^n w) \leq \varphi\Big(d_k\Big(f^{n-1}x, f^{n-1}w\Big)\Big) \leq \cdots \leq \varphi^n(d_k(x, w)). \tag{2.13}$$

This completes the proof. □

Remark 2.2. Theorem 2.1 extends Theorem 2.1 of Bhakta and Choudhury [6] and Theorem 1 of Boyd and Wong [12].

Lemma 2.3 (see [11]). *Let a, b, c, and d be in R, then*

$$\left|\text{opt}\{a, b\} - \text{opt}\{c, d\}\right| \leq \max\{|a - c|, |b - d|\}. \tag{2.14}$$

3. Properties of Solutions

In this section, we assume that $(X, \|\cdot\|)$ and $(Y, \|\cdot\|')$ are real Banach spaces, $S \subseteq X$ is the state space, and $D \subseteq Y$ is the decision space. Define

$$BB(S) = \{f : f : S \longrightarrow R \text{ is bounded on bounded subsets of } S\}. \tag{3.1}$$

For any positive integer k and $f, g \in BB(S)$, let

$$\begin{gathered} d_k(f, g) = \sup\Big\{\left|f(x) - g(x)\right| : x \in \overline{B}(0, k)\Big\}, \\ d(f, g) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(f, g)}{1 + d_k(f, g)}, \end{gathered} \tag{3.2}$$

where $\overline{B}(0, k) = \{x : x \in S \text{ and } \|x\| \leq k\}$, then $\{d_k\}_{k \geq 1}$ is a countable family of pseudometrics on $BB(S)$. It is clear that $(BB(S), d)$ is a complete metric space.

Theorem 3.1. *Let $p : S \times D \to R$ and $H : S \times D \times BB(S) \to R$ be mappings, and let φ be in $\Phi_1 \cap \Phi_2$, such that*

(C1) *for any $k \geq 1$ and $(x, y, u, v) \in \overline{B}(0, k) \times D \times BB(S) \times BB(S)$,*

$$\left|H(x, y, u) - H(x, y, v)\right| \leq \varphi(d_k(u, v)), \tag{3.3}$$

(C2) *for any $k \geq 1$ and $u \in BB(S)$, there exists $\alpha(k, u) > 0$ satisfying*

$$\left|p(x, y)\right| + \left|H(x, y, u)\right| \leq \alpha(k, u), \quad \forall (x, y) \in \overline{B}(0, k) \times D, \tag{3.4}$$

then the functional equation (1.3) *possesses a unique solution* $w \in BB(S)$*, and* $\{G^n g\}_{n\geq 1}$ *converges to* w *for each* $g \in BB(S)$*, where* G *is defined by*

$$Gg(x) = \operatorname*{opt}_{y\in D}\{p(x,y) + H(x,y,g)\}, \quad \forall (x,g) \in S \times BB(S). \tag{3.5}$$

In addition, if φ *is in* Φ_3*, then*

$$d_k(G^n g, w) \leq \varphi^n(d_k(g,w)), \quad \forall g \in BB(S),\ n \geq 1,\ k \geq 1. \tag{3.6}$$

Proof. It follows from (C2) and (3.4) that G maps $BB(S)$ into itself. Given $\varepsilon > 0$, $k \geq 1$, $x \in \overline{B}(0,k)$, and $h, g \in BB(S)$, suppose that $\text{opt}_{y\in D} = \sup_{y\in D}$, then there exist $y, z \in D$ such that

$$\begin{aligned} Gh(x) &< p(x,y) + H(x,y,h) + \varepsilon, \qquad Gg(x) < p(x,z) + H(x,z,g) + \varepsilon, \\ Gh(x) &\geq p(x,z) + H(x,z,h), \qquad Gg(x) \geq p(x,y) + H(x,y,g). \end{aligned} \tag{3.7}$$

In view of (3.3), (3.5), and (3.7), we deduce that

$$\begin{aligned} |Gh(x) - Gg(x)| &< \max\{|H(x,y,h) - H(x,y,g)|, |H(x,z,h) - H(x,z,g)|\} + \varepsilon \\ &\leq \varphi(d_k(h,g)) + \varepsilon, \end{aligned} \tag{3.8}$$

which implies that

$$d_k(Gh, Gg) = \sup\left\{|Gh(x) - Gg(x)| : x \in \overline{B}(0,k)\right\} \leq \varphi(d_k(h,g)) + \varepsilon. \tag{3.9}$$

Similarly, we can show that (3.9) holds for $\text{opt}_{y\in D} = \inf_{y\in D}$. As $\varepsilon \to 0^+$ in (3.9), we get that

$$d_k(Gh, Gg) \leq \varphi(d_k(h,g)). \tag{3.10}$$

Notice that the functional equation (1.3) possesses a unique solution w if and only if the mapping G has a unique fixed point w. Thus, Theorem 3.1 follows from Theorem 2.1. This completes the proof. □

Remark 3.2. The conditions of Theorem 3.1 are weaker than the conditions of Theorem 3.1 of Bhakta and Choudhury [6].

Theorem 3.3. *Let* $r, p_i, q_i, u_i, v_i : S\times D \to R$ *and* $a_i, b_i : S\times D \to S$ *be mappings for* $i = 1, 2, \ldots, m$*. Assume that the following conditions are satisfied:*

(C3) *for each* $k \geq 1$*, there exists* $A(k) > 0$ *such that*

$$|r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)|, |u_i(x,y)|\} \leq A(k), \quad \forall (x,y) \in \overline{B}(0,k) \times D, \tag{3.11}$$

(C4) $\max\{\|a_i(x,y)\|, \|b_i(x,y)\| : i \in \{1,2,\ldots,m\}\} \le \|x\|$, *for all* $(x,y) \in S \times D$,

(C5) *there exists a constant* $\beta \in [0,1)$ *such that*

$$\sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\} \le \beta, \quad \forall (x,y) \in S \times D, \tag{3.12}$$

then the functional equation (1.4) *possesses a unique solution* $w \in BB(S)$, *and* $\{w_n\}_{n\ge1}$ *converges to* w *for each* $w_0 \in BB(S)$, *where* $\{w_n\}_{n\ge1}$ *is defined by*

$$\begin{aligned} w_n(x) = \operatorname*{opt}_{y\in D}\Bigg\{ r(x,y) + \sum_{i=1}^{m} \operatorname{opt}\{&p_i(x,y) + q_i(x,y)w_{n-1}(a_i(x,y)),\\ &u_i(x,y) + v_i(x,y)w_{n-1}(b_i(x,y))\}\Bigg\}, \quad \forall x \in S,\ n \ge 1. \end{aligned} \tag{3.13}$$

Moreover,

$$d_k(w_n, w) \le \beta^n (1-\beta)^{-1} d_k(w_0, w), \quad \forall n \ge 1,\ k \ge 1. \tag{3.14}$$

Proof. Set

$$\begin{aligned} H(x,y,h) = r(x,y) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x,y) + q_i(x,y)h(a_i(x,y)), u_i(x,y) + v_i(x,y)h(b_i(x,y))\}\\ \forall (x,y,h) \in S \times D \times BB(S), \end{aligned} \tag{3.15}$$

$$Gh(x) = \operatorname*{opt}_{y\in D} H(x,y,h), \quad \forall (x,h) \in S \times BB(S). \tag{3.16}$$

It follows from (C3)–(C5) and (3.15) that

$$\begin{aligned} |H(x,y,h)| &\le |r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)| + |q_i(x,y)||h(a_i(x,y))|,\\ &\qquad |u_i(x,y)| + |v_i(x,y)||h(b_i(x,y))|\}\\ &\le |r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)|, |u_i(x,y)|\} + \sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\}\\ &\quad \times \max\{|h(a_i(x,y))|, |h(b_i(x,y))|\}\\ &\le A(k) + \left[\sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\}\right] \sup\left\{|h(t)| : t \in \overline{B}(0,k)\right\}\\ &\le A(k) + \beta \sup\left\{|h(t)| : t \in \overline{B}(0,k)\right\}, \end{aligned} \tag{3.17}$$

for any $k \geq 1$ and $(x, y, h) \in \overline{B}(0, k) \times D \times BB(S)$. Consequently, G is a self mapping on $BB(S)$. By Lemma 2.3, (C4), and (C5), we obtain that for any $k \geq 1$ and $(x, y, g, h) \in \overline{B}(0, k) \times D \times BB(S) \times BB(S)$,

$$
\begin{aligned}
&\left|H(x,y,g) - H(x,y,h)\right| \\
&\quad = \left|\sum_{i=1}^{m} \operatorname{opt}\{p_i(x,y) + q_i(x,y)g(a_i(x,y)), u_i(x,y) + v_i(x,y)g(b_i(x,y))\}\right. \\
&\qquad \left. - \sum_{i=1}^{m} \operatorname{opt}\{p_i(x,y) + q_i(x,y)h(a_i(x,y)), u_i(x,y) + v_i(x,y)h(b_i(x,y))\}\right| \\
&\quad \leq \sum_{i=1}^{m} \max\{|q_i(x,y)||g(a_i(x,y)) - h(a_i(x,y))|, |v_i(x,y)||g(b_i(x,y)) - h(b_i(x,y))|\} \\
&\quad \leq \sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\} \\
&\qquad \times \max\{|g(a_i(x,y)) - h(a_i(x,y))|, |g(b_i(x,y)) - h(b_i(x,y))|\} \\
&\quad \leq \varphi(d_k(g,h)),
\end{aligned}
\tag{3.18}
$$

where $\varphi(t) = \beta t$ for $t \in R^+$. Thus, Theorem 3.3 follows from Theorem 3.1. This completes the proof. □

Remark 3.4. Theorem 2 of Bellman [1, page 121], the result of Bellman and Roosta [5, page 545], Theorem 3.3 of Bhakta and Choudhury [6], and Theorems 3.3 and 3.4 of Liu [8] are special cases of Theorem 3.3. The example below shows that Theorem 3.3 extends properly the results in [1, 5, 6, 8].

Example 3.5. Let $X = Y = S = R$ and $D = R^-$. Put $m = 2$, $\beta = 2/3$, and $A(k) = 3k^3$ for any $k \geq 1$. It follows from Theorem 3.3 that the functional equation

$$
\begin{aligned}
f(x) = \operatorname*{opt}_{y \in D} \Bigg\{ & x^2 \sin(xy + x - y + 1) \\
& + \operatorname{opt}\left\{ x^3\left(1 + \frac{x^2+y^2}{1+(x^2+y^2)^2}\right) + \frac{\sin^2(x-y+x^2)}{3+x^2+y^2} f\left(x\cos\left(x^2+y^2\right)\right),\right. \\
& \qquad \left. x^2 \ln\left(1 + \frac{|xy|}{1+|xy|}\right) + \frac{\cos(xy-2x-1)}{3+|x^2y-1|} f\left(\frac{x}{1+|x|y^2+(x-y)^2}\right)\right\} \\
& + \operatorname{opt}\left\{ \frac{x^3 y}{1+|x|+|y|} + \frac{\cos^2(xy-x^2)}{3+x^2y^2} f\left(x\sin\left(1-xy+x^3y^2\right)\right),\right. \\
& \qquad \left. \frac{x^2\cos(x^2-y^2)}{1+|x|+y^2} + \frac{xy}{4+x^2y^2} f\left(\frac{x}{1+2^{x^2y}}\right)\right\}\Bigg\}, \quad \forall x \in S
\end{aligned}
\tag{3.19}
$$

possesses a unique solution $w \in BB(S)$. However, the results in [1, 5, 6, 8] are not applicable.

Theorem 3.6. *Let $r, p_i, q_i, u_i, v_i : S\times D \to R$ and $a_i, b_i : S\times D \to S$ be mappings for $i = 1, 2, \ldots, m$, and, (φ, ψ) be in Φ_4 satisfying*

(C6) $|r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)|, |u_i(x,y)|\} \le \psi(\|x\|)$, *for all* $(x,y) \in S\times D$,

(C7) $\max\{\|a_i(x,y)\|, \|b_i(x,y)\| : i \in \{1,2,\ldots,m\}\} \le \varphi(\|x\|)$, *for all* $(x,y) \in S\times D$,

(C8) $\sup_{(x,y)\in S\times D} \sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\} \le 1$,

then the functional equation (1.4) possesses a solution $w \in BB(S)$ that satisfies the following conditions:

(C9) *the sequence $\{w_n\}_{n\ge1}$ defined by*

$$\begin{aligned} w_0(x) &= \operatorname*{opt}_{y\in D}\left\{ r(x,y) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x,y), u_i(x,y)\}\right\}, \\ w_n(x) &= \operatorname*{opt}_{y\in D}\left\{ r(x,y) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x,y) + q_i(x,y) w_{n-1}(a_i(x,y)),\right. \\ &\qquad \left. u_i(x,y) + v_i(x,y) w_{n-1}(b_i(x,y))\}\right\} \quad \forall x \in S,\ n \ge 1, \end{aligned} \tag{3.20}$$

converges to w,

(C10) $\lim_{n\to\infty} w(x_n) = 0$ *for any* $x_0 \in S$, $\{y_n\}_{n\ge1} \subset D$ *and* $x_n \in \{a_i(x_{n-1}, y_n), b_i(x_{n-1}, y_n) : i \in \{1,2,\ldots,m\}\}, n \ge 1$,

(C11) *w is unique with respect to condition (C10).*

Proof. Let H and G be defined by (3.15) and (3.16), respectively. We now claim that

$$\varphi(t) < t, \qquad \forall t > 0. \tag{3.21}$$

If not, then there exists some $t > 0$ such that $\varphi(t) \ge t$. On account of $(\varphi, \psi) \in \Phi_4$, we know that for any $n \ge 1$,

$$\psi(\varphi^n(t)) \ge \psi\left(\varphi^{n-1}(t)\right) \ge \cdots \ge \psi(t) > 0, \tag{3.22}$$

whence

$$\lim_{n\to\infty} \psi(\varphi^n(t)) \ge \psi(t) > 0, \tag{3.23}$$

which is a contradiction since $\sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty$.

Next, we assert that the mapping G is nonexpansive on $BB(S)$. Let $k \geq 1$ and $h \in BB(S)$. It is easy to see that

$$\max\{\|a_i(x,y)\|, \|b_i(x,y)\| : i \in \{1,2,\ldots,m\}\} \leq \varphi(\|x\|) < k, \quad \forall (x,y) \in \overline{B}(0,k) \times D, \tag{3.24}$$

by (C7) and (3.21). Consequently, there exists a constant $C(k,h) > 0$ satisfying

$$\max\{|h(a_i(x,y))|, |h(b_i(x,y))| : i \in \{1,2,\ldots,m\}\} \leq C(k,h), \quad \forall (x,y) \in \overline{B}(0,k) \times D. \tag{3.25}$$

In view of (C6), (3.16), and (3.25), we derive that for any $x \in \overline{B}(0,k)$,

$$\begin{aligned} |Gh(x)| &= \left|\operatorname*{opt}_{y\in D} H(x,y,h)\right| \leq \sup_{y\in D} |H(x,y,h)| \\ &\leq \sup_{y\in D}\Bigg\{ |r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)| + |q_i(x,y)||h(a_i(x,y))|, \\ &\qquad\qquad |u_i(x,y)| + |v_i(x,y)||h(b_i(x,y))|\}\Bigg\} \\ &\leq \sup_{y\in D}\Bigg\{ |r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)|, |u_i(x,y)|\} \\ &\qquad + \sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\} \max\{|h(a_i(x,y))|, |h(b_i(x,y))|\}\Bigg\} \\ &\leq \varphi(k) + C(k,h), \end{aligned} \tag{3.26}$$

which yields that G maps $BB(S)$ into itself. Given $\varepsilon > 0$, $k \geq 1$, $x \in \overline{B}(0,k)$, and $h, g \in BB(S)$, suppose that $\operatorname{opt}_{y\in D} = \sup_{y\in D}$, then there exist $y, z \in D$ such that

$$\begin{aligned} Gh(x) &< H(x,y,h) + \varepsilon, \qquad Gg(x) < H(x,z,g) + \varepsilon, \\ Gh(x) &\geq H(x,z,h), \qquad Gg(x) \geq H(x,y,g). \end{aligned} \tag{3.27}$$

Using (C6)–(C8), (3.15) and (3.27), and Lemma 2.3, we deduce that

$$\begin{aligned}
|Gh(x) - Gg(x)| &< \max\{|H(x,y,h) - H(x,y,g)|, |H(x,z,h) - H(x,z,g)|\} + \varepsilon \\
&\le \max\Bigg\{\sum_{i=1}^{m} \max\{|q_i(x,y)||h(a_i(x,y)) - g(a_i(x,y))|, \\
&\qquad |v_i(x,y)||h(b_i(x,y)) - g(b_i(x,y))|\}, \\
&\qquad \sum_{i=1}^{m} \max\{|q_i(x,z)||h(a_i(x,z)) - g(a_i(x,z))|, \\
&\qquad |v_i(x,z)||h(b_i(x,z)) - g(b_i(x,z))|\}\Bigg\} + \varepsilon \\
&\le \max\Bigg\{\sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\}, \\
&\qquad \sum_{i=1}^{m} \max\{|q_i(x,z)|, |v_i(x,z)|\}\Bigg\} d_k(h,g) + \varepsilon \\
&\le d_k(h,g) + \varepsilon,
\end{aligned} \tag{3.28}$$

which means that

$$d_k(Gh, Gg) \le d_k(h,g) + \varepsilon. \tag{3.29}$$

Similarly, we can conclude that the above inequality holds for $\text{opt}_{y\in D} = \inf_{y\in D}$. Letting $\varepsilon \to 0^+$, we get that

$$d_k(Gh, Gg) \le d_k(h,g), \tag{3.30}$$

which implies that

$$d(Gh, Gg) = \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(Gh,Gg)}{1 + d_k(Gh,Gg)} \le \sum_{k=1}^{\infty} \frac{1}{2^k} \cdot \frac{d_k(h,g)}{1 + d_k(h,g)} = d(h,g). \tag{3.31}$$

That is, G is nonexpansive.

We show that for each $n \ge 0$,

$$|w_n(x)| \le \sum_{j=0}^{n} \psi\left(\varphi^j(\|x\|)\right), \quad \forall x \in S. \tag{3.32}$$

In terms of (C6) and (C9), we obtain that

$$|w_0(x)| \le \sup_{y\in D}\left\{ |r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)|, |u_i(x,y)|\} \right\} \le \varphi(\|x\|), \quad \forall x \in S, \tag{3.33}$$

which means that (3.32) holds for $n = 0$. Suppose that (3.32) holds for some $n \ge 0$. It follows from (C6)–(C8) and (3.25) that

$$\begin{aligned}
|w_{n+1}(x)| &= \left| \operatorname*{opt}_{y\in D}\left\{ r(x,y) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x,y) + q_i(x,y)w_n(a_i(x,y)),\right.\right. \\
&\qquad\qquad \left.\left. u_i(x,y) + v_i(x,y)w_n(b_i(x,y))\}\right\}\right| \\
&\le \sup_{y\in D}\left\{ |r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)| + |q_i(x,y)||w_n(a_i(x,y))|,\right. \\
&\qquad\qquad \left. |u_i(x,y)| + |v_i(x,y)||w_n(b_i(x,y))|\}\right\} \\
&\le \sup_{y\in D}\left\{ |r(x,y)| + \sum_{i=1}^{m} \max\{|p_i(x,y)|, |u_i(x,y)|\}\right. \\
&\qquad \left. + \sum_{i=1}^{m} \max\{|q_i(x,y)|, |v_i(x,y)|\} \max\{|w_n(a_i(x,y))|, |w_n(b_i(x,y))|\}\right\} \\
&\le \varphi(\|x\|) + \sup_{y\in D} \max\{|w_n(a_i(x,y))|, |w_n(b_i(x,y))| : i \in \{1,2,\ldots,m\}\} \\
&\le \varphi(\|x\|) + \sup_{y\in D} \max\left\{ \sum_{j=0}^{n} \varphi\Big(\varphi^j(\|a_i(x,y)\|)\Big),\right. \\
&\qquad\qquad \left. \sum_{j=0}^{n} \varphi\Big(\varphi^j(\|b_i(x,y)\|)\Big) : i \in \{1,2,\ldots,m\}\right\} \\
&= \sum_{j=0}^{n+1} \varphi\Big(\varphi^j(\|x\|)\Big).
\end{aligned} \tag{3.34}$$

Therefore, (3.32) holds for any $n \ge 0$.

Next, we prove that $\{w_n\}_{n\ge 0}$ is a Cauchy sequence in $BB(S)$. Given $\varepsilon > 0$, $k \ge 1$, $n \ge 1$, $j \ge 1$, and $x_0 \in \overline{B}(0,k)$, suppose that $\operatorname{opt}_{y\in D} = \sup_{y\in D}$. We select that $y, z \in D$ with

$$\begin{aligned} w_n(x_0) &< r(x_0,y) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,y) + q_i(x_0,y)w_{n-1}(a_i(x_0,y)),\\ &\qquad u_i(x_0,y) + v_i(x_0,y)w_{n-1}(b_i(x_0,y))\} + 2^{-1}\varepsilon,\\ w_{n+j}(x_0) &< r(x_0,z) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,z) + q_i(x_0,z)w_{n+j-1}(a_i(x_0,z)),\\ &\qquad u_i(x_0,z) + v_i(x_0,z)w_{n+j-1}(b_i(x_0,z))\} + 2^{-1}\varepsilon,\\ w_n(x_0) &\ge r(x_0,z) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,z) + q_i(x_0,z)w_{n-1}(a_i(x_0,z)),\\ &\qquad u_i(x_0,z) + v_i(x_0,z)w_{n-1}(b_i(x_0,z))\},\\ w_{n+j}(x_0) &\ge r(x_0,y) + \sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,y) + q_i(x_0,y)w_{n+j-1}(a_i(x_0,y)),\\ &\qquad u_i(x_0,y) + v_i(x_0,y)w_{n+j-1}(b_i(x_0,y))\}. \end{aligned} \tag{3.35}$$

According to (C6)–(C8) and (3.35), we have

$$\begin{aligned} &\left|w_{n+j}(x_0) - w_n(x_0)\right|\\ &\quad< \max\Bigg\{\Bigg|\sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,z) + q_i(x_0,z)w_{n+j-1}(a_i(x_0,z)),\\ &\qquad\qquad u_i(x_0,z) + v_i(x_0,z)w_{n+j-1}(b_i(x_0,z))\}\\ &\qquad\quad - \sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,z) + q_i(x_0,z)w_{n-1}(a_i(x_0,z)),\\ &\qquad\qquad u_i(x_0,z) + v_i(x_0,z)w_{n-1}(b_i(x_0,z))\}\Bigg|,\\ &\qquad\quad \Bigg|\sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,y) + q_i(x_0,y)w_{n+j-1}(a_i(x_0,y)),\\ &\qquad\qquad u_i(x_0,y) + v_i(x_0,y)w_{n+j-1}(b_i(x_0,y))\}\\ &\qquad\quad - \sum_{i=1}^{m} \operatorname{opt}\{p_i(x_0,y) + q_i(x_0,y)w_{n-1}(a_i(x_0,y)),\\ &\qquad\qquad u_i(x_0,y) + v_i(x_0,y)w_{n-1}(b_i(x_0,y))\}\Bigg|\Bigg\} + 2^{-1}\varepsilon \end{aligned}$$

$$
\begin{aligned}
&\le \max\Bigg\{\sum_{i=1}^{m}\max\{|q_i(x_0,z)||w_{n+j-1}(a_i(x_0,z))-w_{n-1}(a_i(x_0,z))|,\\
&\qquad\qquad |v_i(x_0,z)||w_{n+j-1}(b_i(x_0,z))-w_{n-1}(b_i(x_0,z))|\},\\
&\qquad \sum_{i=1}^{m}\max\{|q_i(x_0,y)||w_{n+j-1}(a_i(x_0,y))-w_{n-1}(a_i(x_0,y))|,\\
&\qquad\qquad |v_i(x_0,y)||w_{n+j-1}(b_i(x_0,y))-w_{n-1}(b_i(x_0,y))|\}\Bigg\}+2^{-1}\varepsilon\\
&\le \max\Bigg\{\sum_{i=1}^{m}\max\{|q_i(x_0,z)|,|v_i(x_0,z)|\}\max\{|w_{n+j-1}(a_i(x_0,z))-w_{n-1}(a_i(x_0,z))|,\\
&\qquad\qquad |w_{n+j-1}(b_i(x_0,z))-w_{n-1}(b_i(x_0,z))|\},\\
&\qquad \sum_{i=1}^{m}\max\{|q_i(x_0,y)|,|v_i(x_0,y)|\}\max\{|w_{n+j-1}(a_i(x_0,y))-w_{n-1}(a_i(x_0,y))|,\\
&\qquad\qquad |w_{n+j-1}(b_i(x_0,y))-w_{n-1}(b_i(x_0,y))|\}\Bigg\}+2^{-1}\varepsilon\\
&\le \max\{\max\{|w_{n+j-1}(a_i(x_0,z))-w_{n-1}(a_i(x_0,z))|,\\
&\qquad\qquad |w_{n+j-1}(b_i(x_0,z))-w_{n-1}(b_i(x_0,z))| : i\in\{1,2,\ldots,m\}\},\\
&\qquad \max\{|w_{n+j-1}(a_i(x_0,y))-w_{n-1}(a_i(x_0,y))|,\\
&\qquad\qquad |w_{n+j-1}(b_i(x_0,y))-w_{n-1}(b_i(x_0,y))| : i\in\{1,2,\ldots,m\}\}\}+2^{-1}\varepsilon\\
&= |w_{n+j-1}(x_1)-w_{n-1}(x_1)|+2^{-1}\varepsilon,
\end{aligned}
\tag{3.36}
$$

for some $x_1 \in \{a_i(x_0,y_1), b_i(x_0,y_1) : i \in \{1,2,\ldots,m\}\}$ and $y_1 \in \{y,z\}$. In a similar way, we can conclude that (3.36) holds for $\text{opt}_{y\in D} = \inf_{y\in D}$. Proceeding in this way, we select $y_t \in D$ and $x_t \in \{a_i(x_{t-1},y_t), b_i(x_{t-1},y_t) : i \in \{1,2,\ldots,m\}\}$ for $t \in \{2,3,\ldots,n\}$ such that

$$
\begin{gathered}
|w_{n+j-1}(x_1)-w_{n-1}(x_1)| < |w_{n+j-2}(x_2)-w_{n-2}(x_2)|+2^{-2}\varepsilon,\\
|w_{n+j-2}(x_2)-w_{n-2}(x_2)| < |w_{n+j-3}(x_3)-w_{n-3}(x_3)|+2^{-3}\varepsilon,\\
\vdots\\
|w_{j+1}(x_{n-1})-w_1(x_{n-1})| < |w_j(x_n)-w_0(x_n)|+2^{-n}\varepsilon.
\end{gathered}
\tag{3.37}
$$

In terms of (C7), (3.21), (3.32), (3.36), and (3.37), we know that

$$\begin{aligned}
\left|w_{n+j}(x_0)-w_n(x_0)\right| &< \left|w_j(x_n)-w_0(x_n)\right| + \sum_{i=1}^{n} 2^{-i}\varepsilon \\
&< \left|w_j(x_n)\right| + |w_0(x_n)| + \varepsilon \\
&\le \sum_{i=0}^{j} \psi\left(\varphi^i(\|x_n\|)\right) + \psi(\|x_n\|) + \varepsilon \\
&\le \sum_{i=n-1}^{\infty} \psi\left(\varphi^i(k)\right) + \varepsilon,
\end{aligned} \tag{3.38}$$

which implies that

$$d_k(w_{n+j}, w_n) \le \sum_{i=n-1}^{\infty} \psi\left(\varphi^i(k)\right) + \varepsilon. \tag{3.39}$$

Letting $\varepsilon \to 0^+$ in the above inequality, we have

$$d_k(w_{n+j}, w_n) \le \sum_{i=n-1}^{\infty} \psi\left(\varphi^i(k)\right), \tag{3.40}$$

which means that $\{w_n\}_{n\ge 0}$ is a Cauchy sequence in $(BB(S), d)$ because $\sum_{n=0}^{\infty} \psi(\varphi^n(t)) < \infty$ for each $t > 0$. Let $\lim_{n\to\infty} w_n = w \in BB(S)$. By the nonexpansivity of G, we get that

$$\begin{aligned}
d(w, Gw) &\le d(w, Gw_n) + d(Gw_n, Gw) \\
&\le d(w, w_{n+1}) + d(w_n, w) \longrightarrow 0, \quad \text{as } n \longrightarrow \infty,
\end{aligned} \tag{3.41}$$

which implies that $w = Gw$. That is, w is a solution of the functional equation (1.4).

Now, we show that (C10) holds. Given $\varepsilon > 0$, $x_0 \in S$, $\{y_n\}_{n\ge 1} \subset D$, and $x_n \in \{a_i(x_{n-1}, y_n), b_i(x_{n-1}, y_n) : i \in \{1, 2, \dots, m\}\}$ for $n \ge 1$, set $k = 1 + [\|x_0\|]$. It is easy to verify that there exists a positive integer m satisfying

$$d_k(w, w_n) + \sum_{i=n}^{\infty} \psi\left(\varphi^i(k)\right) < \varepsilon, \quad \text{for } n \ge m. \tag{3.42}$$

Notice that

$$\begin{aligned} \|x_n\| &\le \max\{\|a_i(x_{n-1},y_n)\|,\|b_i(x_{n-1},y_n)\| : i\in\{1,2,\dots,m\}\} \\ &\le \varphi(\|x_{n-1}\|) \le \cdots \le \varphi^n(\|x_0\|) \le \varphi^k(k) < k, \end{aligned} \tag{3.43}$$

for any $n \ge 1$. Consequently, we infer immediately that, for $n \ge m$,

$$\begin{aligned} |w(x_n)| &\le |w(x_n)-w_n(x_n)| + |w_n(x_n)| \le d_k(w,w_n) + \sum_{i=0}^{n}\psi\left(\varphi^i(\|x_n\|)\right) \\ &\le d_k(w,w_n) + \sum_{i=n}^{\infty}\psi\left(\varphi^i(k)\right) < \varepsilon, \end{aligned} \tag{3.44}$$

which yields that $\lim_{n\to\infty} w(x_n) = 0$.

At last, we show that (C11) holds. Suppose that the functional equation (1.4) possesses another solution $h \in BB(S)$, which satisfies (C10). Given $\varepsilon > 0$ and $x_0 \in S$, suppose that $\text{opt}_{y\in D} = \sup_{y\in D}$, then there are $y, z \in S$ satisfying

$$\begin{aligned} w(x_0) &< r(x_0,y) + \sum_{i=1}^{m}\text{opt}\{p_i(x_0,y) + q_i(x_0,y)w(a_i(x_0,y)), \\ &\qquad u_i(x_0,y) + v_i(x_0,y)w(b_i(x_0,y))\} + 2^{-1}\varepsilon, \\ h(x_0) &< r(x_0,z) + \sum_{i=1}^{m}\text{opt}\{p_i(x_0,z) + q_i(x_0,z)h(a_i(x_0,z)), \\ &\qquad u_i(x_0,z) + v_i(x_0,z)h(b_i(x_0,z))\} + 2^{-1}\varepsilon, \\ w(x_0) &\ge r(x_0,z) + \sum_{i=1}^{m}\text{opt}\{p_i(x_0,z) + q_i(x_0,z)w(a_i(x_0,z)), \\ &\qquad u_i(x_0,z) + v_i(x_0,z)w(b_i(x_0,z))\}, \\ h(x_0) &\ge r(x_0,y) + \sum_{i=1}^{m}\text{opt}\{p_i(x_0,y) + q_i(x_0,y)h(a_i(x_0,y)), \\ &\qquad u_i(x_0,y) + v_i(x_0,y)h(b_i(x_0,y))\}, \end{aligned} \tag{3.45}$$

Whence there exists $y_1 \in \{y, z\}$ and $x_1 \in \{a(x_0, y_1), b(x_0, y_1) : i \in \{1, 2, \ldots, m\}\}$ such that

$$\begin{aligned}
&|w(x_0) - h(x_0)| \\
&\quad < \max\Bigg\{ \sum_{i=1}^{m} \big| \mathrm{opt}\{p_i(x_0, y) + q_i(x_0, y)w(a_i(x_0, y)), u_i(x_0, y) + v_i(x_0, y)w(b_i(x_0, y))\} \\
&\qquad\qquad - \mathrm{opt}\{p_i(x_0, y) + q_i(x_0, y)h(a_i(x_0, y)), u_i(x_0, y) + v_i(x_0, y)h(b_i(x_0, y))\}\big|, \\
&\qquad \sum_{i=1}^{m} \big| \mathrm{opt}\{p_i(x_0, z) + q_i(x_0, z)w(a_i(x_0, z)), u_i(x_0, z) + v_i(x_0, z)w(b_i(x_0, z))\} \\
&\qquad\qquad - \mathrm{opt}\{p_i(x_0, z) + q_i(x_0, z)h(a_i(x_0, z)), \\
&\qquad\qquad\qquad u_i(x_0, z) + v_i(x_0, z)h(b_i(x_0, z))\}\big| \Bigg\} + 2^{-1}\varepsilon \\
&\quad \le \max\Bigg\{ \sum_{i=1}^{m} \max\{ |q_i(x_0, y)| |w(a_i(x_0, y)) - h(a_i(x_0, y))|, \\
&\qquad\qquad\qquad |v_i(x_0, y)| |w(b_i(x_0, y)) - h(b_i(x_0, y))| \}, \\
&\qquad \sum_{i=1}^{m} \max\{ |q_i(x_0, z)| |w(a_i(x_0, z)) - h(a_i(x_0, z))|, \\
&\qquad\qquad\qquad |v_i(x_0, z)| |w(b_i(x_0, z)) - h(b_i(x_0, z))| \} \Bigg\} + 2^{-1}\varepsilon \\
&\quad \le \max\Bigg\{ \sum_{i=1}^{m} \max\{ |q_i(x_0, y)|, |v_i(x_0, y)| \}, \sum_{i=1}^{m} \max\{ |q_i(x_0, z)|, |v_i(x_0, z)| \} \Bigg\} \\
&\qquad \times \max\{ |w(a_i(x_0, y)) - h(a_i(x_0, y))|, |w(b_i(x_0, y)) - h(b_i(x_0, y))|, \\
&\qquad\qquad |w(a_i(x_0, z)) - h(a_i(x_0, z))|, |w(b_i(x_0, z)) - h(b_i(x_0, z))| : i \in \{1, 2, \ldots, m\}\} + 2^{-1}\varepsilon \\
&\quad \le |w(x_1) - h(x_1)| + 2^{-1}\varepsilon
\end{aligned} \tag{3.46}$$

by (C8). Proceeding in this way, we select $y_j \in D$ and $x_j \in \{a_i(x_{j-1}, y_j), b_i(x_{j-1}, y_j) : i \in \{1, 2, \ldots, m\}\}$ for $j \in \{2, 3, \ldots, n\}$ satisfying

$$\begin{gathered}
|w(x_1) - h(x_1)| < |w(x_2) - h(x_2)| + 2^{-2}\varepsilon, \\
|w(x_2) - h(x_2)| < |w(x_3) - h(x_3)| + 2^{-3}\varepsilon, \\
\vdots \\
|w(x_{n-1}) - h(x_{n-1})| < |w(x_n) - h(x_n)| + 2^{-n}\varepsilon.
\end{gathered} \tag{3.47}$$

It follows that

$$|w(x_0) - h(x_0)| < |w(x_n) - h(x_n)| + \varepsilon, \tag{3.48}$$

which yields that

$$|w(x_0) - h(x_0)| \le \varepsilon, \tag{3.49}$$

by letting $n \to \infty$. Similarly, (3.49) also holds for $\text{opt}_{y\in D} = \inf_{y\in D}$. As $\varepsilon \to 0^+$, we know that $w(x_0) = h(x_0)$. This completes the proof. □

Remark 3.7. Theorem 3.6 generalizes Theorem 1 of Bellman [1, page 119], Theorem 3.5 of Bhakta and Choudhury [6], Theorem 2.4 of Bhakta and Mitra [7], Theorem 3.5 of Liu [8] and Theorem 3.1 of Liu and Ume [11]. The following example reveals that Theorem 3.6 is indeed a generalization of the results in [1, 6–8, 11].

Example 3.8. Let $X = Y = R$, $S = D = R^+$. Define $\varphi, \psi : R^+ \to R^+$ by

$$\varphi(t) = 2^{-1}t, \quad \psi(t) = 3t^4, \qquad \forall t \in R^+. \tag{3.50}$$

It is easy to verify that the following functional equation:

$$\begin{aligned} f(x) = \operatorname*{opt}_{y\in D}\Bigg\{ & \frac{x^4}{2+\sin(1+x^2y^2)} + \max\left\{ \frac{x^4}{1+xy} + \frac{x\sin(x+y)}{2+4x+y} f\left(\frac{x^2}{2+2x+y}\right), \right. \\ & \left. \frac{x^4}{1+x+y} + \frac{\ln(1+x+y)}{4+x+y} f\left(\frac{x^3y}{1+2x^2y}\right) \right\} \\ & + \max\left\{ \frac{x^4}{1+(x-y)^2} + \frac{y\cos(x-y)}{1+x+4y} f\left(\frac{x^3}{1+2x^2+\sin(x^2+y^2)}\right), \right. \\ & \left. \frac{x^5y}{1+xy} + \frac{\cos(xy+2x-y)}{4+x^y} f\left(\frac{x^4y\sin(x^3+y^3+xy-1)}{1+2x^3y}\right) \right\} \Bigg\}, \quad \forall x \in S \end{aligned} \tag{3.51}$$

satisfies conditions (C6)–(C8). Consequently, Theorem 3.6 ensures that it has a solution $w \in BB(S)$ that satisfies conditions (C9)–(C11). However, Theorem 1 of Bellman [1, page 119], Theorem 3.5 of Bhakta and Choudhury [6], Theorem 2.4 of Bhakta and Mitra [7], Theorem 3.5 of Liu [8], and Theorem 3.1 of Liu and Ume [11] are not applicable.

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