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Research Article

Fixed Point Properties Related to Multivalued Mappings

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We discuss fixed point properties of convex subsets of locally convex linear topological spaces. We derive equivalence among fixed point properties concerning several types of multivalued mappings.

1. Introduction

We present fundamental definitions related to multivalued mappings in order to fix our terminology. We assume Hausdorff separation axiom for all of the topological spaces which appear hereafter. Let X and Y be topological spaces. A multivalued mapping $F : X \twoheadrightarrow Y$ from X to Y is a function which attains a nonempty subset of Y for each point x of X and the subset is denoted by Fx. For any subset B of Y, the *upper inverse* $F^u(B)$ and the *lower inverse* $F^l(B)$ are defined by $F^u(B) = \{x \in X : Fx \subset B\}$ and $F^l(B) = \{x \in X : Fx \cap B \neq \emptyset\}$, respectively. A multivalued mapping $F : X \twoheadrightarrow Y$ is said to be *upper semicontinuous* (*lower semicontinuous*, resp.) if $F^u(G)$ ($F^l(G)$, resp.) is open in X for any open subset G of Y. Moreover, F is said to be *upper demicontinuous* if $F^u(H)$ is open in X for any open half-space H of Y in case Y is a linear topological space.

We are interested in fixed point properties of convex subsets of locally convex linear topological spaces. A topological space is said to have a *fixed point property* if every continuous functions from the topological space to itself has a fixed point. Following to this terminology, we define several fixed point properties depending on types of multivalued mappings we concern.

We always deal with convex-valued multivalued mappings defined on a convex subset of a locally convex topological linear space in this paper. Such situations appear often in arguments on fixed point theory for multivalued mappings, for example, Kakutani fixed point theorem [1], Browder fixed point theorem [2], and so forth. Let X be a convex subset of a locally convex topological linear space and let $F : X \twoheadrightarrow X$ be a convex-valued multivalued mapping from X to X. We call F *Kakutani-type* if F is closed-valued and upper semicontinuous and *weak Kakutani-type* if F is closed-valued and demicontinuous. Similarly F is said to be *Browder-type* if F has open lower sections; that is, $F^{-1}y = \{x \in X : Fx \ni y\}$ is open for all $y \in X$. We call F *open graph-type* if it has an open graph.

A convex subset X of a locally convex linear topological space is said to have a *Kakutani-type fixed point property* if every Kakutani-type multivalued mapping from X to X has a fixed point. Similarly, we define *weak Kakutani-type fixed point property*, *Browder-type fixed point property,* and *open graph-type fixed point property*.

2. Result

Our main result is the following.

Theorem 2.1. *Let X be a paracompact convex subset of a locally convex linear topological space Y. Then each of the following statements is mutually equivalent.*

(1) *X has a fixed point property.*

(2) *X has a Browder-type fixed point property.*

(3) *X has an open graph-type fixed point property.*

(4) *X has a weak Kakutani-type fixed point property.*

(5) *X has a Kakutani-type fixed point property.*

Proof. The proofs of (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5) $\Rightarrow$ (1) are obvious.

(1) $\Rightarrow$ (2). The method of the proof is similar to that of [2, Theorem 1]. Let $F : X \twoheadrightarrow X$ be Browder-type. The family $\{F^{-1}y\}_{y\in X}$ is an open cover of X because any point x of X belongs to an open set $F^{-1}y$ with $y \in Fx$. Therefore, there is a partition of unity $\{f_\alpha\}_{\alpha\in A}$ subordinated to $\{F^{-1}y\}_{y\in X}$. That is, each function $f_\alpha : X \to [0,1]$ is continuous, the family $\{\{x \in X : f_\alpha(x) > 0\}\}_{\alpha\in A}$ of open sets is a locally finite refinement of $\{F^{-1}y\}_{y\in X}$, and $\sum_{\alpha\in A} f_\alpha(x) = 1$ for all $x \in X$. For each $\alpha \in A$, take y such that $\{x \in X : f_\alpha(x) > 0\} \subset F^{-1}y$, and we denote it by y_α. Then define a function $f : X \to X$ by

$$f(x) = \sum_{\alpha\in A} f_\alpha(x) y_\alpha. \tag{2.1}$$

Here the summation $\sum_{\alpha\in A}$ is well defined because there are only a finite number of indices α with $f_\alpha(x) > 0$. The function f is continuous because the family $\{x \in X : f_\alpha(x) > 0\}$ of open sets is locally finite. On the other hand, it follows that $f(X) \subset X$ since X is convex. Thus f has a fixed point $x_0 \in X$ by the hypothesis. That is, we have

$$x_0 = \sum_{\alpha\in A} f_\alpha(x_0) y_\alpha. \tag{2.2}$$

It follows that $x_0 \in F^{-1}y_\alpha$ for each α with $f_\alpha(x_0) > 0$, and hence we have $y_\alpha \in Fx_0$. Since Fx_0 is convex, we have $x_0 \in Fx_0$, and it is proved that x_0 is a fixed point of F.

(3) ⇒ (4). The method of this proof is inspired by the discussions found in [3, 4]. Suppose that $F : X \twoheadrightarrow X$ is weak Kakutani-type but it has no fixed point; that is, $x \notin Fx$ for any $x \in X$. Since Fx is closed and convex, there is a continuous linear functional f on Y which separates x and Fx strictly. Thus there is a real number α such that

$$x \in I_x = \{y \in Y : f(y) < \alpha\}, \qquad Fx \subset J_x = \{y \in Y : f(y) > \alpha\}. \tag{2.3}$$

Put

$$U_x = I_x \cap F^u(J_x). \tag{2.4}$$

Then U_x is a neighborhood of x in X, and we have $F(U_x) \subset J_x$. Since $\{U_x\}_{x\in X}$ is an open cover of X, there is an open cover $\{W_\alpha\}_{\alpha\in A}$ of X such that $\{\overline{W_\alpha}\}_{\alpha\in A}$ is locally finite and refines $\{U_x\}_{x\in X}$ because X is paracompact. For each $\alpha \in A$, take an x such that $\overline{W_\alpha} \subset U_x$ and denote it by x_α. For each $x \in X$, define Gx by

$$Gx = \bigcap_{\overline{W_\alpha} \ni x} (J_{x_\alpha} \cap X). \tag{2.5}$$

Since $x \in U_{x_\alpha}$ for any α with $\overline{W_\alpha} \ni x$, we have $Fx \subset F(U_{x_\alpha}) \subset J_{x_\alpha}$. Thus we have $Fx \subset \bigcap_{\overline{W_\alpha}\ni x} J_{x_\alpha} = Gx$. Therefore, we have $Gx \neq \emptyset$ for all $x \in X$, and the definition of Gx above defines a multivalued mapping $G : X \twoheadrightarrow X$. It is easily seen that G is open and convex valued.

Next we show that G has an open graph. Take any element (x_0, y_0) of the graph Gr(G) of G and fix it. Define

$$M_{x_0} = \bigcap_{x_0 \notin \overline{W_\alpha}} \left(X \setminus \overline{W_\alpha}\right), \tag{2.6}$$

then M_{x_0} is a neighborhood of x_0 because $\{\overline{W_\alpha}\}_{\alpha\in A}$ is locally finite. Thus $M_{x_0} \times Gx_0$ is a neighborhood of (x_0, y_0). We show that $M_{x_0} \times Gx_0 \subset \mathrm{Gr(G)}$. Take any $(x, y) \in M_{x_0} \times Gx_0$. Since $x \in M_{x_0}$, we have $x \notin \overline{W_\alpha}$ for any α with $x_0 \notin \overline{W_\alpha}$. Therefore, we have $\{\alpha \in A : x \in \overline{W_\alpha}\} \subset \{\alpha \in A : x_0 \in \overline{W_\alpha}\}$. From this inclusion, we have

$$y \in Gx_0 \subset Gx. \tag{2.7}$$

That is, $M_{x_0} \times Gx_0 \subset \mathrm{Gr(G)}$. Therefore, G has an open graph.

On the other hand, take any $x \in X$. There is $\alpha \in A$ such that $x \in \overline{W_\alpha}$. Since $x \in U_{x_\alpha}$, we have $x \notin J_{x_\alpha}$, and hence $x \notin Gx$. Thus G has no fixed point and this contradicts the assumption that X has open graph-type fixed point property. □

Klee [5] proved that a convex subset of a locally convex metrizable linear topological space is compact if and only if it has a fixed point property. Since any metrizable topological space is paracompact, we have the following corollary of Theorem 2.1.

Corollary 2.2. *Let X be a convex subset of a locally convex metrizable linear topological space. Then the following statements are mutually equivalent.*

(1) *X is compact.*

(2) *X has a fixed point property.*

(3) *X has a Browder-type fixed point property.*

(4) *X has an open graph-type fixed point property.*

(5) *X has a weak Kakutani-type fixed point property.*

(6) *X has a Kakutani-type fixed point property.*

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