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Research Article

Stability of a Mixed Type Functional Equation on Multi-Banach Spaces: A Fixed Point Approach

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Using fixed point methods, we prove the Hyers-Ulam-Rassias stability of a mixed type functional equation on multi-Banach spaces.

1. Introduction and Preliminaries

The stability problem of functional equations originated from a question of Ulam [1] concerning the stability of group homomorphisms. Hyers [2] gave a first affirmative partial answer to the question of Ulam for Banach spaces. Hyers's theorem was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. The paper of Rassias has provided a lot of influence in the development of what we call generalized Hyers-Ulam-Rassias stability of functional equations. In 1990, Rassias [5] asked whether such a theorem can also be proved for $p \geq 1$. In 1991, Gajda [6] gave an affirmative solution to this question when $p > 1$, but it was proved by Gajda [6] and Rassias and Šemrl [7] that one cannot prove an analogous theorem when $p = 1$. In 1994, a generalization was obtained by Gavruta [8], who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\phi(x, y)$. Beginning around 1980, the stability problems of several functional equations and approximate homomorphisms have been extensively investigated by a number of authors, and there are many interesting results concerning this problem. Some of the open problems in this field were solved in the papers mentioned [9–15].

The notion of multi-normed space was introduced by Dales and Polyakov (see in [16–19]). This concept is somewhat similar to operator sequence space and has some connections with operator spaces and Banach lattices. Motivations for the study of multi-normed spaces and many examples were given in [16]. Let $(E, \|\cdot\|)$ be a complex linear space, and let $K \in \mathbb{N}$, we denote by E^k the linear space $E \oplus \cdots \oplus E$ consisting of k-tuples $(x_1, \ldots, x_k)$, where $x_1, \ldots, x_k \in E$. The linear operations on E^k are defined coordinate-wise. When we write

$(0,\ldots,0,x_i,0,\ldots,0)$ for an element in E^k, we understand that x_i appears in the ith coordinate. The zero elements of either E or E^k are both denoted by 0 when there is no confusion. We denote by $\mathbb{N}_k$ the set $\{1,2,\ldots,k\}$ and by $\mathbb{B}_k$ the group of permutations on $\mathbb{N}_k$.

Definition 1.1. A multi-norm on $\{E^n, n\in\mathbb{N}\}$ is a sequence

$$(\|\cdot\|_n) = (\|\cdot\|_n : n\in\mathbb{N}) \tag{1.1}$$

such that $\|\cdot\|_n$ is a norm on E^n for each $n\in\mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x\in E$, and such that for each $n\in\mathbb{N}$ $(n\geq 2)$, the following axioms are satisfied:

(A_1) $\|(x_{\sigma(1)},\ldots,x_{\sigma(n)})\|_n = \|(x_1,\ldots,x_n)\|_n (\forall\sigma\in B_n, x_1,\ldots,x_n\in E)$;

(A_2) $\|(\alpha_1x_1,\ldots,\alpha_nx_n)\|_n \leq (\max_{i\in\mathbb{N}_n}|\alpha_i|)\|(x_1,\ldots,x_n)\|_n (x_i\in E, \alpha_i\in\mathbb{C}, i=1,\ldots,n)$;

(A_3) $\|(x_1,\ldots,x_{n-1},0)\|_n = \|(x_1,\ldots,x_{n-1})\|_{n-1}\ (x_1,\ldots,x_{n-1}\in E)$;

(A_4) $\|(x_1,\ldots,x_{n-1},x_{n-1})\|_n = \|(x_1,\ldots,x_{n-1})\|_{n-1}(x_1,\ldots,x_{n-1}\in E)$.

In this case, we say that $((E^n,\|\cdot\|_n) : n\in\mathbb{N})$ is a multi-normed space.

Suppose that $((E^n,\|\cdot\|_n) : n\in\mathbb{N})$ is a multi-normed space and $k\in\mathbb{N}$. It is easy to show that

(a) $\|(x,\ldots,x)\|_k = \|x\|(x\in E)$;

(b) $\max_{i\in\mathbb{N}_k}\|x_i\| \leq \|(x_1,\ldots,x_k)\|_k \leq \sum_{i=1}^{k}\|x_i\| \leq k\max_{i\in\mathbb{N}_k}\|x_i\|(x_1,\ldots,x_k\in E)$.

It follows from (b) that if $(E,\|\cdot\|)$ is a Banach space, then $(E^k,\|\cdot\|_k)$ is a Banach space for each $k\in\mathbb{N}$; in this case $((E^k,\|\cdot\|_k) : k\in\mathbb{N})$ is said to be a multi-Banach space.

In the following, we first recall some fundamental result in fixed-point theory.

Let X be a set. A function $d : X\times X \to [0,\infty]$ is called a generalized metric on X if d satisfies

(1) $d(x,y) = 0$ if and only if $x = y$;

(2) $d(x,y) = d(y,x)$ for all $x,y\in X$;

(3) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x,y,z\in X$.

We recall the following theorem of Diaz and Margolis [20].

Theorem 1.2 (see [20]). *let (X,d) be a complete generalized metric space and let $J : X\to X$ be a strictly contractive mapping with Lipschitz constant $0 < L < 1$. Then for each given element $x\in X$, either*

$$d\left(J^nx, J^{n+1}x\right) = \infty \tag{1.2}$$

for all nonnegative integers n or there exists a nonnegative integer n_0 such that

(1) *$d(J^nx, J^{n+1}x) < \infty$ for all $n\geq n_0$;*

(2) *the sequence $\{J^nx\}$ converges to a fixed point y^* of J;*

(3) *y^* is the unique fixed point of J in the set $Y = \{y\in X : d(J^{n_0}x,y) < \infty\}$;*

(4) *$d(y,y^*) \leq 1/(1-L)d(y,Jy)$ for all $y\in Y$.*

Baker [21] was the first author who applied the fixed-point method in the study of Hyers-Ulam stability (see also [22]). In 2003, Cadariu and Radu applied the fixed-point method to the investigation of the Jensen functional equation (see [23, 24]). By using fixed point methods, the stability problems of several functional equations have been extensively investigated by a number of authors (see [25–27]).

In this paper, we will show the Hyers-Ulam-Rassias stability of a mixed type functional equation on multi-Banach spaces using fixed-point methods.

2. A Mixed Type Functional Equation

In this section, we investigate the stability of the following functional equation in multi-Banach spaces:

$$f(x+2y)+f(x-2y)=4f(x+y)+4f(x-y)-6f(x)+f(4y)-4f(3y)+6f(2y)-4f(y). \tag{2.1}$$

Let

$$Df(x,y)=f(x+2y)+f(x-2y)-4f(x+y)-4f(x-y)+6f(x)-f(4y)+4f(3y)-6f(2y)+4f(y). \tag{2.2}$$

First we give some lemma needed later.

Lemma 2.1 (see [28] Lemma 6.1). *If an even function $f : X \to Y$ satisfies* (2.1), *then f is quartic-quadratic function.*

Lemma 2.2 (see [28] Lemma 6.2). *If an odd function $f : X \to Y$ satisfies* (2.1), *then f is cubic-additive function.*

Theorem 2.3. *Let E be a linear space and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$ and let $f : E \to F$ be an even mapping with $f(0) = 0$ for which there exists a positive real number ϵ such that*

$$\sup_{k\in\mathbb{N}}\left\|\left(Df(x_1,y_1),\ldots,Df(x_k,y_k)\right)\right\|_k \le \epsilon \tag{2.3}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k \in E (k \in \mathbb{N})$. Then there exists a unique quadratic mapping $Q_1 : E \to F$ satisfying (2.1) *and*

$$\sup_{k\in\mathbb{N}}\left\|\left(f(2x_1)-16f(x_1)-Q(x_1),\ldots,f(2x_k)-16f(x_k)-Q(x_k)\right)\right\|_k \le 3\epsilon \tag{2.4}$$

for all $x_1,\ldots,x_k \in E$.

Proof. Putting $x_1 = \cdots = x_k = 0$ in (2.3), we have

$$\sup_{k\in\mathbb{N}} \left\| \left(f(4y_1) - 4f(3y_1) + 4f(2y_1) + 4f(y_1), \ldots, f(4y_k) - 4f(3y_k) + 4f(2y_k) + 4f(y_k)\right) \right\|_k \le \epsilon. \tag{2.5}$$

Replacing x_i with y_i in (2.3), we get

$$\sup_{k\in\mathbb{N}} \left\| \left(-f(4y_1) + 5f(3y_1) - 10f(2y_1) + 11f(y_1), \ldots, -f(4y_k) + 5f(3y_k) - 10f(2y_k) + 11f(y_k)\right) \right\|_k \le \epsilon. \tag{2.6}$$

By (2.5) and (2.6), we have

$$\sup_{k\in\mathbb{N}} \left\| \left(f(4x_1) - 20f(2x_1) + 64f(x_1), \ldots, f(4x_k) - 20f(2x_k) + 64f(x_k)\right) \right\|_k \le 9\epsilon. \tag{2.7}$$

Let $J(x) = f(2x) - 16f(x)$ for all $x \in X$. Then we have

$$\sup_{k\in\mathbb{N}} \|(J(2x_1) - 4J(x_1), \ldots, J(2x_k) - 4J(x_k))\|_k \le 9\epsilon. \tag{2.8}$$

Set $X = \{g : E \to F : g(0) = 0\}$ and define a metric d on X by

$$d(g,h) = \inf\left\{ c > 0 : \sup_{k\in\mathbb{N}} \left\| g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k) \right\|_k \le c : x_1, \ldots, x_k \in \mathbb{N},\ k \in \mathbb{N} \right\}. \tag{2.9}$$

Define a map $\Lambda : X \to X$ by $\Lambda(g)(x) = (g(2x))/4$. Let $g, h \in X$ and let $c \in [0, \infty]$ be an arbitrary constant with $d(g,h) \le c$. From the definition of d, we have

$$\sup_{k\in\mathbb{N}} \left\| g(x_1) - h(x_1), \ldots, g(x_k) - h(x_k) \right\|_k \le c \tag{2.10}$$

for $x_1, \ldots, x_k \in \mathbb{N}, k \in \mathbb{N}$. Then

$$\sup_{k\in\mathbb{N}} \left\| (\Lambda g)(x_1) - (\Lambda h)(x_1), \ldots, (\Lambda g)(x_k) - (\Lambda h)(x_k) \right\|_k \le \frac{1}{4} \sup_{k\in\mathbb{N}} \left\| g(2x_1) - h(2x_1), \ldots, g(2x_k) - h(2x_k) \right\|_k \le \frac{c}{4} \tag{2.11}$$

for $x_1, \ldots, x_k \in \mathbb{N}$, $k \in \mathbb{N}$. So

$$d(\Lambda g, \Lambda h) \leq \frac{1}{4} d(g, h). \tag{2.12}$$

Then Λ is a strictly contractive mapping. It follows from (2.8) that

$$\begin{aligned} &\sup_{k\in\mathbb{N}} \|(\Lambda J)(x_1) - J(x_1), \ldots, (\Lambda J)(x_k) - J(x_k)\|_k \\ &\quad \leq \frac{1}{4} \sup_{k\in\mathbb{N}} \|J(2x_1) - 4J(2x_1), \ldots, J(2x_k) - 4J(2x_k)\|_k \leq \frac{9\epsilon}{4} \end{aligned} \tag{2.13}$$

for $x_1, \ldots, x_k \in \mathbb{N}, k \in \mathbb{N}$. Then $d(\Lambda J, J) \leq 9\epsilon/4$. According to Theorem 1.2, the sequence $\{\Lambda^n J\}$ converges to a unique fixed point Q_1 of Λ in X, that is,

$$\begin{aligned} Q_1(x) &= \lim_{n\to\infty} (\Lambda^n J)(x) = \lim_{n\to\infty} \frac{1}{4^n} J(2^n x), \\ d(J, Q_1) &\leq \frac{4}{3} d(\Lambda J, J) = 3\epsilon. \end{aligned} \tag{2.14}$$

Also we have $(Q(2x))/4 = Q(x)$ for all $x \in X$, that is, $Q(2x) = 4Q(x)$ for all $x \in X$. Also we have

$$\begin{aligned} DQ_1(x, y) &= \lim_{n\to\infty} \frac{1}{4^n} \|DJ(2^n x, 2^n y)\| = \lim_{n\to\infty} \frac{1}{4^n} \left\| Df\left(2^{n+1} x, 2^{n+1} y\right) - 16 Df(2^n x, 2^n y) \right\| \\ &\leq \lim_{n\to\infty} \frac{17\epsilon}{4^n} = 0, \end{aligned} \tag{2.15}$$

and Q_1 satisfies (2.1). Since Q_1 is also even and $Q_1(0) = 0$, we have that $Q(2x) - 16Q(x) = -12Q(x)$ is quadratic by Lemma 2.1. Then Q is quadratic. □

Theorem 2.4. *Let E be a linear space and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$ and let $f : E \to F$ be an even mapping with $f(0) = 0$ for which there exists a positive real number ϵ such that* (2.3) *holds for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ $(k \in \mathbb{N})$. Then there exists a unique quartic mapping $Q_2 : E \to F$ satisfying* (2.1) *and*

$$\sup_{k\in\mathbb{N}} \left\| (f(2x_1) - 4f(x_1) - Q_2(x_1), \ldots, f(2x_k) - 4f(x_k) - Q_2(x_k)) \right\|_k \leq \frac{3}{5}\epsilon \tag{2.16}$$

for all $x_1, \ldots, x_k \in E$.

Proof. The proof is similar to that of Theorem 2.3. □

Theorem 2.5. *Let E be a linear space and let $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$ and let $f : E \to F$ be an even mapping with $f(0) = 0$ for which there exists a positive real number ϵ*

such that (2.3) *holds for all* $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ $(k \in \mathbb{N})$*. Then there exist a unique quadratic mapping* $Q_1 : E \to F$ *and a unique quadratic mapping* $Q_2 : E \to F$ *such that*

$$\sup_{k\in\mathbb{N}} \left\| \left(f(x_1) - Q_1(x_1) - Q_2(x_1), \ldots, f(x_k) - Q_1(x_k) - Q_2(x_k)\right) \right\|_k \leq \frac{3\epsilon}{10} \tag{2.17}$$

for all $x_1, \ldots, x_k \in E$*.*

Proof. By Theorems 2.3 and 2.4, there exist a quadratic mapping $Q_1^0 : E \to F$ and a unique quartic mapping $Q_2^0 : E \to f$ such that

$$\begin{gathered} \sup_{k\in\mathbb{N}} \left\| \left(f(2x_1) - 16f(x_1) - Q_1^0(x_1), \ldots, f(2x_k) - 16f(x_k) - Q_1^0(x_k)\right) \right\|_k \leq 3\epsilon \\ \sup_{k\in\mathbb{N}} \left\| \left(f(2x_1) - 4f(x_1) - Q_2^0(x_1), \ldots, f(2x_k) - 4f(x_k) - Q_2^0(x_k)\right) \right\|_k \leq \frac{3}{5}\epsilon \end{gathered} \tag{2.18}$$

for all $x_1, \ldots, x_k \in E$. By (2.18), we have

$$\sup_{k\in\mathbb{N}} \left\| \left(12f(x_1) + Q_1^0(x_1) - Q_2^0(x_1), \ldots, 12f(x_k) + Q_1^0(x_k) - Q_2^0(x_k)\right) \right\|_k \leq \frac{18}{5}\epsilon. \tag{2.19}$$

Let $Q_1(x) = -(1/12)Q_1^0(x)$ and $Q_2(x) = (1/12)Q_2^0(x)$ for all $x \in E$. Then we have (2.17). The uniqueness of Q_1 and Q_2 is easy to show. □

Theorem 2.6. *Let* E *be a linear space and let* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-Banach space. Let* $k \in \mathbb{N}$ *and let* $f : E \to F$ *be an odd mapping for which there exists a positive real number* ϵ *such that* (2.3) *holds for all* $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ $(k \in \mathbb{N})$*. Then there exists a unique additive mapping* $A : E \to F$ *and a unique cubic mapping* $C : E \to F$ *satisfying* (2.1) *and*

$$\begin{gathered} \sup_{k\in\mathbb{N}} \left\| \left(f(2x_1) - 8f(x_1) - A(x_1), \ldots, f(2x_k) - 8f(x_k) - A(x_k)\right) \right\|_k \leq 9\epsilon, \\ \sup_{k\in\mathbb{N}} \left\| \left(f(2x_1) - 2f(x_1) - C(x_1), \ldots, f(2x_k) - f(x_k) - C(x_k)\right) \right\|_k \leq \frac{9}{7}\epsilon \end{gathered} \tag{2.20}$$

for all $x_1, \ldots, x_k \in E$*.*

Proof. The proof is similar to that of Theorems 2.3 and 2.4. □

Theorem 2.7. *Let* E *be a linear space and let* $((F^n, \|\cdot\|_n) : n \in \mathbb{N})$ *be a multi-Banach space. Let* $k \in \mathbb{N}$ *and let* $f : E \to F$ *be an odd mapping for which there exists a positive real number* ϵ *such that* (2.3) *holds for all* $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$ $(k \in \mathbb{N})$*. Then there exists a unique additive mapping* $A : E \to F$ *and a unique cubic mapping* $C : E \to F$ *satisfying* (2.1) *and*

$$\sup_{k\in\mathbb{N}} \left\| \left(f(x_1) - A(x_1) - C(x_1), \ldots, f(x_k) - A(x_k) - C(x_k)\right) \right\|_k \leq \frac{12}{7}\epsilon \tag{2.21}$$

for all $x_1, \ldots, x_k \in E$*.*

Proof. By Theorem 2.6, there is an additive mapping $A_0 : E \to F$ and a cubic mapping $C_0 : E \to F$ such that

$$\sup_{k\in\mathbb{N}}\left\|\left(f(2x_1)-8f(x_1)-A_0(x_1),\ldots,f(2x_k)-8f(x_k)-A_0(x_k)\right)\right\|_k \le 9\epsilon,$$
$$\sup_{k\in\mathbb{N}}\left\|\left(f(2x_1)-2f(x_1)-C_0(x_1),\ldots,f(2x_k)-2f(x_k)-C_0(x_k)\right)\right\|_k \le \frac{9}{7}\epsilon. \tag{2.22}$$

Thus

$$\sup_{k\in\mathbb{N}}\left\|\left(6f(x_1)+A_0(x_1)-C_0(x_1),\ldots,6f(x_k)+A_0(x_k)-C_0(x_k)\right)\right\|_k \le \frac{72}{7}\epsilon \tag{2.23}$$

for all $x_1,\ldots,x_k \in E$. Let $A = -A_0/6$ and $C = C_0/6$. The rest is similar to that of the proof of Theorem 2.5. □

Theorem 2.8. *Let E be a linear space and let $((F^n,\|\cdot\|_n) : n \in \mathbb{N})$ be a multi-Banach space. Let $k \in \mathbb{N}$ and let $f : E \to F$ be an odd mapping satisfying $f(0) = 0$ and there exists a positive real number ϵ such that (2.3) holds for all $x_1,\ldots,x_k,y_1,\ldots,y_k \in E$ ($k \in \mathbb{N}$). Then there exist a unique additive mapping $A : E \to F$, a unique cubic mapping $C : E \to F$, a unique quadratic mapping $Q_1 : E \to F$, and a unique quadratic mapping $Q_2 : E \to F$ such that*

$$\sup_{k\in\mathbb{N}}\left\|\left(f(x_1)-A(x_1)-Q(x_1)-C(x_1)-Q_2(x_1),\ldots,f(x_k)-A(x_k)-Q_1(x_k)\right.\right.$$
$$\left.\left.-C(x_k-Q_2(x_k))\right)\right\|_k \le \frac{141}{70}\epsilon \tag{2.24}$$

for all $x_1,\ldots,x_k \in E$.

Proof. Let $f_e(x) = 1/2(f(x)+f(-x))$ for all $x \in E$. Then $f_e(0) = 0$ and $f_e(-x) = f_e(x)$ and

$$\sup_{k}\left\|Df_e(x_1,y_1),\ldots,Df_e(x_k,y_k)\right\|_k \le \epsilon \tag{2.25}$$

for all $x_1,\ldots,x_k,y_1,\ldots,y_k \in E$. By Theorem 2.5, there are a unique quadratic mapping $Q_1 : E \to F$ and a unique quartic mapping $Q_2 : E \to F$ satisfying

$$\sup_{k\in\mathbb{N}}\left\|\left(f_e(x_1)-Q_1(x_1)-Q_2(x_1),\ldots,f_e(x_k)-Q_1(x_k)-Q_2(x_k)\right)\right\|_k \le \frac{3\epsilon}{10}. \tag{2.26}$$

Let $f_o(x) = 1/2(f(x)-f(-x))$ for all $x \in E$. Then f_o is an odd mapping satisfying

$$\sup_{k}\left\|Df_o(x_1,y_1),\ldots,Df_o(x_k,y_k)\right\|_k \le \epsilon \tag{2.27}$$

for all $x_1, \ldots, x_k, y_1, \ldots, y_k \in E$. By Theorem 2.7, there are a unique additive mapping $A : E \to F$ and a unique quartic mapping $C : E \to F$ satisfying

$$\sup_{k\in\mathbb{N}} \left\| \left(f_o(x_1) - A(x_1) - C(x_1), \ldots, f(x_k) - A(x_k) - C(x_k) \right) \right\|_k \leq \frac{12}{7}\epsilon. \quad (2.28)$$

By (2.26) and (2.28), we have (2.24).This completes the proof. □

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