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Research Article

Weak ψ-Sharp Minima in Vector Optimization Problems

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We present a sufficient and necessary condition for weak ψ-sharp minima in infinite-dimensional spaces. Moreover, we develop the characterization of weak ψ-sharp minima by virtue of a nonlinear scalarization function.

1. Introduction

The notion of a weak sharp minimum in general mathematical program problems was first introduced by Ferris in [1]. It is an extension of sharp minimum in [2]. Weak sharp minima play important roles in the sensitivity analysis [3, 4] and convergence analysis of a wide range of optimization algorithms [5]. Recently, the study of weak sharp solution set covers real-valued optimization problems [5–8] and piecewise linear multiobjective optimization problems [9–11].

Most recently, Bednarczuk [12] defined weak sharp minima of order m for vector-valued mappings under an assumption that the order cone is closed, convex, and pointed and used the concept to prove upper Hölderness and Hölder calmness of the solution set-valued mappings for a parametric vector optimization problem. In [13], Bednarczuk discussed the weak sharp solution set to vector optimization problems and presented some properties in terms of well-posedness of vector optimization problems. In [14], Studniarski gave the definition of weak ψ-sharp local Pareto minimum in vector optimization problems under the assumption that the order cone is convex and presented necessary and sufficient conditions under a variety of conditions. Though the notions in [12, 14] are different for vector optimization problems, they are equivalent for scalar optimization problems. They are a generalization of the weak sharp local minimum of order m.

In this paper, motivated by the work in [14, 15], we present a sufficient and necessary condition of which a point is a weak ψ-sharp minimum for a vector-valued mapping in the infinite-dimensional spaces. In addition, we develop the characterization of weak ψ-sharp minima in terms of a nonlinear scalarization function.

This paper is organized as follows. In Section 2, we recall the definitions of the local Pareto minimizer and weak ψ-sharp local minimizer for vector-valued optimization problems. In Section 3, we present a sufficient and necessary condition for weak ψ-sharp local minimizer of vector-valued optimization problems. We also give an example to illustrate the optimality condition.

2. Preliminary Results

Throughout the paper, X and Y are normed spaces. $B(x,\delta)$ denotes the open ball with center $x \in X$ and radius $\delta > 0$. $\mathcal{N}(x)$ is the family of all neighborhoods of x, and $\operatorname{dist}(x, W)$ is the distance from a point x to a set $W \subset X$. The symbols S^c, $\operatorname{int} S$ and bds denote, respectively, the complement, interior and boundary of S.

Let $D \subset Y$ be a convex cone (containing 0). The cone defines an order structure on Y, that is, a relation "$\leq$" in $Y \times Y$ is defined by $y_1 \leq y_2 \Leftrightarrow y_2 - y_1 \in D$. D is a proper cone if $\{0\} \neq D \neq Y$.

Let Ω be an open subset of X, $S \subset \Omega$. Given a vector-valued map $f : \Omega \to Y$, the following abstract optimization is considered:

$$\operatorname{Min}\{f(x) : x \in S\}. \tag{2.1}$$

In the sequel, we always assume that D is a proper closed and convex cone.

Definition 2.1. One says that x_0 is a local Pareto minimizer for (2.1), denoted by $x_0 \in L\operatorname{Min}(f, S)$, if there exists $U \in \mathcal{N}(x)$ for which there is no $x \in S \cap U$ such that

$$f(x) - f(x_0) \in (-D) \setminus D. \tag{2.2}$$

If one can choose $U = X$, one will say that x_0 is a Pareto minimizer for (2.1), denoted by $x_0 \in \operatorname{Min}(f, S)$.

Note that (2.2) may be replaced by the simple condition $f(x) - f(x_0) \in (-D) \setminus \{0\}$ if we assume that the cone D is pointed.

Definition 2.2 (see [14]). Let $\psi : [0, +\infty) \to [0, +\infty)$ be a nondecreasing function with the property $\psi(t) = 0 \Leftrightarrow t = 0$ (such a family of functions is denoted by Ψ). Let $x_0 \in S$. One says that x_0 is a weak ψ-sharp local Pareto minimizer for (2.1), denoted by $x_0 \in \mathrm{WSL}(\psi, f, S)$, if there exist a constant $\alpha > 0$ and $U \in \mathcal{N}(x_0)$ such that

$$\big(f(x) + D\big) \cap B\big(f(x_0), \alpha\psi(\operatorname{dist}(x, W))\big) = \emptyset, \quad \forall x \in (S \cap U) \setminus W, \tag{2.3}$$

where

$$W := \big\{x \in S : f(x) = f(x_0)\big\}. \tag{2.4}$$

If one can choose $U = X$, one says x_0 is a weak ψ-sharp minimizer for (2.1), denoted by $x_0 \in \mathrm{WS}(\psi, f, S)$. In particular, let $\psi_m(t) := t^m$ for $m = 1, 2, \ldots$. Then, one says that x_0 is a weak ψ-sharp local Pareto minimizer of order m for (2.1) if $x_0 \in \mathrm{WSL}(\psi_m, f, S)$, and one says that x_0 is a weak sharp Pareto minimizer of order m for (2.1) if $x_0 \in \mathrm{WS}(\psi_m, f, S)$.

Remark 2.3. If W is a closed set, condition (2.3) can be expressed as the following equivalent forms:

$$f(x) \in \big(f(x_0) + B\big(0, \alpha\psi(\mathrm{dist}(x, W))\big) - D\big)^c, \quad \forall x \in (S \cap U) \setminus W, \tag{2.5}$$

$$d\big(f(x) - f(x_0), -D\big) \geq \alpha\psi(\mathrm{dist}(x, W)), \quad \forall x \in (S \cap U) \setminus W. \tag{2.6}$$

Remark 2.4. In the Definition 2.2, if $Y = R$, $D = [0, +\infty)$, and $\psi = \psi_m$, then the relation (2.6) becomes the following form:

$$f(x) - f(x_0) \geq \alpha(\mathrm{dist}(x, W))^m, \quad \forall x \in S \cap U, \tag{2.7}$$

which is the well-known definition of a weak sharp minimizer of order m for (2.1); see [16].

3. Main Results

In this section, we first generalize the result of Theorem 1 in Studniarski [14] to infinite-dimensional spaces. Finally, we develop the characterization of weak ψ-sharp minimizer by means of a nonlinear scalarization function.

Let $D \subset Y$ be a proper closed convex cone with $\mathrm{int}\, D \neq \emptyset$. The topological dual space of Y is denoted by Y^*. The polar cone to D is $D^* = \{\lambda \in Y^* : \langle \lambda, y \rangle \geq 0,\ \forall y \in D\}$. It is well known that the cone D^* contains a w^*-compact convex set Λ with $0 \notin \Lambda$ such that

$$D^* = \mathrm{cone}\, \Lambda = \{r\lambda : r \geq 0,\ \lambda \in \Lambda\}. \tag{3.1}$$

The set Λ is called a base for the dual cone D^*. Recall that a point λ is an extremal point of a set Λ if there exist no different points $\lambda_1, \lambda_2 \in \Lambda$ and $t \in (0, 1)$ such that $\lambda = t\lambda_1 + (1 - t)\lambda_2$.

Theorem 3.1. *Suppose that $f : X \to Y$ is a vector-valued map. Let $D \subset Y$ be a proper closed convex cone with* $\mathrm{int}\, D \neq \emptyset$, *$x_0 \in S$, and $\psi \in \Psi$.*

(i) *Let Λ be a w^*-compact convex base of D^* and Q the set of extremal points of Λ. Suppose that W defined by* (2.4) *is a closed set. Then, $x_0 \in WSL(\psi, f, S)$ if and only if there exist $U \in \mathcal{N}(x)$, a constant $\alpha > 0$, a covering $\{S_\lambda : \lambda \in Q\}$ of $S \cap U$, and*

$$\big\langle \lambda, f(x) \big\rangle > \big\langle \lambda, f(x_0) \big\rangle + \alpha\psi(\mathrm{dist}(x, W)), \quad \forall x \in (S_\lambda \cap U) \setminus W,\ \forall \lambda \in Q. \tag{3.2}$$

(ii) *Let $Q \subset D^* \setminus \{0\}$ and assume that $D^* = \mathrm{cl\,cone\,co}\, Q$. Then $x_0 \in L\,\mathrm{Min}(f, S)$ if and only if there exists a covering $\{S_\lambda : \lambda \in Q\}$ of $S \cap U$ such that*

$$\big\langle \lambda, f(x) \big\rangle > \big\langle \lambda, f(x_0) \big\rangle, \quad \forall x \in (S_\lambda \cap U) \setminus W,\ \forall \lambda \in Q. \tag{3.3}$$

Proof. (i) Part "only if": by assumption, there exist $\beta > 0$ and $U \in \mathcal{N}(x_0)$ such that

$$(f(x) - f(x_0) + D) \cap B(0, \beta\varphi(\text{dist}(x, W))) = \emptyset, \quad \forall x \in (S \cap U) \setminus W. \tag{3.4}$$

Let $e \in \text{int}\, D$ be a fixed point. Set $\beta_0 = \inf_{\lambda \in \Lambda}\langle \lambda, e\rangle$. Since Λ is w^*-compact, the infimum is attained at a point of Q. Namely, $\beta_0 = \min_{\lambda \in Q}\langle \lambda, e\rangle$. Clearly, $\langle \lambda, e\rangle > 0$ for any $\lambda \in \Lambda$. Hence, $\beta_0 > 0$.

For each $\lambda \in Q$, we define

$$S_\lambda = \left\{ x \in S \cap U : \langle \lambda, f(x)\rangle \geq \langle \lambda, f(x_0)\rangle + \frac{\beta}{2\|e\|}\varphi(\text{dist}(x, W))\beta_0 \right\}. \tag{3.5}$$

We will show that

$$S \cap U \subset \bigcup_{\lambda \in Q} S_\lambda. \tag{3.6}$$

Let $x \in S \cap U$. If $x \in W$, then $f(x) = f(x_0)$ by (2.4), hence, $x \in S_\lambda$ for all $\lambda \in Q$. If $x \notin W$, suppose that $x \notin S_\lambda$ for any $\lambda \in Q$, then

$$\langle \lambda, f(x)\rangle < \langle \lambda, f(x_0)\rangle + \frac{\beta}{2\|e\|}\varphi(\text{dist}(x, W))\beta_0, \quad \forall \lambda \in Q. \tag{3.7}$$

This relation, together with statement $\langle \lambda, e\rangle \geq \beta_0$ yields

$$\left\langle \lambda, f(x_0) - f(x) + \frac{\beta}{2\|e\|}\varphi(\text{dist}(x, W))e \right\rangle > 0, \quad \forall \lambda \in Q. \tag{3.8}$$

Obviously, for any $\lambda \in D^*$, the above relation becomes the following form:

$$\left\langle \lambda, f(x_0) - f(x) + \frac{\beta}{2\|e\|}\varphi(\text{dist}(x, W))e \right\rangle \geq 0. \tag{3.9}$$

Consequently, by the bipolar theorem, one has

$$d := f(x_0) - f(x) + \frac{\beta}{2\|e\|}\varphi(\text{dist}(x, W))e \in D. \tag{3.10}$$

Therefore,

$$f(x) - f(x_0) + d = \frac{\beta}{2\|e\|}\varphi(\text{dist}(x, W))e, \tag{3.11}$$

and $f(x) - f(x_0) + d \in B(0, \beta\varphi(\text{dist}(x, W)))$, which is a contradiction to (3.4). We have thus proved that S_λ covers $S \cap U$.

Now, let $x \in (S_\lambda \cap U) \setminus W$ and $\lambda \in Q$. From the procedure of the above proof, we see that $(S \cap U) \setminus W \subset \cup_{\lambda \in Q} S_\lambda$. Hence, by (3.5), set $\alpha = \beta\beta_0/(4\|e\|)$, inequality (3.2) is true.

Part "if": we define $\beta_1 = \sup_{\lambda \in \Lambda} \langle \lambda, e \rangle$. The supremum is attained at an extremal point because of the w^*-compactness of Λ. So $\beta_1 = \max_{\lambda \in Q} \langle \lambda, e \rangle > 0$ and $\beta_1^{-1} \langle \lambda, e \rangle \leq 1$ for any $\lambda \in Q$. Hence, by assumption, we have

$$\langle \lambda, f(x) \rangle > \langle \lambda, f(x_0) \rangle + \alpha\psi(\mathrm{dist}(x, W)) \geq \langle \lambda, f(x_0) \rangle + \beta_1^{-1} \alpha\psi(\mathrm{dist}(x, W)) \langle \lambda, e \rangle, \tag{3.12}$$

for $x \in (S_\lambda \cap U) \setminus W$ and $\lambda \in Q$.

Now, suppose that for all $\beta > 0$, (3.4) is false, then there exist $x' \in (S \cap U) \setminus W$ and $d \in D$ such that

$$f(x') - f(x_0) + d \in B\big(0, \beta\psi(\mathrm{dist}(x, W))\big). \tag{3.13}$$

Let $e \in \mathrm{int}\, D$ be a fixed point, and since D is a cone, there is $k > 0$ such that $B(0, 1) \subset ke - D$. Consequently,

$$B\big(0, \beta\psi(\mathrm{dist}(x, W))\big) \subset k\beta\psi(\mathrm{dist}(x, W))e - D. \tag{3.14}$$

Therefore,

$$f(x') - f(x_0) + d \in k\beta\psi(\mathrm{dist}(x, W))e - D. \tag{3.15}$$

There is $d' \in D$ from (3.15) such that

$$f(x') - f(x_0) = k\beta\psi(\mathrm{dist}(x, W))e - \big(d' + d\big). \tag{3.16}$$

Since $x' \in (S \cap U) \setminus W \subset \bigcup_{\lambda \in Q} S_\lambda \setminus W$, there is $\lambda' \in Q$ such that $x' \in S_{\lambda'}$. Moreover, $\Lambda \subset D^*$ and $d + d' \in D$. Hence,

$$\big\langle \lambda', f(x') \big\rangle - \big\langle \lambda', f(x_0) \big\rangle = k\beta\psi\big(\mathrm{dist}(x', W)\big)\big\langle \lambda', e \big\rangle - \big\langle \lambda', d + d' \big\rangle \leq k\beta\psi\big(\mathrm{dist}(x', W)\big)\big\langle \lambda', e \big\rangle. \tag{3.17}$$

By choosing $\beta = \beta_1^{-1}\alpha k^{-1}$, we obtain a contradiction to (3.12).

(ii) Part "only if": for each $\lambda \in Q$, we define,

$$S_\lambda = \big\{x \in S \cap U : \big\langle \lambda, f(x) \big\rangle \geq \big\langle \lambda, f(x_0) \big\rangle \big\}. \tag{3.18}$$

Now, we will check that (3.6) holds true. Pick any $x \in S \cap U$. Suppose that $x \notin S_\lambda$ for any $\lambda \in Q$, then

$$\big\langle \lambda, f(x) - f(x_0) \big\rangle < 0, \quad \forall \lambda \in Q. \tag{3.19}$$

Hence, for any $\lambda \in \operatorname{cl}\operatorname{cone}\operatorname{co} Q = D^*$, $\langle \lambda, f(x) - f(x_0) \rangle \leq 0$. By applying the bipolar theorem, we have

$$f(x) - f(x_0) \in -D, \tag{3.20}$$

Combing it with the assumption, we have

$$f(x) - f(x_0) \in (-D) \cap D, \tag{3.21}$$

which is a contradiction to (3.19). So (3.6) holds and (3.3) is satisfied by the definition of S_λ.

Part "if": suppose that $x_0 \notin L\operatorname{Min}(f, S)$, then there exists $x \in S \cap U$ such that

$$f(x) - f(x_0) \in -D \setminus D. \tag{3.22}$$

Indeed, $x \in S \cap U$ can be replace by $x \in (S \cap U) \setminus W$, because $x \in W$, $f(x) - f(x_0) = 0$, which is contradiction to (3.22). Hence, for $x \in (S \cap U) \setminus W$, we have $\langle \lambda, f(x) - f(x_0) \rangle \leq 0,\ \forall \lambda \in D^*$. In particular,

$$\langle \lambda, f(x) - f(x_0) \rangle \leq 0, \quad \forall \lambda \in Q. \tag{3.23}$$

It follows from the assumption that

$$(\cup_{\lambda \in Q} S_\lambda \cap U) \setminus W \supset (S \cap U) \setminus W. \tag{3.24}$$

Therefore, by (3.3), we obtain

$$\langle \lambda, f(x) - f(x_0) \rangle > 0, \quad \forall \lambda \in Q,\ \forall x \in (S_\lambda \cap U) \setminus W, \tag{3.25}$$

which contradicts relation (3.23). □

Remark 3.2. By taking $U = X$ in part (i) (resp., (ii)) of Theorem 3.1, we obtain a necessary and sufficient condition for x_0 to be in $\mathrm{WS}(\psi, f, S)$ (resp., $\operatorname{Min}(f, S)$). In particular, if we choose $Y = R^p$ and $D = R^p_+$ and $Q = \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$, then, we obtain Theorem 1 in [14].

Finally, we apply the nonlinear scalarization function to discuss the weak ψ-sharp minimizer in vector optimization problems.

Let $D \subset Y$ be a closed and convex cone with nonempty interior $\operatorname{int} D$. Given a fixed point $e \in \operatorname{int} D$ and $y \in Y$, the nonlinear scalarization function $\xi : Y \to R$ is defined by

$$\xi(y) = \inf\{t : te \in y + D\}. \tag{3.26}$$

This function plays an important role in the context of nonconvex vector optimization problems and has excellent properties such as continuousness, convexity, and (strict) monotonicity on Y. More results about the function can be found in [17].

In what follows, we present several properties about the nonlinear scalarization function.

Lemma 3.3 (see [17]). *For any fixed $e \in \operatorname{int} D$, $y \in Y$, and $r \in R$. One has*

(i) $\xi(y) < r \Leftrightarrow re \in y + \operatorname{int} D$,

(ii) $\xi(y) > r \Leftrightarrow re \notin y + D$.

(iii) $\xi(y) = r \Leftrightarrow re \in y + bdD$.

Given a vector-valued map $f : X \to Y$, define $\tilde{f} : X \to Y$ by

$$\tilde{f}(x) = f(x) - f(x_0). \tag{3.27}$$

Next, we consider weak ψ-sharp local minimizer for a vector-valued map f through a weak sharp local minimizer of a scalar function $\xi \circ \tilde{f} : X \to R$.

Theorem 3.4. *Let $x_0 \in S \subset X$. Suppose that W defined by* (2.4) *is a closed set. Then,*

$$x_0 \in WSL(\psi, f, S) \Longleftrightarrow x_0 \in WSL\left(\psi, \xi \circ \tilde{f}, S\right). \tag{3.28}$$

Proof. Part "only if": let us assume that $x_0 \in \mathrm{WSL}(\psi, f, S)$. Thus, there exist $\alpha > 0$ and $U \in \mathcal{N}(x_0)$ such that

$$\left(f(x) - f(x_0) + D\right) \cap B\left(0, \alpha\psi(\operatorname{dist}(x, W))\right) = \emptyset, \quad \forall x \in (S \cap U) \setminus W. \tag{3.29}$$

Note that, when W is a closed set,

$$\frac{\alpha}{4\|e\|}\psi(\operatorname{dist}(x, W))e \in B\left(0, \alpha\psi(\operatorname{dist}(x, W))\right) \quad \forall x \in (S \cap U) \setminus W. \tag{3.30}$$

Therefore,

$$\frac{\alpha}{4\|e\|}\psi(\operatorname{dist}(x, W))e \notin f(x) - f(x_0) + D \quad \forall x \in (S \cap U) \setminus W. \tag{3.31}$$

By using Lemma 3.3(ii), one has

$$\xi\left(f(x) - f(x_0)\right) > \frac{\alpha}{4\|e\|}\psi(\operatorname{dist}(x, W)) \quad \forall x \in (S \cap U) \setminus W. \tag{3.32}$$

According to Lemma 3.3(iii), one has

$$\xi\left(f(x_0) - f(x_0)\right) = 0. \tag{3.33}$$

This relation, together with (3.32) yields

$$\xi(f(x)-f(x_0)) > \xi(f(x_0)-f(x_0)) + \frac{\alpha}{4\|e\|}\psi(\mathrm{dist}(x,W)), \quad \forall x \in (S\cap U)\setminus W. \tag{3.34}$$

Namely,

$$\left(\xi \circ \tilde{f}\right)(x) > \left(\xi \circ \tilde{f}\right)(x_0) + \frac{\alpha}{4\|e\|}\psi(\mathrm{dist}(x,W)), \quad \forall x \in (S\cap U)\setminus W, \tag{3.35}$$

that is, $x_0 \in \mathrm{WSL}(\psi, \xi \circ \tilde{f}, S)$.

Part "if": by assumption, there exist $\beta > 0$ and $U \in \mathcal{N}(x_0)$ such that

$$\xi\left(\tilde{f}(x)\right) > \xi\left(\tilde{f}(x_0)\right) + \beta\psi(\mathrm{dist}(x,W)), \quad \forall x \in (S\cap U)\setminus W. \tag{3.36}$$

In terms of Lemma 3.3(iii), we have

$$\xi\left(\tilde{f}(x_0)\right) = \xi(f(x_0)-f(x_0)) = 0. \tag{3.37}$$

Hence,

$$\xi(f(x)-f(x_0)) > \beta\psi(\mathrm{dist}(x,W)), \quad \forall x \in (S\cap U)\setminus W. \tag{3.38}$$

Once more using Lemma 3.3(ii), one has

$$\beta\psi(\mathrm{dist}(x,W))e \notin f(x)-f(x_0)+D, \quad \forall x \in (S\cap U)\setminus W, \tag{3.39}$$

which implies that

$$\left(\beta\psi(\mathrm{dist}(x,W))e - D\right) \cap \left(f(x)-f(x_0)+D\right) = \emptyset, \quad \forall x \in (S\cap U)\setminus W. \tag{3.40}$$

Since $e \in \mathrm{int}\, D$, there exists some number $\epsilon > 0$ such that $B(0,\epsilon) \subset e - D$. Moreover,

$$B(0,\lambda\epsilon) \subset \lambda e - D, \quad \forall \lambda > 0. \tag{3.41}$$

Hence, it follows from the relation that

$$B\left(0, \epsilon\beta\psi(\mathrm{dist}(x,W))\right) \subset \beta\psi(\mathrm{dist}(x,W))e - D, \quad \forall x \in (S\cap U)\setminus W. \tag{3.42}$$

Combing it with relation (3.40), we deduce that

$$B\left(0, \epsilon\beta\psi(\mathrm{dist}(x,W))\right) \cap \left(f(x)-f(x_0)+D\right) = \emptyset, \quad \forall x \in (S\cap U)\setminus W. \tag{3.43}$$

Let $\alpha = \epsilon\beta$, by the definition of weak ψ-sharp local minimizer, we have $x_0 \in \mathrm{WSL}(\psi, f, S)$.

It is possible to illustrate Theorem 3.4 by means of adapting a simple example given in [14]. □

Example 3.5. Let $n = p = 2$, $S = \Omega = R^2$, and $D = R_+^2$ and let $f = (f_1, f_2) : R^2 \to R^2$ be defined by

$$f_1\left(x^1, x^2\right) := \max\left\{0, \min\left\{x^1, x^2\right\}\right\} = \begin{cases} x^1, & \text{if } x^2 \geq x^1 > 0, \\ x^2, & \text{if } x^1 > x^2 > 0, \\ 0, & \text{if } x^1 \leq 0 \text{ or } x^2 \leq 0, \end{cases}$$
$$f_2\left(x^1, x^2\right) := \max\left\{0, \min\left\{-x^1, x^2\right\}\right\} = \begin{cases} -x^1, & \text{if } x^2 \geq -x^1 > 0, \\ x^2, & \text{if } -x^1 > x^2 > 0, \\ 0, & \text{if } x^1 \geq 0 \text{ or } x^2 \leq 0, \end{cases} \tag{3.44}$$

We choose $U = R^2$. Using Definition 2.2, we derive that $x_0 = (0, 0) \in \mathrm{WS}(\psi_1, f, S)$.

Let $e = (1, 1)$. From Corollary 1.46 in [17], we have $(\xi \circ \tilde{f})(x) = \max_{1 \leq i \leq 2} f_i(x)$. Observe that

$$W = \{x : f(x) = (0, 0)\} = \left\{x : x^2 \leq 0\right\} \cup \left\{x : x^1 = 0\right\}. \tag{3.45}$$

It is easy to verify that $f_i(x) = \mathrm{dist}(x, W)$ for all $x \in S \setminus W$. Using relation (2.7), we show that $x_0 = (0, 0) \in \mathrm{WS}(\psi_1, \xi \circ \tilde{f}, S)$. Hence, condition (3.28) with $\psi = \psi_1$ holds for $\alpha \in (0, 1)$.

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