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Research Article

Some Common Fixed Point Theorems for Weakly Compatible Mappings in Metric Spaces

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We establish a common fixed point theorem for weakly compatible mappings generalizing a result of Khan and Kubiaczyk (1988). Also, an example is given to support our generalization. We also prove common fixed point theorems for weakly compatible mappings in metric and compact metric spaces.

1. Introduction

In the last years, fixed point theorems have been applied to show the existence and uniqueness of the solutions of differential equations, integral equations and many other branches mathematics (see, e.g., [1–3]). Some common fixed point theorems for weakly commuting, compatible, δ-compatible and weakly compatible mappings under different contractive conditions in metric spaces have appeared in [4–15]. Throughout this paper, (X, d) is a metric space.

Following [9, 16], we define,

$$2^X = \{A \subset X : A \text{ is nonempty}\},$$
$$B(X) = \{A \in 2^X : A \text{ is bounded}\}. \tag{1.1}$$

For all $A, B \in B(X)$, we define

$$\delta(A, B) = \sup\{d(a, b) : a \in A,\ b \in B\},$$
$$D(A, B) = \inf\{d(a, b) : a \in A,\ b \in B\}, \tag{1.2}$$
$$H(A, B) = \inf\{r > 0 : A_r \supset B,\ B_r \supset A\},$$

where $A_r = \{x \in X : d(x,a) < r$, for some $a \in A\}$ and $B_r = \{y \in X : d(y,b) < r$, for some $b \in B\}$.

If $A = \{a\}$ for some $a \in A$, we denote $\delta(a,B)$, $D(a,B)$ and $H(a,B)$ for $\delta(A,B)$, $D(A,B)$ and $H(A,B)$, respectively. Also, if $B = \{b\}$, then one can deduce that $\delta(A,B) = D(A,B) = H(A,B) = d(a,b)$.

It follows immediately from the definition of $\delta(A,B)$ that, for every $A, B, C \in B(X)$,

$$\begin{gathered}\delta(A,B) = \delta(B,A) \geq 0, \quad \delta(A,B) \leq \delta(A,C) + \delta(C,B), \quad \delta(A,B) = 0, \\ \text{iff } A = B = \{a\}, \quad \delta(A,A) = \operatorname{diam} A.\end{gathered} \tag{1.3}$$

We need the following definitions and lemmas.

Definition 1.1 (see [16]). A sequence (A_n) of nonempty subsets of X is said to be *convergent* to $A \subseteq X$ if:

(i) each point a in A is the limit of a convergent sequence (a_n), where a_n is in A_n for $n \in \{0\} \cup N$ (N:= the set of all positive integers),

(ii) for arbitrary $\epsilon > 0$, there exists an integer m such that $A_n \subseteq A_\epsilon$ for $n > m$, where A_ϵ denotes the set of all points x in X for which there exists a point a in A, depending on x, such that $d(x,a) < \epsilon$.

A is then said to be the *limit* of the sequence (A_n).

Definition 1.2 (see [9]). A set-valued function $F : X \to 2^X$ is said to be *continuous* if for any sequence (x_n) in X with $\lim_{n\to\infty} x_n = x$, it yields $\lim_{n\to\infty} H(Fx_n, Fx) = 0$.

Lemma 1.3 (see [16]). *If (A_n) and (B_n) are sequences in $B(X)$ converging to A and B in $B(X)$, respectively, then the sequence $(\delta(A_n, B_n))$ converges to $\delta(A,B)$.*

Lemma 1.4 (see [16]). *Let (A_n) be a sequence in $B(X)$ and let y be a point in X such that $\delta(A_n, y) \to 0$. Then the sequence (A_n) converges to the set $\{y\}$ in $B(X)$.*

Lemma 1.5 (see [9]). *For any $A, B, C, D \in B(X)$, it yields that $\delta(A,B) \leq H(A,C) + \delta(C,D) + H(D,B)$.*

Lemma 1.6 (see [17]). *Let $\Psi : [0,\infty) \to [0,\infty)$ be a right continuous function such that $\Psi(t) < t$ for every $t > 0$. Then, $\lim_{n\to\infty}\Psi^n(t) = 0$ for every $t > 0$, where Ψ^n denotes the n-times repeated composition of Ψ with itself.*

Definition 1.7 (see [15]). The mappings $I : X \to X$ and $F : X \to B(X)$ are *weakly commuting on* X if $IFx \in B(X)$ and $\delta(FIx, IFx) \leq \max\{\delta(Ix, Fx), \operatorname{diam} IFx\}$ for all $x \in X$.

Definition 1.8 (see [13]). The mappings $I : X \to X$ and $F : X \to B(X)$ are said to be *δ-compatible* if $\lim_{n\to\infty} \delta(FIx_n, IFx_n) = 0$ whenever (x_n) is a sequence in X such that $IFx_n \in B(X)$, $Fx_n \to \{t\}$ and $Ix_n \to t$ for some $t \in X$.

Definition 1.9 (see [13]). The mappings $I : X \to X$ and $F : X \to B(X)$ are *weakly compatible* if they commute at coincidence points, that is, for each point $u \in X$ such that $Fu = \{Iu\}$, then $FIu = IFu$ (note that the equation $Fu = \{Iu\}$ implies that Fu is a singleton).

If F is a single-valued mapping, then Definition 1.7 (resp., Definitions 1.8 and 1.9) reduces to the concept of weak commutativity (resp., compatibility and weak compatibility) for single-valued mappings due to Sessa [18] (resp., Jungck [11, 12]).

It can be seen that

$$\text{weakly commuting} \Longrightarrow \delta\text{-compatible and } \delta\text{-compatible} \Longrightarrow \text{weakly compatible}, \tag{1.4}$$

but the converse of these implications may not be true (see, [13, 15]).

Throughtout this paper, we assume that Φ is the set of all functions $\phi : [0,\infty)^5 \to [0,\infty)$ satisfying the following conditions:

(i) ϕ is upper semi-continuous continuous at a point 0 from the right, and non-decreasing in each coodinate variable,

(ii) For each $t > 0$, $\Psi(t) = \max\{\phi(t,t,t,t,t), \phi(t,t,t,2t,0), \phi(t,t,t,0,2t)\} < t$.

Theorem 1.10 (see [19]). *Let F, G be mappings of a complete metric space (X,d) into $B(X)$ and I be a mapping of X into itself such that I, F and G are continuous, $F(X) \subseteq J(X)$, $G(X) \subseteq I(X)$, $IF = FI$, $IG = GI$ and for all $x, y \in X$,*

$$\delta(Fx,Gy) \le \phi(d(Ix,Iy), \delta(Ix,Fx), \delta(Iy,Gy), D(Ix,Gy), D(Iy,Fx)), \tag{1.5}$$

where ϕ satisfies (i) and $\phi(t,t,t,at,bt) < t$ for each $t > 0$, and $a \ge 0$, $b \ge 0$ with $a + b \le 2$. Then I, F and G have a unique common fixed point u such that $u = Iu \in Fu \cap Gu$.

In the present paper, we are concerned with the following:

(1) replacing the commutativity of the mappings in Theorem 1.10 by the weak compatibility of a pair of mappings to obtain a common fixed point theorem metric spaces without the continuity assumption of the mappings,

(2) giving an example to support our generalization of Theorem 1.10,

(3) establishing another common fixed point theorem for two families of set-valued mappings and two single-valued mappings,

(4) proving a common fixed point theorem for weakly compatible mappings under a strict contractive condition on compact metric spaces.

2. Main Results

In this section, we establish a common fixed point theorem in metric spaces generalizing Theorems 1.10. Also, an example is introduced to support our generalization. We prove a common fixed point theorem for two families of set-valued mappings and two single-valued mappings. Finally, we establish a common fixed point theorem under a strict contractive condition on compact metric spaces.

First we state and prove the following.

Theorem 2.1. *Let I, J be two sefmaps of a metric space (X, d) and let $F, G : X \to B(X)$ be two set-valued mappings with*

$$\cup F(X) \subseteq J(X), \qquad \cup G(X) \subseteq I(X). \tag{2.1}$$

Suppose that one of $I(X)$ and $J(X)$ is complete and the pairs $\{F, I\}$ and $\{G, J\}$ are weakly compatible. If there exists a function $\phi \in \Phi$ such that for all $x, y \in X$,

$$\delta(Fx, Gy) \leq \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)), \tag{2.2}$$

then there is a point $p \in X$ such that $\{p\} = \{Ip\} = \{Jp\} = Fp = Gp$.

Proof. Let x_0 be an arbitrary point in X. By (2.1), we choose a point x_1 in X such that $Jx_1 \in Fx_0 = Z_0$ and for this point x_1 there exists a point x_2 in X such that $Ix_2 \in Gx_1 = Z_1$. Continuing this manner we can define a sequence (x_n) as follows:

$$Jx_{2n+1} \in Fx_{2n} = Z_{2n}, \qquad Ix_{2n+2} \in Gx_{2n+1} = Z_{2n+1}, \tag{2.3}$$

for $n \in \{0\} \cup N$. For simplicity, we put $V_n = \delta(Z_n, Z_{n+1})$ for $n \in \{0\} \cup N$. By (2.2) and (2.3), we have that

$$\begin{aligned} V_{2n} &= \delta(Z_{2n}, Z_{2n+1}) = \delta(Fx_{2n}, Gx_{2n+1}) \\ &\leq \phi(d(Ix_{2n}, Jx_{2n+1}), \delta(Ix_{2n}, Fx_{2n}), \delta(Jx_{2n+1}, Gx_{2n+1}), D(Ix_{2n}, Gx_{2n+1}), D(Jx_{2n+1}, Fx_{2n})) \\ &\leq \phi(\delta(Z_{2n-1}, Z_{2n}), \delta(Z_{2n-1}, Z_{2n}), \delta(Z_{2n}, Z_{2n+1}), \delta(Z_{2n-1}, Z_{2n}) + \delta(Z_{2n}, Z_{2n+1}), 0) \\ &= \phi(V_{2n-1}, V_{2n-1}, V_{2n}, V_{2n-1} + V_{2n}, 0). \end{aligned} \tag{2.4}$$

If $V_{2n} > V_{2n-1}$, then

$$V_{2n} \leq \phi(V_{2n}, V_{2n}, V_{2n}, 2V_{2n}, 0) \leq \Psi(V_{2n}) < V_{2n}. \tag{2.5}$$

This contradiction demands that

$$V_{2n} \leq \phi(V_{2n-1}, V_{2n-1}, V_{2n-1}, 2V_{2n-1}, 0) \leq \Psi(V_{2n-1}). \tag{2.6}$$

Similarly, one can deduce that

$$V_{2n+1} \leq \phi(V_{2n}, V_{2n}, V_{2n}, 0, 2V_{2n}) \leq \Psi(V_{2n}). \tag{2.7}$$

So, for each $n \in \{0\} \cup N$, we obtain that

$$V_{n+1} \leq \Psi(V_n) \leq \Psi^2(V_{n-1}) \leq \cdots \leq \Psi^n(V_1), \tag{2.8}$$

where $V_1 = \delta(Z_1, Z_2) = \delta(Fx_2, Gx_1) \leq \phi(V_0, V_0, V_0, 0, 2V_0)$. By (2.8) and Lemma 1.6, we obtain that $\lim_{n\to\infty} V_n = \lim_{n\to\infty} \delta(Z_n, Z_{n+1}) = 0$. Since

$$\delta(Z_n, Z_m) \leq \delta(Z_n, Z_{n+1}) + \delta(Z_{n+1}, Z_{n+2}) + \cdots + \delta(Z_{m-1}, Z_m), \tag{2.9}$$

then $\lim_{n,m\to\infty} \delta(Z_n, Z_m) = 0$. Therefore, (Z_n) is a Cauchy sequence.

Let z_n be an arbitrary point in Z_n for $n \in \{0\} \cup N$. Then $\lim_{n,m\to\infty} d(z_n, z_m) \leq \lim_{n,m\to\infty} \delta(Z_n, Z_m) = 0$ and (z_n) is a Cauchy sequence. We assume without loss of generality that $J(X)$ is complete. Let (x_n) be the sequence defined by (2.3). But $Jx_{2n+1} \in Fx_{2n} = Z_{2n}$ for all $n \in \{0\} \cup N$. Hence, we find that

$$d(Jx_{2m-1}, Jx_{2n+1}) \leq \delta(Z_{2m-2}, Z_{2n}) \leq V_{2m-2} + \delta(Z_{2m-1}, Z_{2n}) \longrightarrow 0, \tag{2.10}$$

as $m, n \to \infty$. So, (Jx_{2n+1}) is a Cauchy sequence. Hence, $Jx_{2n+1} \to p = Jv \in J(X)$ for some $v \in X$. But $Ix_{2n} \in Gx_{2n-1} = Z_{2n-1}$ by (2.3), so that $d(Ix_{2n}, Jx_{2n+1}) \leq \delta(Z_{2n-1}, Z_{2n}) = V_{2n-1} \to 0$. Consequently, $Ix_{2n} \to p$. Moreover, we have, for $n \in \{0\} \cup N$, that $\delta(Fx_{2n}, p) \leq \delta(Fx_{2n}, Ix_{2n}) + d(Ix_{2n}, p) \leq V_{2n-1} + d(Ix_{2n}, p)$. Therefore, $\delta(Fx_{2n}, p) \to 0$. So, we have by Lemma 1.4 that $Fx_{2n} \to \{p\}$. In like manner it follows that $\delta(Gx_{2n+1}, p) \to 0$ and $Gx_{2n+1} \to \{p\}$.

Since, for $n \in \{0\} \cup N$,

$$\begin{aligned} \delta(Fx_{2n}, Gv) &\leq \phi(d(Ix_{2n}, Jv), \delta(Ix_{2n}, Fx_{2n}), \delta(Jv, Gv), D(Ix_{2n}, Gv), D(Jv, Fx_{2n})) \\ &\leq \phi(d(Ix_{2n}, Jv), \delta(Ix_{2n}, Fx_{2n}), \delta(Jv, Gv), \delta(Ix_{2n}, Gv), \delta(Jv, Fx_{2n})), \end{aligned} \tag{2.11}$$

and $\delta(Ix_{2n}, Gv) \to \delta(p, Gv)$ as $n \to \infty$, we get from Lemma 1.3 that

$$\delta(p, Gv) \leq \phi(0, 0, \delta(p, Gv), \delta(p, Gv), 0) \leq \Psi(\delta(p, Gv)) < \delta(p, Gv). \tag{2.12}$$

This is absurd. So, $\{p\} = Gv = \{Jv\}$. But $\cup G(X) \subseteq I(X)$, so $\exists u \in X$ such that $\{Iu\} = Gv = \{Jv\}$. If $Fu \neq Gv$, $\delta(Fu, Gv) \neq 0$, then we have

$$\begin{aligned} \delta(Fu, p) &= \delta(Fu, Gv) \\ &\leq \phi(d(Iu, Jv), \delta(Iu, Fu), \delta(Jv, Gv), D(Iu, Gv), D(Jv, Fu)) \\ &\leq \phi(d(Iu, Jv), \delta(Iu, Fu), \delta(Jv, Gv), \delta(Iu, Gv), \delta(Jv, Fu)) \\ &= \phi(0, \delta(Fu, p), 0, 0, \delta(Fu, p)) \\ &\leq \Psi(\delta(Fu, p)) < \delta(Fu, p). \end{aligned} \tag{2.13}$$

We must conclude that$\{p\} = Fu = Gv = \{Iu\} = \{Jv\}$.

Since $Fu = \{Iu\}$ and the pair $\{F, I\}$ is weakly compatible, so $Fp = FIu = IFu = \{Ip\}$. Using the inequality (2.2), we have

$$\begin{aligned}\delta(Fp,p) &\leq \delta(Fp,Gv)\\ &\leq \phi(d(Ip,Jv),\delta(Ip,Fp),\delta(Jv,Gv),D(Ip,Gv),D(Jv,Fp))\\ &\leq \phi(\delta(Fp,p),0,0,\delta(Fp,p),\delta(Fp,p))\\ &\leq \Psi(\delta(Fp,p))\\ &< \delta(Fp,p).\end{aligned} \tag{2.14}$$

This contradiction demands that $\{p\} = Fp = \{Ip\}$. Similarly, if the pair $\{G, J\}$ is weakly compatible, one can deduce that $\{p\} = Gp = \{Jp\}$. Therefore, we get that $\{p\} = Fp = Gp = \{Ip\} = \{Jp\}$.

The proof, assuming the completeness of $I(X)$, is similar to the above.

To see that p is unique, suppose that $\{q\} = Fq = Gq = \{Iq\} = \{Jq\}$. If $p \neq q$, then

$$d(p,q) = \delta(Fp,Gq) \leq \phi(d(p,q),0,0,d(p,q),d(p,q)) \leq \Psi(d(p,q)) < d(p,q), \tag{2.15}$$

which is inadmissible. So, $p = q$.

Now, we give an example to show the greater generality of Theorem 2.1 over Theorem 1.10. □

Example 2.2. Let $X = [0,1]$ endowed with the Euclidean metric d. Assume that $\phi(t_1,t_2,t_3,t_4,t_5) = t_1/3$ for every $t_1,t_2,t_3,t_4,t_5 \in [0,\infty)$. Define $F, G : X \to B(X)$ and $I, J : X \to X$as follows:

$$\begin{gathered} Fx = \left\{\frac{1}{2}\right\} \text{ if } x \in X, \qquad Gx = \left\{\frac{1}{2}\right\} \text{ if } x \in \left[0,\frac{1}{2}\right], \qquad Gx = \left(\frac{3}{8},\frac{1}{2}\right] \text{ if } x \in \left(\frac{1}{2},1\right], \\ Ix = \frac{1}{2} \text{ if } x \in \left[0,\frac{1}{2}\right], \qquad Ix = \frac{x+1}{4} \text{ if } x \in \left(\frac{1}{2},1\right], \qquad Jx = 1-x \text{ if } x \in \left[0,\frac{1}{2}\right], \\ Jx = 0 \text{ if } x \in \left(\frac{1}{2},1\right]. \end{gathered} \tag{2.16}$$

We have that $\cup F(X) = \{1/2\} = \{J(1/2)\} \subseteq J(X)$ and $\cup G(X) = (3/8, 1/2] = I(X)$. Moreover, $\delta(Fx, Gy) = 0$ if $y \in [0, 1/2]$. If $y \in (1/2, 1]$, then $\delta(Fx, Gy) \leq 1/8$ and $d(Ix, Jy) \geq 3/8$. So, we obtain that

$$\delta(Fx,Gy) \leq \frac{1}{3}d(Ix,Jy) = \frac{1}{3}\phi(d(Ix,Jy),\delta(Ix,Fx),\delta(Jy,Gy),D(Ix,Gy),D(Jy,Fx)), \tag{2.17}$$

for all $x, y \in X$. It is clear that X is a complete metric space. Since $J(X) = [1/2, 1] \cup \{0\}$ is a closed subset of X, so $J(X)$ is complete. We note that $\{F, I\}$ is a δ-compatible pair and therefore a weakly compatible pair. Also, $G(1/2) = \{J(1/2)\}$ and $GJ(1/2) = JG(1/2) = \{1/2\}$, that is, G and J are weakly compatible. On the other hand, if $x_n = 1/2 - 2^{-n}$, so that $\delta(GJx_n, JGx_n) \to 1/8 \neq 0$ even though $Gx_n, \{Jx_n\} \to \{1/2\}$, that is, $\{G, J\}$ is not a δ-compatible pair. We know that $1/2$ is the unique common fixed point of I, J, F and G. Hence the hypotheses of Theorem 2.1 are satisfied. Theorem 1.10 is not applicable because $GJx \neq JGx$ for all $x \in X$, and the maps I, J and G are not continuous at $x = 1/2$.

In Theorem 2.1, if the mappings F and G are replaced by F_α and G_α, $\alpha \in \Lambda$ where Λ is an index set, we obtain the following.

Theorem 2.3. *Let (X, d) be a metric space, and let I, J be selfmaps of X, and for $\alpha \in \Lambda$, $F_\alpha, G_\alpha : X \to B(X)$ be set-valued mappings with $\cup[\cup_{\alpha\in\Lambda} F_\alpha(X)] \subseteq J(X)$ and $\cup[\cup_{\alpha\in\Lambda} G_\alpha(X)] \subseteq I(X)$. Suppose that one of $I(X)$ and $J(X)$ is complete and for $\alpha \in \Lambda$ the pairs $\{F_\alpha, I\}$ and $\{G_\alpha, J\}$ are weakly compatible. If there exists a function $\phi \in \Phi$ such that, for all $x, y \in X$,*

$$\delta(F_\alpha x, G_\alpha y) \leq \phi(d(Ix, Jy), \delta(Ix, F_\alpha x), \delta(Jy, G_\alpha y), D(Ix, G_\alpha y), D(Jy, F_\alpha x)), \tag{2.18}$$

then there is a point $p \in X$ such that $\{p\} = \{Ip\} = \{Jp\} = F_\alpha p = G_\alpha p$ for each $\alpha \in \Lambda$.

Proof. Using Theorem 2.1, we obtain for any $\alpha \in \Lambda$, there is a unique point $z_\alpha \in X$ such that $Iz_\alpha = Jz_\alpha = z_\alpha$ and $F_\alpha z_\alpha = G_\alpha z_\alpha = \{z_\alpha\}$. For all $\alpha, \beta \in \Lambda$,

$$\begin{aligned} d(z_\alpha, z_\beta) &\leq \delta(F_\alpha z_\alpha, G_\beta z_\beta) \\ &\leq \phi(d(Iz_\alpha, Jz_\beta), \delta(Iz_\alpha, F_\alpha z_\alpha), \delta(Jz_\beta, G_\beta z_\beta), D(Iz_\alpha, G_\beta z_\beta), D(Jz_\beta, F_\alpha z_\alpha)) \\ &\leq \phi(d(z_\alpha, z_\beta), 0, 0, d(z_\alpha, z_\beta), d(z_\beta, z_\alpha)) \\ &\leq \Psi(d(z_\alpha, z_\beta)) < d(z_\alpha, z_\beta). \end{aligned} \tag{2.19}$$

This yields that $z_\alpha = z_\beta$.

Inspired by the work of Chang [9], we state the following theorem on compact metric spaces. □

Theorem 2.4. *Let (X, d) be a compact metric space, I, J selfmaps of X, $F, G : X \to B(X)$ set-valued functions with $\cup F(X) \subseteq J(X)$ and $\cup G(X) \subseteq I(X)$. Suppose that the pairs $\{F, I\}$, $\{G, J\}$ are weakly compatible and the functions F, I are continuous. If there exists a function $\phi \in \Phi$, and for all $x, y \in X$, the following inequality:*

$$\delta(Fx, Gy) < \phi(d(Ix, Jy), \delta(Ix, Fx), \delta(Jy, Gy), D(Ix, Gy), D(Jy, Fx)), \tag{2.20}$$

holds whenever the right-hand side of (2.20) is positive, then there is a unique point u in X such that $Fu = Gu = \{u\} = \{Iu\} = \{Ju\}$.

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