Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 751090, 6 pages
doi:10.1155/2009/751090

Research Article

On T-Stability of Picard Iteration in Cone Metric Spaces

M. Asadi,[1] H. Soleimani,[1] S. M. Vaezpour,[2,3] and B. E. Rhoades[4]

[1] *Department of Mathematics, Science and Research Branch, Islamic Azad University (IAU), Tehran 14778 93855, Iran*
[2] *Department of Mathematics, Amirkabir University of Technology, Tehran 15916 34311, Iran*
[3] *Department of Mathematics, Newcastle University, Newcastle, NSW 2308, Australia*
[4] *Department of Mathematics, Indiana University, Bloomington, IN 46205, USA*

Correspondence should be addressed to S. M. Vaezpour, vaez@aut.ac.ir

Received 28 March 2009; Revised 28 September 2009; Accepted 19 October 2009

Recommended by Brailey Sims

The aim of this work is to investigate the T-stability of Picard's iteration procedures in cone metric spaces and give an application.

1. Introduction and Preliminary

Let E be a real Banach space. A nonempty convex closed subset $P \subset E$ is called a cone in E if it satisfies the following:

(i) P is closed, nonempty, and $P \neq \{0\}$,

(ii) $a, b \in \mathbb{R}$, $a, b \geq 0$, and $x, y \in P$ imply that $ax + by \in P$,

(iii) $x \in P$ and $-x \in P$ imply that $x = 0$.

The space E can be partially ordered by the cone $P \subset E$; by defining, $x \leq y$ if and only if $y - x \in P$. Also, we write $x \ll y$ if $y - x \in \operatorname{int} P$, where $\operatorname{int} P$ denotes the interior of P.

A cone P is called normal if there exists a constant $K > 0$ such that $0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

In the following we always suppose that E is a real Banach space, P is a cone in E, and $\leq$ is the partial ordering with respect to P.

Definition 1.1 (see [1]). Let X be a nonempty set. Assume that the mapping $d : X \times X \to E$ satisfies the following:

(i) $0 \leq d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$,

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$,

(iii) $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Definition 1.2. Let $T : X \to X$ be a map for which there exist real numbers a, b, c satisfying $0 < a < 1$, $0 < b < 1/2$, $0 < c < 1/2$. Then T is called a *Zamfirescu operator* if, for each pair $x, y \in X$, T satisfies at least one of the following conditions:

(1) $d(Tx, Ty) \leq ad(x, y)$,

(2) $d(Tx, Ty) \leq b(d(x, Tx) + d(y, Ty))$,

(3) $d(Tx, Ty) \leq c(d(x, Ty) + d(y, Tx))$.

Every Zamfirescu operator T satisfies the inequality:

$$d(Tx, Ty) \leq \delta d(x, y) + 2\delta d(x, Tx) \tag{1.1}$$

for all $x, y \in X$, where $\delta = \max\{a, b/(1-b), c/(1-c)\}$, with $0 < \delta < 1$. For normed spaces see [2].

Lemma 1.3 (see [3]). *Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences satisfying the following inequality:*

$$a_{n+1} \leq (1 - \lambda_n)a_n + b_n, \tag{1.2}$$

where $\lambda_n \in (0,1)$, for all $n \geq n_0$, $\sum_{n=1}^{\infty} \lambda_n = \infty$, and $b_n/\lambda_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Remark 1.4. Let $\{a_n\}$ and $\{b_n\}$ be nonnegative real sequences satisfying the following inequality:

$$a_{n+1} \leq \lambda a_{n-m} + b_n, \tag{1.3}$$

where $\lambda \in (0,1)$, for all $n \geq n_0$ and for some positive integer number m. If $b_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Lemma 1.5. *Let P be a normal cone with constant K, and let $\{a_n\}$ and $\{b_n\}$ be sequences in E satisfying the following inequality:*

$$a_{n+1} \leq ha_n + b_n, \tag{1.4}$$

where $h \in (0,1)$ and $b_n \to 0$ as $n \to \infty$. Then $\lim_{n\to\infty} a_n = 0$.

Proof. Let m be a positive integer such that $h^m K < 1$. By recursion we have

$$a_{n+1} \leq b_n + hb_{n-1} + \cdots + h^m b_{n-m} + h^{m+1} a_{n-m}, \tag{1.5}$$

so

$$\|a_{n+1}\| \le K\|b_n + hb_{n-1} + \cdots + h^m b_{n-m}\| + h^{m+1}K\|a_{n-m}\|, \tag{1.6}$$

and then by Remark 1.4 $\|a_n\| \to 0$. Therefore $a_n \to 0$. □

2. T-Stability in Cone Metric Spaces

Let (X, d) be a cone metric space, and T a self-map of X. Let x_0 be a point of X, and assume that $x_{n+1} = f(T, x_n)$ is an iteration procedure, involving T, which yields a sequence $\{x_n\}$ of points from X.

Definition 2.1 (see [4]). The iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T-stable with respect to T if $\{x_n\}$ converges to a fixed point q of T and whenever $\{y_n\}$ is a sequence in X with $\lim_{n\to\infty} d(y_{n+1}, f(T, y_n)) = 0$ we have $\lim_{n\to\infty} y_n = q$.

In practice, such a sequence $\{y_n\}$ could arise in the following way. Let x_0 be a point in X. Set $x_{n+1} = f(T, x_n)$. Let $y_0 = x_0$. Now $x_1 = f(T, x_0)$. Because of rounding or discretization in the function T, a new value y_1 approximately equal to x_1 might be obtained instead of the true value of $f(T, x_0)$. Then to approximate y_2, the value $f(T, y_1)$ is computed to yield y_2, an approximation of $f(T, y_1)$. This computation is continued to obtain $\{y_n\}$ an approximate sequence of $\{x_n\}$.

One of the most popular iteration procedures for approximating a fixed point of T is Picard's iteration defined by $x_{n+1} = Tx_n$. If the conditions of Definition 2.1 hold for $x_{n+1} = Tx_n$, then we will say that Picard's iteration is T-stable.

Recently Qing and Rhoades [5] established a result for the T-stability of Picard's iteration in metric spaces. Here we are going to generalize their result to cone metric spaces and present an application.

Theorem 2.2. *Let (X, d) be cone metric space, P a normal cone, and $T : X \to X$ with $F(T) \neq \emptyset$. If there exist numbers $a \ge 0$ and $0 \le b < 1$, such that*

$$d(Tx, q) \le ad(x, Tx) + bd(x, q) \tag{2.1}$$

for each $x \in X$, $q \in F(T)$ and in addition, whenever $\{y_n\}$ is a sequence with $d(y_n, Ty_n) \to 0$ as $n \to \infty$, then Picard's iteration is T-stable.

Proof. Suppose $\{y_n\} \subseteq X$, $c_n = d(y_{n+1}, Ty_n)$ and $c_n \to 0$. We shall show that $y_n \to q$. Since

$$d(y_{n+1}, q) \le d(y_{n+1}, Ty_n) + d(Ty_n, q) \le c_n + ad(y_n, Ty_n) + bd(y_n, q), \tag{2.2}$$

if we put $a_n := d(Ty_n, q)$ and $b_n := c_n + ad(y_n, Ty_n)$ in Lemma 1.5, then we have $y_n \to q$.

Note that the fixed point q of T is unique. Because if p is another fixed point of T, then

$$d(p, q) = d(Tp, q) \le ad(p, Tp) + bd(p, q) = bd(p, q), \tag{2.3}$$

which implies $p = q$. □

Corollary 2.3. *Let (X,d) be a cone metric space, P a normal cone, and $T : X \to X$ with $q \in F(T)$. If there exists a number $\lambda \in [0,1)$, such that $d(Tx,Ty) \leq \lambda d(x,y)$, for each $x,y \in X$, then Picard's iteration is T-stable.*

Corollary 2.4. *Let (X,d) be a cone metric space, P a normal cone, and $T : X \to X$ is a Zamfirescu operator with $F(T) \neq \emptyset$ and whenever $\{y_n\}$ is a sequence with $d(y_n,Ty_n) \to 0$ as $n \to \infty$, then Picard's iteration is T-stable.*

Definition 2.5 (see [6]). Let (X,d) be a cone metric space. A map $T : X \to X$ is called a quasicontraction if for some constant $\lambda \in (0,1)$ and for every $x,y \in X$, there exists $u \in C(T;x,y) \equiv \{d(x,y), d(x,Tx), d(y,Ty), d(x,Ty), d(y,Tx)\}$, such that $d(Tx,Ty) \leq \lambda u$.

Lemma 2.6. *If T is a quasicontraction with $0 < \lambda < 1/2$, then T is a Zamfirescu operator and so satisfies* (2.1).

Proof. Let $\lambda \in (0,1/2)$ for every $x,y \in X$ we have $d(Tx,Ty) \leq \lambda u$ for some $u \in C(T;x,y)$. In the case that $u = d(x,Ty)$ we have

$$d(Tx,Ty) \leq \lambda d(x,Ty) \leq \lambda d(x,Tx) + \lambda d(Tx,Ty). \tag{2.4}$$

So

$$d(Tx,Ty) \leq \frac{\lambda}{1-\lambda} d(x,Tx) \leq 2\frac{\lambda}{1-\lambda} d(x,Tx) + \frac{\lambda}{1-\lambda} d(x,y). \tag{2.5}$$

Put $\delta := \lambda/(1-\lambda)$ so $0 < \delta < 1$. The other cases are similarly proved. Therefore T is a Zamfirescu operator. □

Theorem 2.7. *Let (X,d) be a nonempty complete cone metric space, P be a normal cone, and T a quasicontraction and self map of X with some $0 < \lambda < 1/2$. Then Picard's iteration is T-stable.*

Proof. By [6, Theorem 2.1], T has a unique fixed point $q \in X$. Also T satisfies (2.1). So by Theorem 2.2 it is enough to show that $d(y_n,Ty_n) \to 0$. We have

$$d(y_n,Ty_n) \leq d(y_n,Ty_{n-1}) + d(Ty_{n-1},Ty_n). \tag{2.6}$$

Put $b_n := d(y_n,Ty_n)$, $c_n := d(y_{n+1},Ty_n)$ and $d_n := d(Ty_{n-1},Ty_n)$. Therefore $c_n \to 0$ as $n \to \infty$ and

$$b_n \leq c_{n-1} + d_n \leq c_{n-1} + \lambda u_n, \tag{2.7}$$

where

$$u_n \in C(T,y_{n-1},y_n) = \{d(y_{n-1},y_n), d(y_{n-1},Ty_{n-1}), d(y_n,Ty_n), d(y_{n-1},Ty_n), d(y_n,Ty_{n-1})\}. \tag{2.8}$$

Hence we have $u_n = b_n$ or $u_n \leq sb_{n-1} + lc_{n-1}$ where $s = 0, 1$ or $1/(1-\lambda)$ and $l = 1$ or $1+\lambda$. Therefore by (2.7), $b_n \leq (\lambda l + 1)c_{n-1} + \lambda s b_{n-1}$ by $0 \leq \lambda s < 1$. Now by Lemma 1.5 we have $b_n \to 0$. □

3. An Application

Theorem 3.1. *Let* $X := (C[0,1], \mathbb{R})$ *with* $\|f\|_\infty := \sup_{0\leq x\leq 1} |f(x)|$ *for* $f \in X$ *and let* T *be a self map of* X *defined by* $Tf(x) = \int_0^1 F(x, f(t))dt$ *where*

(a) $F : [0,1] \times \mathbb{R} \to \mathbb{R}$ *is a continuous function,*

(b) *the partial derivative* F_y *of* F *with respect to* y *exists and* $|F_y(x,y)| \leq L$ *for some* $L \in [0,1)$,

(c) *for every real number* $0 \leq a < 1$ *one has* $ax \leq F(x, ay)$ *for every* $x, y \in [0,1]$.

Let $P := \{(x,y) \in \mathbb{R}^2 \mid x, y \geq 0\}$ *be a normal cone and* (X, d) *the complete cone metric space defined by* $d(f,g) = (\|f-g\|_\infty, \alpha\|f-g\|_\infty)$ *where* $\alpha \geq 0$. *Then,*

(i) *Picard's iteration is* T*-stable if* $0 \leq L < 1/2$,

(ii) *Picard's iteration fails to be* T*-stable if* $1/2 \leq L < 1$ *and* $\int_0^1 F(x,t)dt \neq x$.

Proof. (i) We have T being a continuous quasicontraction map with $0 \leq \lambda := L < 1/2$; so by Theorem 2.7, Picard's iteration is T-stable.

(ii) Put $y_n(x) := nx/(n+1)$ so $y_n \in X$ and $d(y_n, h) \to 0$, where $h(x) = x$. Also $d(y_{n+1}, Ty_n) \to 0$, since

$$\begin{aligned} \|y_{n+1} - Ty_n\|_\infty &= \sup_{0\leq x\leq 1} \left| \frac{n+1}{n+2}x - \int_0^1 F\left(x, \frac{nt}{n+1}\right)dt \right| \\ &\leq \sup_{0\leq x\leq 1} \left| \frac{n+1}{n+2}x - \frac{nx}{n+1} \right| \longrightarrow 0, \end{aligned} \tag{3.1}$$

as $n \to \infty$. But $y_n \to h$ and h is not a fixed point for T. Therefore Picard's iteration is not T-stable. □

Example 3.2. Let $F_1(x,y) := x + y/4$ and $F_2(x,y) := x + y/2$. Therefore F_1 and F_2 satisfy the hypothesis of Theorem 3.1 where F_1 has property (i) and F_2 has property (ii). So the self maps T_1, T_2 of X defined by $T_1 f(x) = x + (1/4)\int_0^1 f(t)dt$ and $T_2 f(x) = x + (1/2)\int_0^1 f(t)dt$ have unique fixed points but Picard's iteration is T-stable for T_1 but not T-stable for T_2.

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