Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 707406, 6 pages
doi:10.1155/2009/707406

Research Article

A Note on Implicit Functions in Locally Convex Spaces

Marianna Tavernise and Alessandro Trombetta

Dipartimento di Matematica, Università degli Studi della Calabria, 87036 Arcavacata di Rende (CS), Italy

Correspondence should be addressed to Marianna Tavernise, tavernise@mat.unical.it

Received 27 February 2009; Accepted 19 October 2009

Recommended by Fabio Zanolin

An implicit function theorem in locally convex spaces is proved. As an application we study the stability, with respect to a parameter λ, of the solutions of the Hammerstein equation $x = \lambda KFx$ in a locally convex space.

1. Introduction

Implicit function theorems are an important tool in nonlinear analysis. They have significant applications in the theory of nonlinear integral equations. One of the most important results is the classic Hildebrandt-Graves theorem. The main assumption in all its formulations is some differentiability requirement. Applying this theorem to various types of Hammerstein integral equations in Banach spaces, it turned out that the hypothesis of existence and continuity of the derivative of the operators related to the studied equation is too restrictive. In [1] it is introduced an interesting linearization property for parameter dependent operators in Banach spaces. Moreover, it is proved a generalization of the Hildebrandt-Graves theorem which implies easily the second averaging theorem of Bogoljubov for ordinary differential equations on the real line.

Let $X = (X, \|\cdot\|_X)$ and $Y = (Y, \|\cdot\|_Y)$ be Banach spaces, Λ an open subset of the real line $\mathbb{R}$ or of the complex plane $\mathbb{C}$, A an open subset of the product space $\Lambda \times X$ and $\mathcal{L}(X, Y)$ the space of all continuous linear operators from X into Y. An operator $\Phi : A \to Y$ and an operator function $L : \Lambda \to \mathcal{L}(X, Y)$ are called *osculating* at $(\lambda_0, x_0) \in A$ if there exists a function $\sigma : \mathbb{R}^2 \to [0, +\infty)$ such that $\lim_{(\rho,r)\to(0,0)}\sigma(\rho, r) = 0$ and

$$\|\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)\|_Y \le \sigma(\rho, r)\|x_1 - x_2\|_X, \tag{1.1}$$

when $|\lambda - \lambda_0| \le \rho$ and $\|x_1 - x_0\|_X, \|x_2 - x_0\|_X \le r$.

The notion of osculating operators has been considered from different points of view (see [2, 3]). In this note we reformulate the definition of osculating operators. Our setting is a locally convex topological vector space. Moreover, we present a new implicit function theorem and, as an example of application, we study the solutions of an Hammerstein equation containing a parameter.

2. Preliminaries

Before providing the main results, we need to introduce some basic facts about locally convex topological vector spaces. We give these definitions following [4–6]. Let X be a Hausdorff locally convex topological vector space over the field $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. A family of continuous seminorms P which induces the topology of X is called a *calibration* for X. Denote by $\mathcal{P}(X)$ the set of all calibrations for X. A *basic calibration* for X is $P \in \mathcal{P}(X)$ such that the collection of all

$$U(\varepsilon, p) = \{x \in X : p(x) \leq \varepsilon\}, \quad \varepsilon > 0, \ p \in P \tag{2.1}$$

is a neighborhood base at 0. Observe that $P \in \mathcal{P}(X)$ is a basic calibration for X if and only if for each $p_1, p_2 \in P$ there is $p_0 \in P$ such that $p_i(x) \leq p_0(x)$ for $i = 1, 2$ and $x \in X$. Given $P \in \mathcal{P}(X)$, the family of all maxima of finite subfamily of P is a basic calibration.

A linear operator L on X is called *P-bounded* if there exists a constant $C > 0$ such that

$$p(Lx) \leq Cp(x), \quad x \in X, \ p \in P. \tag{2.2}$$

Denote by $\mathcal{L}(X)$ the space of all continuous linear operators on X and by $\mathcal{B}_P(X)$ the space of all P-bounded linear operators L on X. We have $\mathcal{B}_P(X) \subset \mathcal{L}(X)$. Moreover, the space $\mathcal{B}_P(X)$ is a unital normed algebra with respect to the norm

$$\|L\|_P = \sup\{p(Lx) : x \in X, \ p \in P, \ p(x) = 1\}. \tag{2.3}$$

We say that a family $\{L_\alpha : \alpha \in I\} \subset \mathcal{B}_P(X)$ is *uniformly P-bounded* if there exists a constant $C > 0$ such that

$$p(L_\alpha x) \leq Cp(x), \quad x \in X, \ p \in P \tag{2.4}$$

for any $\alpha \in I$.

In the following we will assume that X is a complete Hausdorff locally convex topological vector space and that $P \in \mathcal{P}(X)$ is a basic calibration for X.

3. Main Result

Let Λ be an open subset of the real line $\mathbb{R}$ or of the complex plane $\mathbb{C}$. Consider the product space $\Lambda \times X$ of Λ and X provided with the product topology. Let A be an open subset of $\Lambda \times X$ and $(\lambda_0, x_0) \in A$. Consider a nonlinear operator $\Phi : A \to X$ and the related equation

$$\Phi(\lambda, x) = 0. \tag{3.1}$$

Assume that (λ_0, x_0) is a solution of the above equation. A fundamental problem in nonlinear analysis is to study solutions (λ, x) of (3.1) for λ close to λ_0.

We say that an operator $\Phi : A \to X$ and an operator $L : \Lambda \to \mathcal{L}(X)$ are called *P-osculating* at (λ_0, x_0) if there exist a function $\sigma : \mathbb{R}^2 \to [0, +\infty)$ and $q \in P$ such that $\lim_{(\rho,r)\to(0,0)} \sigma(\rho, r) = 0$ and for any $p \in P$

$$p(\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)) \le \sigma(\rho, r) p(x_1 - x_2), \tag{3.2}$$

when $|\lambda - \lambda_0| \le \rho$ and $x_1, x_2 \in x_0 + U(r, q)$.

Now we prove our main result.

Theorem 3.1. *Suppose that $\Phi : A \to X$ and (λ_0, x_0) satisfy the following conditions:*

(a) (λ_0, x_0) *is a solution of* (3.1) *and the operator* $\Phi(\cdot, x_0)$ *is continuous at* λ_0*;*

(b) *there exists an operator function* $L : \Lambda \to \mathcal{L}(X)$ *such that* Φ *and* L *are P-osculating at* (λ_0, x_0)*;*

(c) *the linear operator* $L(\lambda)$ *is invertible and* $L(\lambda)^{-1} \in \mathcal{B}_P(X)$ *for each* $\lambda \in \Lambda$*. Moreover the family* $\{L(\lambda)^{-1} : \lambda \in \Lambda\}$ *is uniformly P-bounded.*

Then there are $\varepsilon > 0$, $q \in P$ *and* $\delta > 0$ *such that, for each* $\lambda \in \Lambda$ *with* $|\lambda - \lambda_0| \le \delta$, (3.1) *has a unique solution* $x(\lambda) \in x_0 + U(\varepsilon, q)$.

Proof. Let Φ and $L : \Lambda \to L(X)$ be P-osculating at (λ_0, x_0). Consider the operator $T : A \to X$ defined by

$$T(\lambda, x) = x - L(\lambda)^{-1}\Phi(\lambda, x). \tag{3.3}$$

Let $p \in P$. By the assumption (c) there exists $C > 0$ such that

$$p(T(\lambda, x_1) - T(\lambda, x_2)) \le C p(\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)) \tag{3.4}$$

for any $(\lambda, x_1), (\lambda, x_2) \in A$. Moreover, since Φ and L are P-osculating at (λ_0, x_0), there are a function $\sigma : R^2 \to [0, +\infty)$ and $q \in P$ such that

$$p(\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)) \le \sigma(\rho, r) p(x_1 - x_2) \tag{3.5}$$

for $|\lambda - \lambda_0| \leq \rho$ and $x_1, x_2 \in x_0 + U(r, q)$. Hence

$$p(T(\lambda, x_1) - T(\lambda, x_2)) \leq C\sigma(\rho, r)p(x_1 - x_2) \tag{3.6}$$

for $|\lambda - \lambda_0| \leq \rho$ and $x_1, x_2 \in x_0 + U(r, q)$.

Choose $\varepsilon > 0$ such that

$$p(T(\lambda, x_1) - T(\lambda, x_2)) \leq \frac{1}{2}p(x_1 - x_2) \tag{3.7}$$

for $|\lambda - \lambda_0| \leq \varepsilon$ and $x_1, x_2 \in x_0 + U(\varepsilon, q)$. Therefore, for each $\lambda \in \Lambda$ such that $|\lambda - \lambda_0| \leq \varepsilon$, the operator $T(\lambda, \cdot)$ from $x_0 + U(\varepsilon, q)$ into X is a contraction in the sense of [7].

Since $\Phi(\cdot, x_0)$ is continuous at λ_0, we may further find $\delta' > 0$ such that

$$\begin{gathered} p(\Phi(\lambda, x_0)) \leq \frac{\varepsilon}{2C}, \\ p(T(\lambda, x_0) - x_0) \leq Cp(\Phi(\lambda, x_0)) \leq \frac{\varepsilon}{2} \end{gathered} \tag{3.8}$$

for $|\lambda - \lambda_0| \leq \delta'$. Set $\delta := \min\{\varepsilon, \delta'\}$ we have

$$p(T(\lambda, x) - x_0) \leq p(T(\lambda, x) - T(\lambda, x_0)) + p(T(\lambda, x_0) - x_0) \leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \tag{3.9}$$

for $|\lambda - \lambda_0| \leq \delta$ and $x \in x_0 + U(\varepsilon, q)$. This shows that

$$T(\lambda, \cdot)(x_0 + U(\varepsilon, q)) \subseteq x_0 + U(\varepsilon, q) \tag{3.10}$$

for each λ such that $|\lambda - \lambda_0| \leq \delta$. Then, by [7, Theorem 1.1], when $|\lambda - \lambda_0| \leq \delta$, the operator $T(\lambda, \cdot)$ has a unique fixed point $x(\lambda) \in x_0 + U(\varepsilon, q)$, which is obviously a solution of (3.1). □

4. An Application

As an example of application of our main result, we study the stability of the solutions of an operator equation with respect to a parameter.

Consider in X the Hammerstein equation

$$x = \lambda KFx, \tag{4.1}$$

containing a parameter $\lambda \in \Lambda$. In our case K is a continuous linear operator on X and $F : X \to X$ is the so-called superposition operator. We have the following theorem.

Theorem 4.1. *Let K be P-bounded. Suppose that for each $x \in X$ there exists $q \in P$ such that the operator F satisfies the Lipschitz condition*

$$p(Fx_1 - Fx_2) \leq \omega(r)p(x_1 - x_2) \tag{4.2}$$

for any $p \in P$ and $x_1, x_2 \in x + U(r,q)$, where $\lim_{r\to 0}\omega(r) = 0$. If $x_0 \in X$ is a solution of (4.1) *for $\lambda = \lambda_0$, then there exist $\varepsilon > 0$ and $\delta > 0$ such that, for each $\lambda \in \Lambda$ with $|\lambda - \lambda_0| \leq \delta$,* (4.1) *has a unique solution $x(\lambda) \in x_0 + U(\varepsilon, q)$.*

Proof. Since the linear operator K is P-bounded, we can find a constant $C > 0$ such that

$$p(Kx) \leq Cp(x), \quad x \in X, \ p \in P. \tag{4.3}$$

If $\lambda = 0$, then $x_0 = 0$ is clearly a solution of (4.1). Consider the operator $\Phi_0 : \Lambda \times X \to X$ defined by

$$\Phi_0(\lambda, x) = x - \lambda KFx, \tag{4.4}$$

and set $L_0(\lambda)x = x$ for any $\lambda \in \Lambda$ and $x \in X$. Clearly the operator $\Phi(\cdot, 0)$ is continuous at 0. By the hypothesis made on the operator F, there exists $q \in P$ such that

$$p(\Phi_0(\lambda, x_1) - \Phi_0(\lambda, x_2) - L_0(\lambda)(x_1 - x_2)) \leq C\rho\omega(r)p(x_1 - x_2) \tag{4.5}$$

for any $p \in P$; when $|\lambda| \leq \rho$ and $x_1, x_2 \in U(r,q)$, the operators Φ_0 and L_0 are P-osculating at $(0,0)$. Moreover, for each $\lambda \in \Lambda$, we have $L_0(\lambda)^{-1} = L_0(\lambda)$ and $p(L_0(\lambda)^{-1}x) = p(x)$ for any $x \in X$ and $p \in P$. Then the result follows by Theorem 3.1. Now assume that $x_0 \in X$ is a solution of (4.1) for some $\lambda_0 \neq 0$. Let $\Phi : \Lambda \times X \to X$ be defined by

$$\Phi(\lambda, x) = \frac{x}{\lambda} - KFx, \tag{4.6}$$

and set $L(\lambda)x = x/\lambda$ for any $\lambda \in \Lambda$ and $x \in X$. The operator $\Phi(\cdot, x_0)$ is continuous at λ_0 and there exists $q \in P$ such that

$$p(\Phi(\lambda, x_1) - \Phi(\lambda, x_2) - L(\lambda)(x_1 - x_2)) \leq C\omega(r)p(x_1 - x_2) \tag{4.7}$$

for any $p \in P$, when $\lambda \in \Lambda$ and $x_1, x_2 \in x_0 + U(r,q)$. So the operators Φ and L are P-osculating at (λ_0, x_0). Further, assuming $|\lambda - \lambda_0| \leq a$ for some $a > 0$, we can find $b > 0$ such that $p(L(\lambda)^{-1}x) \leq bp(x)$ for any $p \in P$ and $x \in X$. As before, the proof is completed by appealing to Theorem 3.1. □

References

[1] P. P. Zabreĭko, Ju. S. Kolesov, and M. A. Krasnosel'skij, "Implicit functions and the averaging principle of N. N. Bogoljubov and N. M. Krylov," *Doklady Akademii Nauk SSSR*, vol. 184, no. 3, pp. 526–529, 1969.
[2] A. Trombetta, "An implicit function theorem in complete F-normed spaces," *Atti del Seminario Matematico e Fisico dell'Università di Modena*, vol. 48, no. 2, pp. 527–533, 2000.
[3] A. Trombetta, "t-osculating operators in a space of continuous functions and applications," *Journal of Mathematical Analysis and Applications*, vol. 256, no. 1, pp. 304–311, 2001.
[4] E. Kramar, "Invariant subspaces for some operators on locally convex spaces," *Commentationes Mathematicae Universitatis Carolinae*, vol. 38, no. 4, pp. 635–644, 1997.

[5] R. T. Moore, “Banach algebras of operators on locally convex spaces,” *Bulletin of the American Mathematical Society*, vol. 75, pp. 68–73, 1969.

[6] L. Narici and E. Beckenstein, *Topological Vector Spaces*, vol. 95 of *Monographs and Textbooks in Pure and Applied Mathematics*, Marcel Dekker, New York, NY, USA, 1985.

[7] E. Tarafdar, “An approach to fixed-point theorems on uniform spaces,” *Transactions of the American Mathematical Society*, vol. 191, pp. 209–225, 1974.