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Research Article

A New Approximation Scheme Combining the Viscosity Method with Extragradient Method for Mixed Equilibrium Problems

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We introduce a new approximation scheme combining the viscosity method with extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem and the set of fixed points of a finite family of nonexpansive mappings and the set of the variational inequality for a monotone, Lipschitz continuous mapping. We obtain a strong convergence theorem for the sequences generated by these processes in Hilbert spaces. Based on this result, we also get some new and interesting results. The results in this paper generalize, extend, and improve some well-known results in the literature.

1. Introduction

Let H be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and induced norm $\|\cdot\|$ and let C be a nonempty closed convex subset of H. Let $\varphi : C \to R\cup\{+\infty\}$ be a function and let F be a bifunction from $C\times C$ to R, where R is the set of real numbers. Ceng and Yao [1] and Bigi et al. [2] considered the following mixed equilibrium problem:

$$\text{Find } x\in C \quad \text{such that } F(x,y)+\varphi(y)\geq\varphi(x),\quad \forall y\in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $\mathrm{MEP}(F,\varphi)$. It is easy to see that x is a solution of problem (1.1) implies that $x\in\operatorname{dom}\varphi=\{x\in C\mid \varphi(x)<+\infty\}$.

If $\varphi = 0$, then the mixed equilibrium problem (1.1) becomes the following equilibrium problem:

$$\text{Finding } x \in C \quad \text{such that } F(x, y) \geq 0, \quad \forall y \in C. \tag{1.2}$$

The set of solutions of (1.2) is denoted by $EP(F)$.

If $F(x, y) = 0$ for all $x, y \in C$, the mixed equilibrium problem (1.1) becomes the following minimization problem:

$$\text{Finding } x \in C \quad \text{such that } \varphi(y) \geq \varphi(x), \quad \forall y \in C. \tag{1.3}$$

The set of solutions of (1.3) is denoted by $\operatorname{Argmin}(\varphi)$.

The problem (1.1) is very general in the sense that it includes, as special cases, optimization problems, variational inequalities, minimax problems, Nash equilibrium problem in noncooperative games, and others; see, for instance, [1–4].

Recall that a mapping S of a closed convex subset C into itself is nonexpansive [5] if there holds that

$$\|Sx - Sy\| \leq \|x - y\| \quad \forall x, y \in C. \tag{1.4}$$

We denote the set of fixed points of S by $\operatorname{Fix}(S)$. Ceng and Yao [1] introduced an iterative scheme for finding a common element of the set of solutions of problem (1.1) and the set of common fixed points of a finite family of nonexpansive mappings in a Hilbert space and obtained a strong convergence theorem.

Some methods have been proposed to solve the problem (1.2); see, for instance, [3, 4, 6–12] and the references therein. Recently, Combettes and Hirstoaga [6] introduced an iterative scheme of finding the best approximation to the initial data when $\operatorname{EP}(F)$ is nonempty and proved a strong convergence theorem. Takahashi and Takahashi [7] introduced an iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space and proved a strong convergence theorem. Su et al. [8] introduced the following iterative scheme by the viscosity approximation method for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality problem for an α-inverse strongly monotone mapping in a Hilbert space. Starting with an arbitrary $x_1 \in H$, define sequences $\{x_n\}$ and $\{u_n\}$ by

$$\begin{gathered} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1 - \alpha_n) S P_C(u_n - \lambda_n A u_n), \quad \forall n \in N. \end{gathered} \tag{1.5}$$

They proved that under certain appropriate conditions imposed on $\{\alpha_n\}$, $\{r_n\}$, and $\{\lambda_n\}$, the sequences $\{x_n\}$ and $\{u_n\}$ generated by (1.5) converge strongly to $z \in \operatorname{Fix}(S) \cap \operatorname{EP}(F) \cap \operatorname{VI}(C, A)$, where $z = P_{\mathrm{Fix}(S)\cap \mathrm{EP}(F)\cap \mathrm{VI}(C,A)} f(z)$. Tada and Takahashi [9] introduced two iterative schemes for finding a common element of the set of solutions of problem (1.2) and the set of fixed points of a nonexpansive mapping in a Hilbert space and obtained both strong convergence theorem and weak convergence theorem.

On the other hand, for solving the variational inequality problem in the finite-dimensional Euclidean R^n, Korpelevich [13] introduced the following so-called extragradient method:

$$\begin{aligned} x_1 &= x \in C, \\ y_n &= P_C(x_n - \lambda A x_n), \\ x_{n+1} &= P_C\left(x_n - \lambda A y_n\right) \end{aligned} \tag{1.6}$$

for every $n = 0, 1, 2, \ldots$, where $\lambda \in (0, 1/k)$. She showed that if $\mathrm{VI}(C, A)$ is nonempty, then the sequences $\{x_n\}$ and $\{y_n\}$, generated by (1.6), converge to the same point $z \in \mathrm{VI}(C, A)$. The idea of the extragradient iterative process introduced by Korpelevich was successfully generalized and extended not only in Euclidean but also in Hilbert and Banach spaces; see, for example, the recent papers of He et al. [14], Gárciga Otero and Iuzem [15], and Solodov and Svaiter [16], Solodov [17]. Moreover, Zeng and Yao [18] and Nadezhkina and Takahashi [19] introduced iterative processes based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for a monotone, Lipschitz continuous mapping. Yao and Yao [20] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping. Plubtieng and Punpaeng [11] introduced an iterative process based on the extragradient method for finding the common element of the set of fixed points of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of variational inequality problem for α-inverse strongly monotone mappings. Chang et al. [12] introduced some iterative processes based on the extragradient method for finding the common element of the set of fixed points of a infinite family of nonexpansive mappings, the set of solutions of an equilibrium problem, and the set of solutions of variational inequality problem for an α-inverse strongly monotone mapping. Peng et al. [21] introduced a new approximation scheme combining the viscosity method with parallel method for finding a common element of the set of solutions of a generalized equilibrium problem and the set of fixed points of a finite family of strict pseudocontractions and obtain a strong convergence theorem for the sequences generated by these processes in Hilbert spaces.

In the present paper, we introduce a new approximation scheme combining the viscosity method with extragradient method for finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a finite family of nonexpansive mappings, and the set of solutions of the variational inequality for a monotone, Lipschitz continuous mapping. We obtain a strong convergence theorem for the sequences generated by these processes. Based on this result, we also get some new and interesting results. The results in this paper generalize and improve some well-known results in the literature.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$. Let C be a nonempty closed convex subset of H. Let symbols $\rightarrow$ and $\rightharpoonup$ denote strong and weak convergence, respectively. In a real Hilbert space H, it is well known that

$$\left\| \lambda x + (1 - \lambda) y \right\|^2 = \lambda \|x\|^2 + (1 - \lambda) \left\| y \right\|^2 - \lambda (1 - \lambda) \left\| x - y \right\|^2 \tag{2.1}$$

for all $x, y \in H$ and $\lambda \in [0, 1]$.

For any $x \in H$, there exists a unique nearest point in C, denoted by $P_C(x)$, such that $\|x - P_C(x)\| \le \|x - y\|$ for all $y \in C$. The mapping P_C is called the metric projection of H onto C. We know that P_C is a nonexpansive mapping from H onto C. It is also known that $P_C x \in C$ and

$$\left\langle x - P_C(x), P_C(x) - y \right\rangle \ge 0 \tag{2.2}$$

for all $x \in H$ and $y \in C$.

It is easy to see that (2.2) is equivalent to

$$\left\| x - y \right\|^2 \ge \|x - P_C(x)\|^2 + \left\| y - P_C(x) \right\|^2 \tag{2.3}$$

for all $x \in H$ and $y \in C$.

A mapping A of C into H is called monotone if

$$\left\langle Ax - Ay, x - y \right\rangle \ge 0 \tag{2.4}$$

for all $x, y \in C$. A mapping A of C into H is called α-inverse strongly monotone if there exists a positive real number α such that

$$\langle x - y, Ax - Ay \rangle \ge \alpha \left\| Ax - Ay \right\|^2 \tag{2.5}$$

for all $x, y \in C$. A mapping $A : C \rightarrow H$ is called k-Lipschitz continuous if there exists a positive real number k such that

$$\left\| Ax - Ay \right\| \le k \left\| x - y \right\| \tag{2.6}$$

for all $x, y \in C$. It is easy to see that if A is α-inverse strongly monotone mappings, then A is monotone and Lipschitz continuous. The converse is not true in general. The class of α-inverse strongly monotone mappings does not contain some important classes of mappings even in a finite-dimensional case. For example, if the matrix in the corresponding linear complementarity problem is positively semidefinite, but not positively definite, then the mapping A is monotone and Lipschitz continuous, but not α-inverse strongly monotone.

Let A be a monotone mapping of C into H. In the context of the variational inequality problem the characterization of projection (2.2) implies the following:

$$u \in \mathrm{VI}(C,A) \Longrightarrow u = P_C(u - \lambda Au), \quad \lambda > 0,$$
$$u = P_C(u - \lambda Au) \quad \text{for some } \lambda > 0 \Longrightarrow u \in \mathrm{VI}(C,A). \tag{2.7}$$

It is also known that H satisfies Opial's condition [22], that is, for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\| \tag{2.8}$$

holds for every $y \in H$ with $x \neq y$.

A set-valued mapping $T : H \to 2^H$ is called monotone if for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x,f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y,g) \in G(T)$ implies $f \in Tx$. Let A be a monotone, k-Lipschitz continuous mapping of C into H and let $N_C v$ be normal cone to C at $v \in C$, that is, $N_C v = \{w \in H : \langle v - u, w\rangle \geq 0, \forall u \in C\}$. Define

$$Tv = \begin{cases} Av + N_C v & \text{if } v \in C, \\ \emptyset & \text{if } v \notin C. \end{cases} \tag{2.9}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $v \in \mathrm{VI}(C,A)$ (see [23]).

For solving the mixed equilibrium problem, let us give the following assumptions for the bifunction F, φ and the set C:

(A1) $F(x,x) = 0$ for all $x \in C$;

(A2) F is monotone, that is, $F(x,y) + F(y,x) \leq 0$ for any $x, y \in C$;

(A3) for each $y \in C$, $x \mapsto F(x,y)$ is weakly upper semicontinuous;

(A4) for each $x \in C$, $y \mapsto F(x,y)$ is convex;

(A5) for each $x \in C$, $y \mapsto F(x,y)$ is lower semicontinuous;

(B1) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z); \tag{2.10}$$

(B2) C is a bounded set;

(B3) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$\varphi(y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < \varphi(z); \tag{2.11}$$

(B4) for each $x \in H$ and $r > 0$, there exist a bounded subset $D_x \subseteq C$ and $y_x \in C$ such that for any $z \in C \setminus D_x$,

$$F(z, y_x) + \frac{1}{r}\langle y_x - z, z - x\rangle < 0. \tag{2.12}$$

We will use the following results in the sequel.

Lemma 2.1 (see [21, 24]). *Let C be a nonempty closed convex subset of H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Assume that either* (B1) *or* (B2) *holds. For $r > 0$ and $x \in H$, define a mapping $T_r : H \to C$ as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \varphi(y) + \frac{1}{r}\langle y - z, z - x\rangle \geq \varphi(z),\ \forall y \in C \right\} \tag{2.13}$$

for all $x \in H$. Then the following conclusions hold:

(1) *for each $x \in H$, $T_r(x) \neq \emptyset$;*

(2) *T_r is single-valued;*

(3) *T_r is firmly nonexpansive, that is, for any $x, y \in H$,*

$$\|T_r(x) - T_r(y)\|^2 \leq \langle T_r(x) - T_r(y), x - y\rangle; \tag{2.14}$$

(4) $\mathrm{Fix}(T_r) = \mathrm{MEP}(F, \varphi)$;

(5) $\mathrm{MEP}(F, \varphi)$ *is closed and convex.*

Lemma 2.2 (see [25, 26]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n, \tag{2.15}$$

where γ_n is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty} \gamma_n = \infty$;

(ii) $\limsup_{n \to \infty} \delta_n / \gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then, $\lim_{n \to \infty} \alpha_n = 0$.

Lemma 2.3. *In a real Hilbert space H, there holds the following inequality:*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle \tag{2.16}$$

for all $x, y \in H$.

Lemma 2.4 (see [21]). *Let $\{x_n\}$ and $\{w_n\}$ be bounded sequences in a Banach space, and let $\{\beta_n\}$ be a sequence of real numbers such that $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$ for all $n = 0, 1, 2, \ldots$. Suppose that $x_{n+1} = (1-\beta_n)w_n + \beta_n x_n$ for all $n = 0, 1, 2, \ldots$ and $\limsup_{n\to\infty}\|w_{n+1} - w_n\| + \|x_{n+1} - x_n\| \le 0$. Then, $\lim_{n\to\infty}\|w_n - x_n\| = 0$.*

Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 \le \lambda_i \le 1$ for every $i = 1, 2, \ldots, N$. We define a mapping W of C into itself as follows:

$$\begin{gathered} U_1 = \lambda_1 T_1 + (1-\lambda_1)I, \\ U_2 = \lambda_2 T_2 U_1 + (1-\lambda_2)I, \\ \vdots \\ U_{N-1} = \lambda_{N-1} T_{N-1} U_{N-2} + (1-\lambda_{N-1})I, \\ W := U_N = \lambda_N T_N U_{N-1} + (1-\lambda_N)I. \end{gathered} \tag{2.17}$$

Such a mapping W is called the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. It is easy to see that nonexpansivity of each T_i ensures the nonexpansivity of W. The concept of W-mappings was introduced in [27, 28]. It is now one of the main tools in studying convergence of iterative methods for approaching a common fixed point of nonlinear mappings; more recent progresses can be found in [10, 29, 30] and the references cited therein.

Lemma 2.5 (see [29]). *Let C be a nonempty closed convex set of a strictly convex Banach space. Let $T_1, T_2, \ldots, T_N$ be nonexpansive mappings of C into itself such that $\bigcap_{i=1}^N F(T_i) \ne \emptyset$ and let $\lambda_1, \lambda_2, \ldots, \lambda_N$ be real numbers such that $0 < \lambda_i < 1$ for every $i = 1, 2, \ldots, N-1$ and $0 < \lambda_N \le 1$. Let W be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. Then $\mathrm{Fix}(W) = \bigcap_{i=1}^N \mathrm{Fix}(T_i)$.*

Lemma 2.6 (see [10]). *Let C be a nonempty convex subset of a Banach space. Let $\{T_i\}_{i=1}^N$ be a finite family of nonexpansive mappings of C into itself and let $\{\lambda_{n,1}\}\{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\}$ be sequences in $[0,1]$ such that $\lambda_{n,i} \to \lambda_i$ $(i = 1, \ldots, N)$. Moreover for every integer $n \ge 1$, let W and W_n be the W-mappings generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$ and $T_1, T_2, \ldots, T_N$ and $\{\lambda_{n,1}\}\{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\}$, respectively. Then for every $x \in C$, it follows that*

$$\lim_{n\to\infty}\|W_n x - Wx\| = 0. \tag{2.18}$$

3. Strong Convergence Theorems

In this section, we show a strong convergence of an iterative algorithm based on both viscosity approximation method and extragradient method which solves the problem of finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points of a finite family of nonexpansive mappings, and the set of solutions of the variational inequality for a monotone, Lipschitz continuous mapping in a Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5) *and let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of C into H such that $\Omega = \bigcap_{n=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \cap \mathrm{MEP}(F, \varphi) \neq \emptyset$. Let $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let W_n be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Assume that either* (B1) *or* (B2) *holds. Let f be a contraction of H into itself and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$
\begin{gathered}
x_1 = x \in C, \\
F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\
y_n = P_C(u_n - \gamma_n A u_n), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \gamma_n A y_n)
\end{gathered}
\tag{3.1}
$$

for every $n = 1, 2, \ldots$ where $\{\gamma_n\}$, $\{r_n\}$, $\{\alpha_n\}$, $\{\lambda_{n1}\}$, $\{\lambda_{n2}\}, \ldots, \{\lambda_{nN}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$;

(C3) $\lim_{n\to\infty}\gamma_n = 0$;

(C4) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(C5) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_\Omega f(w)$.

Proof. We show that $P_\Omega f$ is a contraction of C into itself. In fact, there exists $a \in [0, 1)$ such that $\|f(x) - f(y)\| \leq a\|x - y\|$ for all $x, y \in C$. So, we have

$$
\left\|P_\Omega f(x) - P_\Omega f(y)\right\| \leq \left\|f(x) - f(y)\right\| \leq a\left\|x - y\right\|
\tag{3.2}
$$

for all $x, y \in C$. Since H is complete, there exists a unique element $u_0 \in C$ such that $u_0 = P_\Omega f(u_0)$.

Put $t_n = P_C(u_n - \gamma_n A y_n)$ for every $n = 1, 2, \ldots$. Let $u \in \Omega$ and let $\{T_{r_n}\}$ be a sequence of mappings defined as in Lemma 2.1. Then $u = P_C(u - \gamma_n A u)$. From $u_n = T_{r_n}(x_n) \in C$, we have

$$\|u_n - u\| = \|T_{r_n}(x_n) - T_{r_n}(u)\| \leq \|x_n - u\|. \tag{3.3}$$

From (2.3), the monotonicity of A, and $u \in \mathrm{VI}(C, A)$, we have

$$\begin{aligned}
&\|t_n - u\|^2 \\
&\quad \leq \left\|u_n - \gamma_n A y_n - u\right\|^2 - \left\|u_n - \gamma_n A y_n - t_n\right\|^2 \\
&\quad = \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\gamma_n \langle A y_n, u - t_n\rangle \\
&\quad = \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\gamma_n \left(\langle A y_n - Au, u - y_n\rangle + \langle Au, u - y_n\rangle + \langle A y_n, y_n - t_n\rangle\right) \\
&\quad \leq \|u_n - u\|^2 - \|u_n - t_n\|^2 + 2\gamma_n \langle A y_n, y_n - t_n\rangle \\
&\quad \leq \|u_n - u\|^2 - \left\|u_n - y_n\right\|^2 - 2\langle u_n - y_n, y_n - t_n\rangle - \left\|y_n - t_n\right\|^2 + 2\gamma_n \langle A y_n, y_n - t_n\rangle \\
&\quad = \|u_n - u\|^2 - \left\|u_n - y_n\right\|^2 - \left\|y_n - t_n\right\|^2 + 2\langle u_n - \gamma_n A y_n - y_n, t_n - y_n\rangle.
\end{aligned} \tag{3.4}$$

Further, Since $y_n = P_C(u_n - \gamma_n A u_n)$ and A is k-Lipschitz continuous, we have

$$\begin{aligned}
\langle u_n - \gamma_n A y_n - y_n, t_n - y_n\rangle &= \langle u_n - \gamma_n A u_n - y_n, t_n - y_n\rangle + \langle \gamma_n A u_n - \gamma_n A y_n, t_n - y_n\rangle \\
&\leq \langle \gamma_n A u_n - \gamma_n A y_n, t_n - y_n\rangle \leq \gamma_n k \left\|u_n - y_n\right\| \left\|t_n - y_n\right\|.
\end{aligned} \tag{3.5}$$

So, it follows from (C3) that the following inequality holds for $n \geq n_0$, where n_0 is a positive integer:

$$\begin{aligned}
\|t_n - u\|^2 &\leq \|u_n - u\|^2 - \left\|u_n - y_n\right\|^2 - \left\|y_n - t_n\right\|^2 + 2\gamma_n k \left\|u_n - y_n\right\| \left\|t_n - y_n\right\| \\
&\leq \|u_n - u\|^2 - \left\|u_n - y_n\right\|^2 - \left\|y_n - t_n\right\|^2 + {\gamma_n}^2 k^2 \left\|u_n - y_n\right\|^2 + \left\|t_n - y_n\right\|^2 \\
&= \|u_n - u\|^2 + \left({\gamma_n}^2 k^2 - 1\right) \left\|u_n - y_n\right\|^2 \\
&\leq \|u_n - u\|^2.
\end{aligned} \tag{3.6}$$

Put $M_0 = \max\{\|x_1 - u\|, (1/(1-a))\|f(u) - u\|\}$. It is obvious that $\|x_1 - u\| \leq M_0$. Suppose $\|x_n - u\| \leq M_0$. By Lemma 2.5, we know that W_n is nonexpansive and $\mathrm{Fix}(W_n) = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$.

From (3.3), (3.6) and $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)W_n t_n$, we have $u = W_n u$ and

$$\begin{aligned}
\|x_{n+1}-u\| &= \left\|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)W_n t_n - u\right\| \\
&\le \alpha_n\left\|f(x_n)-f(u)\right\| + \alpha_n\left\|f(u)-u\right\| + \beta_n\|x_n-u\| + (1-\alpha_n-\beta_n)\|W_n t_n - u\| \\
&\le \alpha_n a\|x_n-u\| + \alpha_n\left\|f(u)-u\right\| + \beta_n\|x_n-u\| + \left(1-\alpha_n-\beta_n\right)\|W_n t_n - u\| \\
&\le \alpha_n a\|x_n-u\| + \alpha_n\left\|f(u)-u\right\| + \beta_n\|x_n-u\| + \left(1-\alpha_n-\beta_n\right)\|t_n - u\| \\
&\le \alpha_n a\|x_n-u\| + \alpha_n\left\|f(u)-u\right\| + \beta_n\|x_n-u\| + \left(1-\alpha_n-\beta_n\right)\|x_n - u\| \\
&= (1-a)\alpha_n\frac{\left\|f(u)-u\right\|}{1-a} + (1-(1-a)\alpha_n)\|x_n-u\| \\
&\le (1-a)\alpha_n M_0 + (1-(1-a)\alpha_n)M_0 = M_0
\end{aligned} \tag{3.7}$$

for every $n = 1,2,\ldots$. Therefore, $\{x_n\}$ is bounded. From (3.3) and (3.6), we also obtain that $\{t_n\}$ and $\{u_n\}$ are bounded.

From $y_n = P_C(u_n - \gamma_n A u_n)$ and the monotonicity and the Lipschitz continuity of A, we have

$$\begin{aligned}
\left\|y_n-u\right\|^2 &= \left\|P_C(u_n-\gamma_n A u_n) - P_C(u-\gamma_n A u)\right\|^2 \\
&\le \left\|u_n-\gamma_n A u_n - (u-\gamma_n A u)\right\|^2 \\
&= \|u_n-u\|^2 - 2\gamma_n\langle A u_n - A u, u_n - u\rangle + \gamma_n^2\|A u_n - A u\|^2 \\
&\le \|u_n-u\|^2 + \gamma_n^2 k^2\|u_n-u\|^2 \\
&= \left(1+\gamma_n^2 k^2\right)\|u_n-u\|^2.
\end{aligned} \tag{3.8}$$

Hence, we obtain that $\{y_n\}$ is bounded. It follows from the Lipschitz continuity of A that $\{A x_n\}$, $\{A u_n\}$, and $\{A y_n\}$ are bounded. Since f and W_n are nonexpansive, we know that $\{f(x_n)\}$ and $\{W_n t_n\}$ are also bounded. From the definition of t_n, we get

$$\begin{aligned}
\|t_{n+1}-t_n\| &= \left\|P_C(u_{n+1}-\gamma_{n+1} A y_{n+1}) - P_C(u_n-\gamma_n A y_n)\right\| \\
&\le \left\|(u_{n+1}-\gamma_{n+1} A y_{n+1}) - (u_n-\gamma_n A y_n)\right\| \\
&\le \left\|(u_{n+1}-\gamma_{n+1} A u_{n+1}) - (u_n-\gamma_{n+1} A u_n) + \gamma_{n+1}(A u_{n+1} - A y_{n+1} - A u_n) + \gamma_n A y_n\right\| \\
&\le \|u_{n+1}-u_n\| + \gamma_{n+1}\|A u_{n+1} - A u_n\| + \gamma_{n+1}\left\|A u_{n+1} - A y_{n+1} - A u_n\right\| + \gamma_n\left\|A y_n\right\| \\
&\le \|u_{n+1}-u_n\| + k\gamma_{n+1}\|u_{n+1} - u_n\| + \gamma_{n+1}\left\|A u_{n+1} - A y_{n+1} - A u_n\right\| + \gamma_n\left\|A y_n\right\| \\
&\le \|u_{n+1}-u_n\| + (\gamma_{n+1}+\gamma_n)M_1,
\end{aligned} \tag{3.9}$$

where M_1 is an approximate constant such that

$$M_1 \geq \sup_{n\geq 1}\{k\|u_{n+1}-u_n\| + \|Au_{n+1}-Ay_{n+1}-Au_n\| + \|Ay_n\|\}. \tag{3.10}$$

On the other hand, from $u_n = T_{r_n}(x_n)$ and $u_{n+1} = T_{r_{n+1}}(x_{n+1})$, we have

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y-u_n, u_n-x_n\rangle \geq 0, \quad \forall y \in C, \tag{3.11}$$

$$F(u_{n+1}, y) + \varphi(y) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle y-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0, \quad \forall y \in C. \tag{3.12}$$

Putting $y = u_{n+1}$ in (3.11) and $y = u_n$ in (3.12), we have

$$\begin{aligned} F(u_n, u_{n+1}) + \varphi(u_{n+1}) - \varphi(u_n) + \frac{1}{r_n}\langle u_{n+1}-u_n, u_n-x_n\rangle \geq 0\ , \\ F(u_{n+1}, u_n) + \varphi(u_n) - \varphi(u_{n+1}) + \frac{1}{r_{n+1}}\langle u_n-u_{n+1}, u_{n+1}-x_{n+1}\rangle \geq 0. \end{aligned} \tag{3.13}$$

So, from the monotonicity of F, we get

$$\left\langle u_{n+1}-u_n, \frac{u_n-x_n}{r_n} - \frac{u_{n+1}-x_{n+1}}{r_{n+1}}\right\rangle \geq 0, \tag{3.14}$$

and hence

$$\left\langle u_{n+1}-u_n, u_n-u_{n+1}+u_{n+1}-x_n - \frac{r_n}{r_{n+1}}(u_{n+1}-x_{n+1})\right\rangle \geq 0. \tag{3.15}$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in N$. Then,

$$\begin{aligned} \|u_{n+1}-u_n\|^2 &\leq \left\langle u_{n+1}-u_n, x_{n+1}-x_n + \left(1-\frac{r_n}{r_{n+1}}\right)(u_{n+1}-x_{n+1})\right\rangle \\ &\leq \|u_{n+1}-u_n\|\left\{\|x_{n+1}-x_n\| + \left|1-\frac{r_n}{r_{n+1}}\right|\|u_{n+1}-x_{n+1}\|\right\}, \end{aligned} \tag{3.16}$$

and hence

$$\begin{aligned} \|u_{n+1}-u_n\| &\leq \|x_{n+1}-x_n\| + \frac{1}{r_{n+1}}|r_{n+1}-r_n|\|u_{n+1}-x_{n+1}\| \\ &\leq \|x_{n+1}-x_n\| + \frac{1}{b}|r_{n+1}-r_n|M_2, \end{aligned} \tag{3.17}$$

where $M_2 = \sup\{\|u_n-x_n\| : n \in N\}$.

It follows from (3.9) and (3.7) that

$$\|t_{n+1} - t_n\| \le \|x_{n+1} - x_n\| + \frac{1}{b}|r_{n+1} - r_n| M_2 + (\gamma_{n+1} + \gamma_n) M_1, \tag{3.18}$$

Define a sequence $\{v_n\}$ such that

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) v_n, \quad \forall n \ge 1. \tag{3.19}$$

Then, we have

$$\begin{aligned} v_{n+1} - v_n &= \frac{x_{n+2} - \beta_{n+1} x_{n+1}}{1 - \beta_{n+1}} - \frac{x_{n+1} - \beta_n x_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1} f(x_{n+1}) + (1 - \alpha_{n+1} - \beta_{n+1}) W_{n+1} t_{n+1}}{1 - \beta_{n+1}} - \frac{\alpha_n f(x_n) + (1 - \alpha_n - \beta_n) W_n t_n}{1 - \beta_n} \\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}} f(x_{n+1}) - \frac{\alpha_n}{1 - \beta_n} f(x_n) + W_{n+1} t_{n+1} - W_n t_n \\ &\quad + \frac{\alpha_n}{1 - \beta_n} W_n t_n - \frac{\alpha_{n+1}}{1 - \beta_{n+1}} W_{n+1} t_{n+1}, \end{aligned} \tag{3.20}$$

Next we estimate $\|W_{n+1} t_{n+1} - W_n t_n\|$. It follows from the definition of W_n that

$$\begin{aligned} \|W_{n+1} t_n - W_n t_n\| &= \|\lambda_{n+1,N} T_N U_{n+1,N-1} t_n + (1 - \lambda_{n+1,N}) t_n - \lambda_{n,N} T_N U_{n,N-1} t_n - (1 - \lambda_{n,N}) t_n\| \\ &\le |\lambda_{n+1,N} - \lambda_{n,N}| \|t_n\| + \|\lambda_{n+1,N} T_N U_{n+1,N-1} t_n - \lambda_{n,N} T_N U_{n,N-1} t_n\| \\ &\le |\lambda_{n+1,N} - \lambda_{n,N}| \|t_n\| + \|\lambda_{n+1,N} (T_N U_{n+1,N-1} t_n - T_N U_{n,N-1} t_n)\| \\ &\quad + |\lambda_{n+1,N} - \lambda_{n,N}| \|T_N U_{n,N-1} t_n\| \\ &\le 2M_3 |\lambda_{n+1,N} - \lambda_{n,N}| + \lambda_{n+1,N} \|U_{n+1,N-1} t_n - U_{n,N-1} t_n\|, \end{aligned} \tag{3.21}$$

where M_3 is an approximate constant such that

$$M_3 \ge \max\left\{ \sup_{n \ge 1}\{\|t_n\|\}, \sup_{n \ge 1}\{\|T_k U_{n,k-1} t_n\|\} \mid k = 1, 2, \ldots, N \right\}. \tag{3.22}$$

Since $0 < \lambda_{n_i} \leq 1$ for all $n \geq 1$ and $i = 1, 2, \ldots, N$, we have

$$\begin{aligned}
&\|U_{n+1,N-1}t_n - U_{n,N-1}t_n\| \\
&\quad = \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}t_n + (1-\lambda_{n+1,N-1})t_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}t_n - (1-\lambda_{n,N-1})t_n\| \\
&\quad \leq |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|t_n\| + \|\lambda_{n+1,N-1}T_{N-1}U_{n+1,N-2}t_n - \lambda_{n,N-1}T_{N-1}U_{n,N-2}t_n\| \\
&\quad \leq |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|t_n\| + \lambda_{n+1,N-1}\|T_{N-1}U_{n+1,N-2}t_n - T_{N-1}U_{n,N-2}t_n\| \\
&\qquad + |\lambda_{n+1,N-1} - \lambda_{n,N-1}|\|T_{N-1}U_{n,N-2}t_n\| \\
&\quad \leq 2M_3|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + \|U_{n+1,N-2}t_n - U_{n,N-2}t_n\|.
\end{aligned} \tag{3.23}$$

It follows that

$$\begin{aligned}
&\|U_{n+1,N-1}t_n - U_{n,N-1}t_n\| \\
&\quad \leq 2M_3|\lambda_{n+1,N-1} - \lambda_{n,N-1}| + 2M_3|\lambda_{n+1,N-2} - \lambda_{n,N-2}| + \|U_{n+1,N-3}t_n - U_{n,N-3}t_n\| \\
&\quad \leq 2M_3\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| + \|U_{n+1,1}t_n - U_{n,1}t_n\| \\
&\quad = 2M_3\sum_{i=2}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| + \|\lambda_{n+1,1}T_1t_n + (1-\lambda_{n+1,1})t_n - \lambda_{n,1}T_1t_n - (1-\lambda_{n,1})t_n\| \\
&\quad \leq 2M_3\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned} \tag{3.24}$$

Substituting (3.24) into (3.21) yields that

$$\begin{aligned}
\|W_{n+1}t_n - W_nt_n\| &\leq 2M_3|\lambda_{n+1,N} - \lambda_{n,N}| + 2\lambda_{n+1,N}M_3\sum_{i=1}^{N-1}|\lambda_{n+1,i} - \lambda_{n,i}| \\
&\leq 2M_3\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned} \tag{3.25}$$

Hence, we have

$$\begin{aligned}
\|W_{n+1}t_{n+1} - W_nt_n\| &\leq \|W_{n+1}t_{n+1} - W_{n+1}t_n\| + \|W_{n+1}t_n - W_nt_n\| \\
&\leq \|t_{n+1} - t_n\| + 2M_3\sum_{i=1}^{N}|\lambda_{n+1,i} - \lambda_{n,i}|.
\end{aligned} \tag{3.26}$$

From (3.20), (3.26), and (3.18), we have

$$\begin{aligned}
&\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|f(x_{n+1})\right\|+\|W_{n+1}t_{n+1}\|\right)+\frac{\alpha_n}{1-\beta_n}\left(\left\|f(x_n)\right\|+\|W_nt_n\|\right) \\
&\qquad+\|W_{n+1}t_{n+1}-W_nt_n\|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|f(x_{n+1})\right\|+\|W_{n+1}t_{n+1}\|\right)+\frac{\alpha_n}{1-\beta_n}\left(\left\|f(x_n)\right\|+\|W_nt_n\|\right) \\
&\qquad+\|t_{n+1}-t_n\|+2M_3\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}|-\|x_{n+1}-x_n\| \\
&\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\left(\left\|f(x_{n+1})\right\|+\|W_{n+1}t_{n+1}\|\right)+\frac{\alpha_n}{1-\beta_n}\left(\left\|f(x_n)\right\|+\|W_nt_n\|\right)+2M_3\sum_{i=1}^{N}|\lambda_{n+1,i}-\lambda_{n,i}| \\
&\qquad+\frac{1}{b}|r_{n+1}-r_n|M_2+\left(\gamma_{n+1}+\gamma_n\right)M_1.
\end{aligned} \tag{3.27}$$

It follows from (C1)–(C5) that

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|)\le 0. \tag{3.28}$$

Hence by Lemma 2.4, we have $\lim_{n\to\infty}\|v_n-x_n\|=0$. Consequently

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=\lim_{n\to\infty}(1-\beta_n)\|v_n-x_n\|=0. \tag{3.29}$$

Since $x_{n+1}=\alpha_n f(x_n)+\beta_n x_n+(1-\alpha_n-\beta_n)W_nt_n$, we have

$$\begin{aligned}
\|x_n-W_nt_n\| &\le \|x_{n+1}-x_n\|+\|x_{n+1}-W_nt_n\| \\
&\le \|x_{n+1}-x_n\|+\alpha_n\left\|f(x_n)-W_nt_n\right\|+\beta_n\|x_n-W_nt_n\|,
\end{aligned} \tag{3.30}$$

and thus

$$\|x_n-W_nt_n\|\le\frac{1}{1-\beta_n}\left(\|x_{n+1}-x_n\|+\alpha_n\left\|f(x_n)-W_nt_n\right\|\right). \tag{3.31}$$

It follows from (C1) and (C2) that $\lim_{n\to\infty}\|x_n-W_nt_n\|=0$.

Since $x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)W_n t_n$, for $u \in \Omega$, it follows from (3.3) and (3.6) that

$$\begin{aligned}\|x_{n+1}-u\|^2 &= \left\|\alpha_n f(x_n) + \beta_n x_n + (1-\alpha_n-\beta_n)W_n t_n - u\right\|^2 \\ &\le \alpha_n \left\|f(x_n)-u\right\|^2 + \beta_n\|x_n-u\|^2 + (1-\alpha_n-\beta_n)\|W_n t_n - u\|^2 \\ &\le \alpha_n \left\|f(x_n)-u\right\|^2 + \beta_n\|x_n-u\|^2 + (1-\alpha_n-\beta_n)\|t_n - u\|^2 \\ &\le \alpha_n \left\|f(x_n)-u\right\|^2 + \beta_n\|x_n-u\|^2 + (1-\alpha_n-\beta_n)\left[\|u_n-u\|^2 + \left(\gamma_n{}^2k^2-1\right)\left\|u_n-y_n\right\|^2\right] \\ &\le \alpha_n \left\|f(x_n)-u\right\|^2 + (1-\alpha_n)\|x_n-u\|^2 + (1-\alpha_n-\beta_n)\left(\gamma_n{}^2k^2-1\right)\left\|u_n-y_n\right\|^2\end{aligned} \tag{3.32}$$

from which it follows that

$$\begin{aligned}\left\|u_n-y_n\right\|^2 &\le \frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}\left(\left\|f(x_n)-u\right\|^2 - \|x_n-u\|^2\right) \\ &\quad + \frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}\left(\|x_n-u\|^2 - \|x_{n+1}-u\|^2\right) \\ &\le \frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}\left(\left\|f(x_n)-u\right\|^2 - \|x_n-u\|^2\right) \\ &\quad + \frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}(\|x_n-u\| - \|x_{n+1}-u\|)\|x_{n+1}-x_n\|.\end{aligned} \tag{3.33}$$

It follows from (C1)–(C3) and $\|x_{n+1}-x_n\| \to 0$ that $\|u_n-y_n\| \to 0$.

By the same argument as in (3.6), we also have

$$\begin{aligned}\|t_n-u\|^2 &\le \|u_n-u\|^2 - \left\|u_n-y_n\right\|^2 - \left\|y_n-t_n\right\|^2 + 2\gamma_n k\left\|u_n-y_n\right\|\left\|t_n-y_n\right\| \\ &\le \|u_n-u\|^2 - \left\|u_n-y_n\right\|^2 - \left\|y_n-t_n\right\|^2 + \left\|u_n-y_n\right\|^2 + \gamma_n{}^2k^2\left\|t_n-y_n\right\|^2 \\ &= \|u_n-u\|^2 + \left(\gamma_n{}^2k^2-1\right)\left\|y_n-t_n\right\|^2.\end{aligned} \tag{3.34}$$

Combining the above inequality and (3.32), we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \alpha_n \left\|f(x_n)-u\right\|^2 + \beta_n\|x_n-u\|^2 + (1-\alpha_n-\beta_n)\|t_n - u\|^2 \\ &\le \alpha_n \left\|f(x_n)-u\right\|^2 + \beta_n\|x_n-u\|^2 + (1-\alpha_n-\beta_n)\left[\|u_n-u\|^2 + (\gamma_n{}^2k^2-1)\left\|y_n-t_n\right\|^2\right] \\ &\le \alpha_n \left\|f(x_n)-u\right\|^2 + (1-\alpha_n)\|x_n-u\|^2 + (1-\alpha_n-\beta_n)\left(\gamma_n{}^2k^2-1\right)\left\|y_n-t_n\right\|^2,\end{aligned} \tag{3.35}$$

and thus

$$\begin{aligned}\|t_n - y_n\|^2 &\le \frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}\left(\|f(x_n)-u\|^2 - \|x_n-u\|^2\right)\\&\quad+\frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}\left(\|x_n-u\|^2-\|x_{n+1}-u\|^2\right)\\&\le \frac{\alpha_n}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}\left(\|f(x_n)-u\|^2 - \|x_n-u\|^2\right)\\&\quad+\frac{1}{(1-\alpha_n-\beta_n)(1-\gamma_n{}^2k^2)}(\|x_n-u\|-\|x_{n+1}-u\|)\|x_{n+1}-x_n\|,\end{aligned} \tag{3.36}$$

which implies that $\|t_n - y_n\| \to 0$.

From $\|u_n - t_n\| \le \|u_n - y_n\| + \|y_n - t_n\|$ we also have $\|u_n - t_n\| \to 0$. As A is k-Lipschitz continuous, we have $\|Ay_n - At_n\| \to 0$.

For $u \in \Omega$, we have, from Lemma 2.1,

$$\begin{aligned}\|u_n-u\|^2 &= \|T_{r_n}x_n - T_{r_n}u\|^2 \le \langle T_{r_n}x_n - T_{r_n}u, x_n-u\rangle\\&= \langle u_n-u, x_n-u\rangle = \frac{1}{2}\left\{\|u_n-u\|^2+\|x_n-u\|^2-\|x_n-u_n\|^2\right\}.\end{aligned} \tag{3.37}$$

Hence,

$$\|u_n-u\|^2 \le \|x_n-u\|^2 - \|x_n-u_n\|^2. \tag{3.38}$$

By(3.3), (3.6), (3.32), and (3.38), we have

$$\begin{aligned}\|x_{n+1}-u\|^2 &\le \alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+(1-\alpha_n-\beta_n)\|t_n-u\|^2\\&\le \alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+(1-\alpha_n-\beta_n)\|u_n-u\|^2\\&\le \alpha_n\|f(x_n)-u\|^2+\beta_n\|x_n-u\|^2+(1-\alpha_n-\beta_n)\left[\|x_n-u\|^2-\|x_n-u_n\|^2\right]\\&\le \alpha_n\|f(x_n)-u\|^2+(1-\alpha_n)\|x_n-u\|^2-(1-\alpha_n-\beta_n)\|x_n-u_n\|^2.\end{aligned} \tag{3.39}$$

Hence,

$$\begin{aligned}(1-\alpha_n-\beta_n)\|x_n-u_n\|^2 &\le \alpha_n\|f(x_n)-u\|^2-\alpha_n\|x_n-u\|^2+\|x_n-u\|^2-\|x_{n+1}-u\|^2\\&\le \alpha_n\|f(x_n)-u\|^2-\alpha_n\|x_n-u\|^2+(\|x_n-u\|+\|x_{n+1}-u\|)\|x_n-x_{n+1}\|.\end{aligned} \tag{3.40}$$

It follows from (C1), (C2), and $\|x_n - x_{n+1}\| \to 0$ that $\lim_{n\to\infty}\|x_n-u_n\| = 0$.

Since

$$\begin{aligned}\|W_n y_n - y_n\| &\le \|W_n y_n - W_n t_n\| + \|W_n t_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\| \\ &\le \|y_n - t_n\| + \|W_n t_n - x_n\| + \|x_n - u_n\| + \|u_n - y_n\|.\end{aligned} \tag{3.41}$$

It follows that

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{3.42}$$

Next we show that

$$\limsup_{n\to\infty} \langle f(u_0) - u_0, x_n - u_0 \rangle \le 0, \tag{3.43}$$

where $u_0 = P_\Omega f(u_0)$. To show this inequality, we can choose a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ such that

$$\lim_{n\to\infty} \left\langle f(u_0) - u_0, x_{n_j} - u_0 \right\rangle = \limsup_{n\to\infty} \langle f(u_0) - u_0, x_n - u_0 \rangle. \tag{3.44}$$

Since $\{x_{ni}\}$ is bounded, there exists a subsequence $\{x_{nij}\}$ of $\{x_{ni}\}$ which converges weakly to w. Without loss of generality, we can assume that $\{x_{ni}\} \rightharpoonup w$. From $\|x_n - u_n\| \to 0$, we obtain that $u_{ni} \rightharpoonup w$. From $\|u_n - y_n\| \to 0$, we also obtain that $y_{n_i} \rightharpoonup w$. From $\|u_n - t_n\| \to 0$, we also obtain that $t_{ni} \rightharpoonup w$. Since $\{u_{ni}\} \subset C$ and C is closed and convex, we obtain $w \in C$.

In order to show that $w \in \Omega$, we first show $w \in \mathrm{MEP}(F, \varphi)$. By $u_n = T_{r_n} x_n$, we know that

$$F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C. \tag{3.45}$$

It follows from (A2) that

$$\varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \ge F(y, u_n), \quad \forall y \in C. \tag{3.46}$$

Hence,

$$\varphi(y) - \varphi(u_{n_i}) + \left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}} \right\rangle \ge F(y, u_{n_i}), \quad \forall y \in C. \tag{3.47}$$

It follows from (A4), (A5), and the weakly lower semicontinuity of φ, $(u_{n_i} - x_{n_i})/r_{n_i} \to 0$ and $u_{n_i} \rightharpoonup w$ that

$$F(y, w) + \varphi(w) - \varphi(y) \le 0, \quad \forall y \in C. \tag{3.48}$$

For t with $0 < t \leq 1$ and $y \in C$, let $y_t = ty + (1-t)w$. Since $y \in C$ and $w \in C$, we obtain $y_t \in C$ and hence $F(y_t, w) + \varphi(w) - \varphi(y_t) \leq 0$. So by (A4) and the convexity of φ, we have

$$\begin{aligned} 0 &= F(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq tF(y_t, y) + (1-t)F(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &\leq t\left[F(y_t, y) + \varphi(y) - \varphi(y_t)\right]. \end{aligned} \tag{3.49}$$

Dividing by t, we get

$$F(y_t, y) + \varphi(y) - \varphi(y_t) \geq 0. \tag{3.50}$$

Letting $t \to 0$, it follows from (A3) and the weakly lower semicontinuity of φ that

$$F(w, y) + \varphi(y) - \varphi(w) \geq 0 \tag{3.51}$$

for all $y \in C$ and hence $w \in \mathrm{MEP}(F, \varphi)$.

Now we show that $w \in \mathrm{VI}(C, A)$. Put

$$Tw_1 = \begin{cases} Aw_1 + N_C w_1 & \text{if } w_1 \in C, \\ \emptyset & \text{if } w_1 \notin C, \end{cases} \tag{3.52}$$

where $N_C w_1$ is the normal cone to C at $w_1 \in C$. We have already mentioned that in this case the mapping T is maximal monotone, and $0 \in Tw_1$ if and only if $w_1 \in \mathrm{VI}(C, A)$. Let $(w_1, g) \in G(T)$. Then $Tw_1 = Aw_1 + N_C w_1$ and hence $g - Aw_1 \in N_C w_1$. So, we have $\langle w_1 - t, g - Aw_1 \rangle \geq 0$ for all $t \in C$. On the other hand, from $t_n = P_C(u_n - \lambda_n A y_n)$ and $w_1 \in C$ we have

$$\langle u_n - \lambda_n A y_n - t_n, t_n - w_1 \rangle \geq 0, \tag{3.53}$$

and hence

$$\left\langle w_1 - t_n, \frac{t_n - u_n}{\lambda_n} + A y_n \right\rangle \geq 0. \tag{3.54}$$

Therefore, we have

$$\begin{aligned}
\left\langle w_1 - t_{n_i}, g\right\rangle &\geq \langle w_1 - t_{n_i}, Aw_1\rangle \\
&\geq \langle w_1 - t_{n_i}, Aw_1\rangle - \left\langle w_1 - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}} + Ay_{n_i}\right\rangle \\
&= \left\langle w_1 - t_{n_i}, Aw_1 - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \left\langle w_1 - t_{n_i}, Aw_1 - At_{n_i} + At_{n_i} - Ay_{n_i} - \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\
&= \langle w_1 - t_{n_i}, Aw_1 - At_{n_i}\rangle + \langle w_1 - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle v - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle \\
&\geq \langle w_1 - t_{n_i}, At_{n_i} - Ay_{n_i}\rangle - \left\langle w_1 - t_{n_i}, \frac{t_{n_i} - u_{n_i}}{\lambda_{n_i}}\right\rangle.
\end{aligned} \tag{3.55}$$

Hence we obtain $\langle w_1 - w, g\rangle \geq 0$ as $i \to \infty$. Since T is maximal monotone, we have $w \in T^{-1}0$ and hence $w \in \mathrm{VI}(C, A)$.

We next show that $w \in \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. To see this, we observe that we may assume (by passing to a further subsequence if necessary) $\lambda_{n_i,k} \to \lambda_k$ for $k = 1, 2, \ldots, N$. Let W be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_1, \lambda_2, \ldots, \lambda_N$. By Lemma 2.5, we know that W is nonexpansive and $\bigcap_{i=1}^{N} \mathrm{Fix}(T_i) = \mathrm{Fix}(W)$. it follows from Lemma 2.6 that

$$W_{n_i}x \longrightarrow Wx, \quad \forall x \in C. \tag{3.56}$$

Assume $w \notin \mathrm{Fix}(W)$. Since $x_{n_i} \rightharpoonup w$ and $w \neq Ww$, it follows from the Opial condition, (3.42), and (3.56) that

$$\begin{aligned}
\liminf_{i\to\infty}\|x_{n_i} - w\| &< \liminf_{i\to\infty}\|x_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty}\{\|x_{n_i} - W_{n_i}x_{n_i}\| + \|W_{n_i}x_{n_i} - Wx_{n_i}\| + \|Wx_{n_i} - Ww\|\} \\
&\leq \liminf_{i\to\infty}\|x_{n_i} - w\|,
\end{aligned} \tag{3.57}$$

which is a contradiction. Hence, we have $w \in \mathrm{Fix}(W) = \bigcap_{i=1}^{N} \mathrm{Fix}(T_i)$. This implies $w \in \Omega$. Therefore, we have

$$\limsup_{n\to\infty}\left\langle f(u_0) - u_0, x_n - u_0\right\rangle = \lim_{j\to\infty}\left\langle f(u_0) - u_0, x_{n_j} - u_0\right\rangle = \left\langle f(u_0) - u_0, w - u_0\right\rangle \leq 0. \tag{3.58}$$

Finally, we show that $x_n \to u_0$, where $u_0 = P_\Omega f(u_0)$.

From Lemma 2.3, we have

$$\begin{aligned}
\|x_{n+1}-u_0\|^2 &= \left\|\alpha_n(f(x_n)-u_0)+\beta_n(x_n-u_0)+(1-\alpha_n-\beta_n)(W_nt_n-u_0)\right\|^2 \\
&\le \left\|\beta_n(x_n-u_0)+(1-\alpha_n-\beta_n)(W_nt_n-u_0)\right\|^2+2\alpha_n\langle f(x_n)-u_0,x_{n+1}-u_0\rangle \\
&\le (1-\alpha_n-\beta_n)\|W_nt_n-u_0\|^2+\beta_n\|x_n-u_0\|^2+2\alpha_n\langle f(x_n)-u_0,x_{n+1}-u_0\rangle \\
&\le (1-\alpha_n-\beta_n)\|W_nt_n-u_0\|^2+\beta_n\|x_n-u_0\|^2+2\alpha_n\langle f(x_n)-f(u_0),x_{n+1}-u_0\rangle \\
&\quad +2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle \\
&\le (1-\alpha_n-\beta_n)\|t_n-u_0\|^2+\beta_n\|x_n-u_0\|^2+2\alpha_n a\|x_n-u_0\|\|x_{n+1}-u_0\| \\
&\quad +2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle \\
&\le (1-\alpha_n)\|x_n-u_0\|^2+\alpha_n a\left(\|x_n-u_0\|^2+\|x_{n+1}-u_0\|^2\right)+2\alpha_n\langle f(u_0)-u_0,x_{n+1}-u_0\rangle,
\end{aligned} \tag{3.59}$$

and thus

$$\|x_{n+1}-u_0\|^2 \le \left(1-\frac{\alpha_n}{1-a\alpha_n}\right)\|x_n-u_0\|^2+\frac{\alpha_n}{1-a\alpha_n}\langle 2f(u_0)-2u_0,x_{n+1}-u_0\rangle. \tag{3.60}$$

It follows from Lemma 2.2, (3.58), and (3.60) that $\lim_{n\to\infty}\|x_n-u_0\|=0$. From $\|x_n-u_n\|\to 0$ and $\|y_n-u_n\|\to 0$, we have $u_n\to u_0$ and $y_n\to u_0$. The proof is now complete. □

4. Applications

By Theorem 3.1, we can obtain some new and interesting strong convergence theorems as follows.

Theorem 4.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C\times C$ to R satisfying* (A1)–(A5) *and let $\varphi: C\to R\cup\{+\infty\}$ be a proper lower semicontinuous and convex function. Let $T_1,T_2,\ldots,T_N$ be a finite family of nonexpansive mappings of C into H such that $\Sigma=\bigcap_{n=1}^N \mathrm{Fix}(T_i)\cap \mathrm{MEP}(F,\varphi)\neq\emptyset$. Let $\{\lambda_{n,1}\},\{\lambda_{n,2}\},\ldots,\{\lambda_{n,N}\}$ be sequences in $[\varepsilon_1,\varepsilon_2]$ with $0<\varepsilon_1\le\varepsilon_2<1$. Let W_n be the W-mapping generated by $T_1,T_2,\ldots,T_N$ and $\lambda_{n,1},\lambda_{n,2},\ldots,\lambda_{n,N}$. Assume that either* (B1) *or* (B2) *holds. Let f be a contraction of H into itself and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{gathered}
x_1=x\in C,\\
F(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r_n}\langle y-u_n,u_n-x_n\rangle\ge 0,\quad \forall y\in C,\\
x_{n+1}=\alpha_nf(x_n)+\beta_nx_n+(1-\alpha_n-\beta_n)W_nu_n
\end{gathered} \tag{4.1}$$

for every $n = 1, 2, \dots$ where $\{r_n\}$, $\{\alpha_n\}$, $\{\lambda_{n1}\}$, $\{\lambda_{n2}\}, \dots, \{\lambda_{nN}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$;

(C4) $\liminf_{n\to\infty}r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(C5) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \dots, N$.

Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_{\Sigma}f(w)$.

Proof. Putting $A = 0$, by Theorem 3.1 we obtain the desired result. □

Theorem 4.2. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let F be a bifunction from $C \times C$ to R satisfying* (A1)–(A5). *Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $T_1, T_2, \dots, T_N$ be a finite family of nonexpansive mappings of C into H such that $\Lambda = \bigcap_{n=1}^{N}\operatorname{Fix}(T_i) \cap \operatorname{VI}(C, A) \cap EP(F) \neq \emptyset$. Let $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \dots, \{\lambda_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let W_n be the W-mapping generated by $T_1, T_2, \dots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \dots, \lambda_{n,N}$. Assume that either* (B4) *or* (B2) *holds. Let f be a contraction of H into itself and let $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ be sequences generated by*

$$\begin{aligned} x_1 &= x \in C, \\ F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle &\geq 0, \quad \forall y \in C, \\ y_n &= P_C(u_n - \gamma_n A u_n), \\ x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \gamma_n A y_n) \end{aligned} \tag{4.2}$$

for every $n = 1, 2, \dots$ where $\{\gamma_n\}$, $\{r_n\}$, $\{\alpha_n\}$, $\{\lambda_{n1}\}$, $\{\lambda_{n2}\}, \dots, \{\lambda_{nN}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the following conditions:

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty}\alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$;

(C3) $\lim_{n\to\infty}\gamma_n = 0$;

(C4) $\liminf_{n\to\infty}r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(C5) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \dots, N$.

Then, $\{x_n\}$, $\{u_n\}$, and $\{y_n\}$ converge strongly to $w = P_{\Lambda}f(w)$.

Proof. Putting $\varphi = 0$, by Theorem 3.1 we obtain the desired result. □

Theorem 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $\varphi : C \to R \cup \{+\infty\}$ be a proper lower semicontinuous and convex function. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $T_1, T_2, \dots, T_N$ be a finite family of nonexpansive mappings of C into H such that $\Theta = \bigcap_{n=1}^{N}\operatorname{Fix}(T_i) \cap \operatorname{VI}(C, A) \cap \operatorname{Argmin}(\varphi) \neq \emptyset$. Let $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \dots, \{\lambda_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let W_n be the W-mapping generated by $T_1, T_2, \dots, T_N$ and*

$\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. *Assume that either* (B3) *or* (B2) *holds. Let* f *be a contraction of* H *into itself and let* $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *be sequences generated by*

$$\begin{gathered} x_1 = x \in C, \\ \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \gamma_n A u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(u_n - \gamma_n A y_n) \end{gathered} \tag{4.3}$$

for every $n = 1, 2, \ldots$. *where* $\{\gamma_n\}$, $\{r_n\}$, $\{\alpha_n\}$, $\{\lambda_{n1}\}$, $\{\lambda_{n2}\}, \ldots$, $\{\lambda_{nN}\}$, *and* $\{\beta_n\}$ *are sequences of numbers satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$;

(C3) $\lim_{n\to\infty}\gamma_n = 0$;

(C4) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$;

(C5) $\lim_{n\to\infty}|\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *converge strongly to* $w = P_\Theta f(w)$.

Proof. Let $F(x, y) = 0$ for all $x, y \in C$, by Theorem 3.1 we obtain the desired result. □

Theorem 4.4. *Let* C *be a nonempty closed convex subset of a real Hilbert space* H. *Let* F *be a bifunction from* $C \times C$ *to* R *satisfying* (A1)–(A5) *and let* $\varphi : C \to R \cup \{+\infty\}$ *be a proper lower semicontinuous and convex function. Let* A *be a monotone and* k*-Lipschitz continuous mapping of* C *into* H. *Let* S *be a nonexpansive mapping of* C *into* H *such that* $\mathrm{Fix}(S) \cap \mathrm{VI}(C, A) \cap \mathrm{MEP}(F, \varphi) \neq \emptyset$. *Assume that either* (B1) *or* (B2) *holds. Let* f *be a contraction of* H *into itself and let* $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *be sequences generated by*

$$\begin{gathered} x_1 = x \in C, \\ F(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n \rangle \geq 0, \quad \forall y \in C, \\ y_n = P_C(u_n - \gamma_n A u_n), \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) S P_C(u_n - \gamma_n A y_n) \end{gathered} \tag{4.4}$$

for every $n = 1, 2, \ldots$ *where* $\{\gamma_n\}$, $\{r_n\}$, $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences of numbers satisfying the following conditions:*

(C1) $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty}\beta_n \geq \liminf_{n\to\infty}\beta_n > 0$;

(C3) $\lim_{n\to\infty}\gamma_n = 0$;

(C4) $\liminf_{n\to\infty} r_n > 0$ *and* $\lim_{n\to\infty}|r_{n+1} - r_n| = 0$.

Then, $\{x_n\}$, $\{u_n\}$, *and* $\{y_n\}$ *converge strongly to* $w = P_{\mathrm{Fix}(S)\cap \mathrm{VI}(C,A)\cap \mathrm{MEP}(F,\varphi)} f(w)$.

Proof. Let $W_n = S$, by Theorem 3.1 we obtain the desired result. □

Theorem 4.5. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let A be a monotone and k-Lipschitz continuous mapping of C into H. Let $T_1, T_2, \ldots, T_N$ be a finite family of nonexpansive mappings of C into H such that $\Gamma = \bigcap_{n=1}^{N} \mathrm{Fix}(T_i) \cap \mathrm{VI}(C, A) \neq \emptyset$. Let $\{\lambda_{n,1}\}, \{\lambda_{n,2}\}, \ldots, \{\lambda_{n,N}\}$ be sequences in $[\varepsilon_1, \varepsilon_2]$ with $0 < \varepsilon_1 \leq \varepsilon_2 < 1$. Let W_n be the W-mapping generated by $T_1, T_2, \ldots, T_N$ and $\lambda_{n,1}, \lambda_{n,2}, \ldots, \lambda_{n,N}$. Let $\{x_n\}$ and $\{y_n\}$ be sequences generated by*

$$
\begin{aligned}
x_1 &= x \in C, \\
y_n &= P_C(x_n - \gamma_n A x_n), \\
x_{n+1} &= \alpha_n f(x_n) + \beta_n x_n + (1 - \alpha_n - \beta_n) W_n P_C(x_n - \gamma_n A y_n)
\end{aligned}
\tag{4.5}
$$

for every $n = 1, 2, \ldots$ where $\{\gamma_n\}$, $\{\alpha_n\}$, $\{\lambda_{n1}\}$, $\{\lambda_{n2}\}, \ldots$, $\{\lambda_{nN}\}$, and $\{\beta_n\}$ are sequences of numbers satisfying the following conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C2) $1 > \limsup_{n\to\infty} \beta_n \geq \liminf_{n\to\infty} \beta_n > 0$;

(C3) $\lim_{n\to\infty} \gamma_n = 0$;

(C5) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$ *for all* $i = 1, 2, \ldots, N$.

Then, $\{x_n\}$ and $\{y_n\}$ converge strongly to $w = P_\Gamma f(w)$.

Proof. Let $\varphi = 0$ and let $F(x, y) = 0$ for all $x, y \in C$. Then $u_n = P_C x_n = x_n$. By Theorem 3.1 we obtain the desired result. □

Remark 4.6. (1) Since the α-inverse-strongly-monotonicity of A has been weakened by the monotonicity and Lipschitz continuity of A. Theorems 3.1, 4.2, and 4.4 generalize and improve Theorem 3.1 in [11], Theorem 3.1 in [12], and Theorem 3.1 in [8] and the main results in [31]. Theorem 4.5 improves Theorem 3.1 in [20].

(2) It is easy to see that Theorems 3.1, 4.2, and 4.4 also generalize and improve Theorems 3.1, and 4.2 in [9].

(3) It is clear that Theorem 4.5 generalizes, extends, and improves Theorem 3.1 in [18] and Theorem 3.1 in [19].

(4) Theorem 3.1 improves and extends Theorem 3.1 in [1].

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