Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2009, Article ID 194671, 9 pages
doi:10.1155/2009/194671

Research Article

On Some Generalized Ky Fan Minimax Inequalities

Xianqiang Luo

Department of Mathematics, Wuyi University, Jangmen, 529020, China

Correspondence should be addressed to Xianqiang Luo, luoxq1978@126.com

Received 31 October 2008; Revised 26 March 2009; Accepted 21 April 2009

Recommended by Naseer Shahzad

Some generalized Ky Fan minimax inequalities for vector-valued mappings are established by applying the classical Browder fixed point theorem and the Kakutani-Fan-Glicksberg fixed point theorem.

1. Introduction

It is well known that Ky Fan minimax inequality [1] plays a very important role in various fields of mathematics, such as variational inequality, game theory, mathematical economics, fixed point theory, control theory. Many authors have got some interesting achievements in generalization of the inequality in various ways. For example, Ferro [2] obtained a minimax inequality by a separation theorem of convex sets. Tanaka [3] introduced some quasiconvex vector-valued mappings to discuss minimax inequality. Li and Wang [4] obtained a minimax inequality by using some scalarization functions. Tan [5] obtained a minimax inequality by the generalized G-KKM mapping. Verma [6] obtained a minimax inequality by an R-KKM mapping. Li and Chen [7] obtained a set-valued minimax inequality by a nonlinear separation function $\xi_{k,a}$. Ding [8, 9] obtained a minimax inequality by a generalized R-KKM mapping. Some other results can be found in [10–16].

In this paper, we will establish some generalized Ky Fan minimax inequalities forvector-valued mappings by the classical Browder fixed point theorem and the Kakutani-Fan-Glicksberg fixed point theorem.

2. Preliminaries

Now, we recall some definitions and preliminaries needed. Let X and Y be two nonempty sets, and let $T : X \to 2^Y$ be a nonempty set-valued mapping, $x \in T^{-1}(y)$ if and only if $y \in T(x)$, $T(X) = \bigcup_{x\in X} T(x)$. Throughout this paper, assume that every space is Hausdorff.

Definition 2.1 (see [10]). For topological spaces X and Y, a mapping $T : X \to 2^Y$ is said to be

(i) upper semicontinuous (usc), if for each open set $B \subset Y$, the set $T^{-1}(B) = \{x \in X : T(x) \subset B\}$ is open subset of X;

(ii) lower semicontinuous (lsc), if for each closed set $B \subset Y$, the set $T^{-1}(B) = \{x \in X : T(x) \subset B\}$ is closed subset of X;

(iii) continuous, if it is both (usc) and (lsc);

(iv) compact-valued, if $T(x)$ is compact in Y for any $x \in X$.

Definition 2.2 (see [11]). Let Z be a topological vector space and $C \subset Z$ be a pointed convex cone with a nonempty interior $\operatorname{int} C$, and let B be a nonempty subset of Z. A point $z \in B$ is said to be

(i) a minimal point of B if $B \cap (z - C) = \{z\}$;

(ii) a weakly minimal point of B if $B \cap (z - \operatorname{int} C) = \emptyset$;

(iii) a maximal point of B if $B \cap (z + C) = \{z\}$;

(iv) a weakly maximal point of B if $B \cap (z + \operatorname{int} C) = \emptyset$.

By $\min B$, $\min_w B$, $\max B$, $\max_w B$, we denote, respectively, the set of all minimal points, the set of all weakly minimal points, the set of all maximal points, the set of all weakly maximal points of B.

Lemma 2.3 (see [11]). *Let B be a nonempty compact subset of a topological vector space Z with a closed pointed convex cone C. Then*

(i) $\min B \neq \emptyset$;

(ii) $B \subset \min B + C \subset \min_w B + C$;

(iii) $\max B \neq \emptyset$;

(iv) $B \subset \max B - C \subset \max_w B - C$.

Lemma 2.4 (see [11]). *Let E and Z be two topological vector spaces, $\emptyset \neq X \subset E$, and let $F : X \to 2^Z$ be a set-valued mapping. If X is compact, and F is upper semicontinuous and compact-valued, then $F(X) = \bigcup_{x\in X} F(x)$ is compact set.*

Lemma 2.5 (see [2, Theorem 3.1]). *Let E be a topological vector space, let Z be a topological vector space with a closed pointed convex cone C, $\operatorname{int} C \neq \emptyset$, let X and Y be two nonempty compact subsets of E, and let $f : X \times Y \to Z$ be a continuous mapping. Then both $F_1 : X \to 2^Z$ defined by $F_1(x) = \max_w f(x, Y)$ and $F_2 : X \to 2^Z$ defined by $F_2(x) = \min_w f(x, Y)$ are upper semicontinuous and compact-valued.*

Definition 2.6. Let Z be a topological vector space and let C be a closed pointed convex cone in Z, $\operatorname{int} C \neq \emptyset$. Given $e \in \operatorname{int} C$ and $a \in Z$, the function $h_{e,a}$ and $g_{e,a} : Z \to R$ are, respectively, defined by $h_{e,a}(z) = \min\{t \in R : z \in a + te - C\}$, and $g_{e,a}(z) = \max\{t \in R : z \in a + te + C\}$.

We quote some of their properties as follows (see [12]):

(i) $h_{e,a}(z) < r \Leftrightarrow z \in a + re - \operatorname{int} C$; $g_{e,a}(z) > r \Leftrightarrow z \in a + re + \operatorname{int} C$;

(ii) $h_{e,a}(z) \leq r \Leftrightarrow z \in a + re - C$; $g_{e,a}(z) \geq r \Leftrightarrow z \in a + re + C$;

(iii) $h_{e,a}(z) > r \Leftrightarrow z \notin a + re - C$; $g_{e,a}(z) < r \Leftrightarrow z \notin a + re + C$;

(iv) $h_{e,a}(z) \geq r \Leftrightarrow z \notin a + re - \operatorname{int} C$; $g_{e,a}(z) \leq r \Leftrightarrow z \notin a + re + \operatorname{int} C$;

(v) $h_{e,a}$ is a continuous and convex function; $g_{e,a}$ is a continuous and concave function;

(vi) $h_{e,a}$ and $g_{e,a}$ are strictly monotonically increasing (monotonically increasing), that is, if $z_1 - z_2 \in \operatorname{int} C \Rightarrow f(z_1) > f(z_2)$ $(z_1 - z_2 \in C \Rightarrow f(z_1) \geq f(z_2))$, where f denotes $h_{e,a}$ or $g_{e,a}$.

Definition 2.7 (see [3]). Let E be a topological vector space, let X be a nonempty convex subsets of E, and let Z be a topological vector space with a pointed convex cone C, $\operatorname{int} C \neq \emptyset$. A vector-valued mapping $f : X \to Z$ is said to be

(i) C-quasiconcave if for each $z \in Z$, the set $\{x \in X : f(x) \in z + C\}$ is convex;

(ii) properly C-quasiconcave if for any $x, y \in X$ and $t \in [0,1]$, either $f(tx + (1-t)y) \in f(x) + C$ or $f(tx + (1-t)y) \in f(y) + C$.

The following two propositions are very important in proving Proposition 2.10.

Proposition 2.8 (see [4]). *Let Z be a topological vector space and let C be a closed pointed convex cone in Z, $\operatorname{int} C \neq \emptyset$, $f : X \to Z$:*

(i) *f is C-quasiconcave if and only if for all $e \in \operatorname{int} C$ and for all $a \in Z$, $g_{e,a}(f)$ is quasiconcave;*

(ii) *if f is properly C-quasiconcave.*

Then $h_{e,a}(f)$ is quasiconcave.

Proposition 2.9. *Let E be a topological vector space and let X be a nonempty convex subset of E, $f : X \to R$. Then the following two statements are equivalent:*

(i) *for any $r \in R$, $\{x \in X : f(x) \geq r\}$ is convex;*

(ii) *for any $t \in R$, $\{x \in X : f(x) > t\}$ is convex.*

Proof. (i)⇒(ii) For any $t \in R$, $x_1, x_2 \in \{x \in X : f(x) > t\}$. Let $r = \min\{f(x_1), f(x_2)\} > t$, then $x_1, x_2 \in \{x \in X : f(x) \geq r\}$. By (i), we have $\{x \in X : f(x) \geq r\}$ is convex, then $\operatorname{co}(\{x_1, x_2\}) \subset \{x \in X : f(x) \geq r > t\}$. Thus, $\operatorname{co}(\{x_1, x_2\}) \subset \{x \in X : f(x) > t\}$ is convex.

(ii)⇒(i) For any $r \in R$, $x_1, x_2 \in \{x \in X : f(x) \geq r\}$, then for all $\varepsilon > 0$, $x_1, x_2 \in \{x \in X : f(x) > r - \varepsilon\}$. By (ii), we have $\{x \in X : f(x) > r - \varepsilon\}$ is convex, that is, $\operatorname{co}(\{x_1, x_2\}) \subset \{x \in X : f(x) > r - \varepsilon\}$. Since ε is arbitrary, then $\operatorname{co}(\{x_1, x_2\}) \subset \{x \in X : f(x) \geq r\}$ is convex. □

Proposition 2.10. *Let E be a topological vector space, let Z be a topological vector space with a closed pointed convex cone C, $\operatorname{int} C \neq \emptyset$, and let X be a nonempty compact convex subset of E, $f : X \to Z$ be a vector mapping. Then the following two statements are equivalent:*

(i) *for any $z \in Z$, $\{x \in X : f(x) \in z + C\}$ is convex, that is, $f(x)$ is C-quasiconcave;*

(ii) *for any $z \in Z$, $\{x \in X : f(x) \in z + \operatorname{int} C\}$ is convex.*

Proof. (i)⇒(ii) for all $z \in Z$ and for all $e \in \operatorname{int} C$, let $a = z - e$. By Proposition 2.8, we have $g_{e,a}(f(x))$ is quasiconcave, that is, for any $r \in R$, $\{x \in X : g_{e,a}(f(x)) \geq r\}$ is convex, then by Proposition 2.9, we have for any $t \in R$, $\{x \in X : g_{e,a}(f(x)) > t\}$ is convex. Thus, $\{x \in X : g_{e,a}(f(x)) > 1\}$ is convex. Therefore, we have $\{x \in X : f(x) \in z + \operatorname{int} C\}$ is convex since $\{x \in X : f(x) \in z + \operatorname{int} C\} = \{x \in X : g_{e,a}(f(x)) > 1\}$ by property (i) of $g_{e,a}$.

(ii)⇒(i) By Proposition 2.8, we need only prove for all $e \in \operatorname{int} C$ and for all $a \in Z$, $g_{e,a}(f(x))$ is quasiconcave, that is, for any $r \in R$, $\{x \in X : g_{e,a}(f(x)) \geq r\}$ is convex.

For any $t \in R$, let $z = a + te$. By property (i) of $g_{e,a}$, we have

$$\{x \in X : f(x) \in z + \operatorname{int} C\} = \{x \in X : g_{e,a}(f(x)) > t\}. \tag{2.1}$$

Thus, for any $t \in R$, $\{x \in X : g_{e,a}(f(x)) > t\}$ is convex since $\{x \in X : f(x) \in z + \operatorname{int} C\}$ is convex by (ii). Therefore, by Proposition 2.9, we have for any $r \in R$, $\{x \in X : g_{e,a}(f(x)) \geq r\}$ is convex. □

3. Generalized Ky Fan Minimax Inequalities

In this section, we will establish some generalized Ky Fan minimax inequalities and a corollary by Propositions 1.1, 1.3 and Lemmas 3.1, 3.2.

Lemma 3.1 (see [13]). *Let E be a topological vector space, let $X \subset E$ be a nonempty compact and convex set, and let $T : X \to 2^X$, such that*

(i) *for each $x \in X$, $T(x)$ is nonempty and convex;*

(ii) *for each $x \in X$, $T^{-1}(x)$ is open.*

Then T has a fixed point.

Lemma 3.2 (see [11], Kakutani-Fan-Glicksberg fixed point theorem). *Let E be a locally convex topological vector space and let $X \subset E$ be a nonempty compact and convex set. If $T : X \to 2^X$ is upper semicontinuous, and for any $x \in X$, $T(x)$ is a nonempty, closed and convex subset, then T has a fixed point.*

Theorem 3.3. *Let E be a topological vector space, let Z be a topological vector space with a closed pointed convex cone C, $\operatorname{int} C \neq \emptyset$, let X be a nonempty compact convex subset of E, and let $f : X \times X \to Z$ be a continuous mapping, such that*

(i) *for all $z \in (\max_w)_{t \in X} f(t,t)$, for any $x \in X$, $\{y \in X : f(x,y) \in z + \operatorname{int} C\}$ is convex.*

Then

$$\max_{t \in X}{}_w f(t,t) \subset \min_{x \in X} \max_{y \in X}{}_w f(x,y) + Z \setminus (-\operatorname{int} C). \tag{3.1}$$

Proof. Let $z \in (\max_w)_{t \in X} f(t,t)$, then by the definition of the weakly maximal point, we have

$$\text{for any } x \in X, \quad f(x,x) \notin z + \operatorname{int} C. \tag{$*$}$$

For each $x \in X$, let

$$T(x) = \{y \in X : f(x,y) \in z + \operatorname{int} C\}. \tag{3.2}$$

Now, we prove that there exists $x_0 \in X$, such that $T(x_0) = \emptyset$.

Supposed for each $x \in X$, $T(x) \neq \emptyset$, then by condition (i), we have for each $x \in X$, $T(x)$ is nonempty and convex. In addition, we have for each $y \in X$, $T^{-1}(y)$ is open since f is continuous.

Thus, by Lemma 3.1, there exists $x' \in X$, such that $x' \in T(x')$, that is, $f(x', x') \in z + \operatorname{int} C$, which contradicts $(*)$.

Therefore, there exists $x_0 \in X$, such that $T(x_0) = \emptyset$, that is, for any $y \in X$,

$$z \notin f(x_0, y) - \operatorname{int} C. \tag{3.3}$$

Since $\max_w f(x_0, X) \neq \emptyset$, then $z \in \max_w f(x_0, X) + Z \setminus (-\operatorname{int} C) \subset \bigcup_{x \in X} \max_w f(x, X) + Z \setminus (-\operatorname{int} C) = \min_{x \in X}(\max_w)_{y \in X} f(x, y) + Z \setminus (-\operatorname{int} C)$ (because of $Z \setminus (-\operatorname{int} C) = Z \setminus (-\operatorname{int} C) + C$, and Lemma 2.3). □

Remark 3.4. By Proposition 2.10, in the above Theorem 3.3, the condition (i) can be replaced by "for each $x \in X$, $f(x, y)$ is C-quasiconcave in y".

Theorem 3.5. *Let E be a topological vector space, let Z be a topological vector space with a closed convex pointed cone C, int $C \neq \emptyset$, let X be a nonempty compact convex subset of E, and let $f : X \times X \to Z$ be a continuous mapping, such that*

(i) *for each $x \in X$, $f(x, y)$ is properly C-quasiconcave in y.*

Then

$$\min_{x \in X}{}_w \max_{y \in X}{}_w f(x, y) \subset \max_{t \in X} f(t, t) + Z \setminus \operatorname{int} C. \tag{3.4}$$

Proof. Since X is compact, and f is continuous, then by Lemma 2.3, we have for any $x \in X$, $\max_w f(x, X) \neq \emptyset$ and $(\min_w)_{x \in X} (\max_w)_{y \in X} f(x, y) \neq \emptyset$.

For any $x \in X$, there exists $y_x \in X$, such that $f(x, y_x) \in \max_w f(x, X)$. Let $z \in (\min_w)_{x \in X} (\max_w)_{y \in X} f(x, y)$, by the definition of the weakly minimal point, we have $f(x, y_x) \notin z - \operatorname{int} C$. Thus, for each $x \in X$, let

$$T(x) = \{y \in X : f(x, y) \notin z - \operatorname{int} C\} \neq \emptyset. \tag{3.5}$$

Now, we prove that there exists $x_0 \in X$, such that $x_0 \in T(x_0)$.

For all $e \in \operatorname{int} C$, let $a = z - e \in Z$, the function $h_{e,a} : Z \to R$ is defined by

$$h_{e,a}(z) = \min\{t \in R : z \in a + te - C\}. \tag{3.6}$$

Let $g(x, y) = h_{e,a}(f(x, y))$, then $g(x, y)$ is continuous since both $h_{e,a}$ and f are continuous. By property (iv) of $h_{e,a}$, we have

$$T(x) = \{y \in X : f(x, y) \notin z - \operatorname{int} C\} = \{y \in X : g(x, y) \geq 1\}. \tag{$**$}$$

For any $n \in N$, let $T_n(x) = \{y \in X : g(x, y) > 1 - 1/n\}$, then it satisfies the all conditions of Lemma 3.1.

In fact, firstly, by $T(x) \subset T_n(x)$, we have $T_n(x) \neq \emptyset$, and for each $y \in X$, $T_n^{-1}(y)$ is open since $g(x,y)$ is continuous. Secondly, by condition (i) and Proposition 2.8, we have $g(x,y)$ is quasiconcave in y, that is, for any $r \in R$, $\{y \in X : g(x,y) \geq r\}$ is convex. Thus, by Proposition 2.9, $T_n(x) = \{y \in X : g(x,y) > 1 - 1/n\}$ is convex.

By Lemma 3.1, there exists $x_n \in X$, such that $x_n \in T_n(x)$, that is,

$$g(x_n, x_n) > 1 - \frac{1}{n}. \tag{3.7}$$

Since X is compact, then $\{x_n\}$ has a subnet converging to $x_0 \in X$. Let $n \to \infty$ in the above expression, together with $(**)$, yields

$$g(x_0, x_0) \geq 1 \Longleftrightarrow x_0 \in T(x_0). \tag{3.8}$$

Thus,

$$z \notin f(x_0, x_0) + \operatorname{int} C. \tag{3.9}$$

Therefore, for all $z \in (\min_w)_{x\in X}(\max_w)_{y\in X} f(x,y)$, we have

$$z \in f(x_0, x_0) + Z \setminus \operatorname{int} C \subset \max_{t\in X} f(t,t) - C + Z \setminus \operatorname{int} C = \max_{t\in X} f(t,t) + Z \setminus \operatorname{int} C. \tag{3.10}$$

□

Theorem 3.6. *Let E be a locally convex topological vector space, let Z be a topological vector space with a closed convex pointed cone C, $\operatorname{int} C \neq \emptyset$, let X be a nonempty compact and convex subset of E, let $f : X \times X \to Z$ be a continuous mapping, and let $z_0 \in Z$ such that*

(i) *for each $x \in X$, $T(x) = \{y \in X : f(x,y) \in z_0 + C\}$ is nonempty convex.*

Then

$$z_0 \in \max_{x\in X} f(x,x) - C. \tag{3.11}$$

Proof. For each $x \in X$, we define $T : X \to 2^X$ by

$$T(x) = \{y_x \in X : f(x, y_x) \in z_0 + C\}. \tag{3.12}$$

Now, we prove that T has a fixed point.

(1) By the condition (i), we have for each $x \in X$, $T(x) \neq \emptyset$ is closed and convex since f is continuous and C is closed.

(2) T is upper semicontinuous mapping.

For each $x \in X$, $T(x)$ is compact since X is compact and $T(x) \subset X$ is closed. We only need to prove T has a closed graph.

In fact, Let $(x', y') \in \overline{\mathrm{Gr}(T)}$, and a net (x_α, y_α) in $\mathrm{Gr}(T)$ converging to (x', y'). Since f is continuous and $z_0 + C$ is closed, then

$$f(x_\alpha, y_\alpha) \longrightarrow f(x', y') \in z_0 + C. \tag{3.13}$$

Thus,

$$y' \in T(x') \Longrightarrow (x', y') \in \mathrm{Gr}(T). \tag{3.14}$$

Therefore, by Lemma 3.2 (KFG fixed point theorem), T has a fixed point x_3 such that

$$x_3 \in T(x_3). \tag{3.15}$$

Then

$$z_0 \in f(x_3, x_3) - C \subset \bigcup_{x\in X} f(x, x) - C \subset \max_{x\in X} f(x, x) - C. \tag{3.16}$$

□

Remark 3.7. If for each $x \in X$, $f(x, y)$ is C-quasiconcave in y and $z_0 \subset f(x, X) - C$, then the condition (i) holds. Thus, we can obtain the following corollary.

Corollary 3.8. *Let E be a locally convex topological vector space, let Z be a topological vector space with a closed convex pointed cone C, $\operatorname{int} C \neq \emptyset$, let X be a nonempty compact and convex subset of E, and let $f : X \times X \to Z$ be a continuous mapping such that*

(i) $f(x, y)$ *is C-quasiconcave in y for each $x \in X$;*

(ii) $(\min_w)_{x\in X} (\max_w)_{y\in X} f(x, y) \subset f(x, X) - C$ *for each $x \in X$.*

Then

$$\min_{x\in X}{}_w \max_{y\in X}{}_w f(x, y) \subset \max_{x\in X} f(x, x) - C. \tag{3.17}$$

Proof. Let $z_0 \in (\min_w)_{x\in X} (\max_w)_{y\in X} f(x, y)$, and for each $x \in X$, let $T(x) = \{y_x \in X : f(x, y_x) \in z_0 + C\}$. By condition (ii), $T(x)$ is nonempty. And by condition (i), $T(x)$ is convex. Thus, by Theorem 3.6, the conclusion holds. □

Remark 3.9. By Definition 2.7, the condition (i) can be replaced by "(i) $f(x, y)$ is properly C-quasiconcave in y for each $x \in X$."

Example 3.10. Let $E = R$, $X = [0, 1]$, $Z = R^2$, $C = \{(x, y) \in R \times R : |x| \leq y\}$. Given a fixed $x \in X$, for each $y \in X$, we define $f : X \times X \to Z$ by

$$f(x, y) = \begin{cases} (x, y), & \text{if } y \leq x \\ (y, y), & \text{if } y \geq x. \end{cases} \tag{3.18}$$

In Figure 1, the red line denotes the graph of $f(x, y)$ for each $x \in X$.

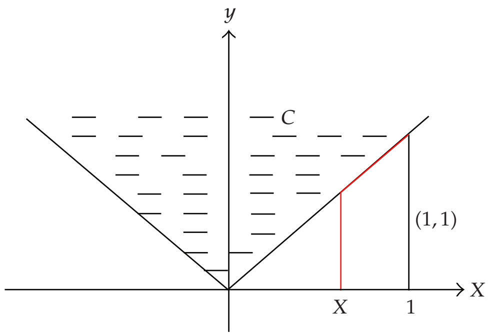

Figure 1: The function's graph.

Now we prove f satisfies the conditions of Corollary 3.8:

(i) f is a continuous.

Let $B \subset Z$ is closed, let $(x_\alpha, y_\alpha) \subset f^{-1}(B) = \{(x, y) : f(x, y) \in B\}$, and $(x_\alpha, y_\alpha) \to (x', y')$. Then by the definition of f, we have

$$f(x_\alpha, y_\alpha) = \begin{cases} (x_\alpha, y_\alpha), & \text{if } y_\alpha \leq x_\alpha \\ (y_\alpha, y_\alpha), & \text{if } y_\alpha \geq x_\alpha. \end{cases} \tag{3.19}$$

Thus there exists a subnet yet denoted by (x_α, y_α), and $y_\alpha \leq x_\alpha$, such that $f(x_\alpha, y_\alpha) = (x_\alpha, y_\alpha) \to (x', y') \in B$ since B is closed. Hence, $y' \leq x'$, and $f(x', y') = (x', y') \in B \Rightarrow (x', y') \in f^{-1}(B)$. Therefore, $f^{-1}(B)$ is closed.

(ii) From Figure 1, we can check that $f(x, y)$ is properly C-quasiconcave in y for each $x \in \mathrm{X}$.

(iii) From Figure 1, we can check that $(\min_w)_{x\in X}(\max_w)_{y\in X} f(x, y) = \{(x, x) : x \in [0, 1]\} \subset (1, 1) - C \subset \max_w f(x, \mathrm{X}) = \{(y, y) : y \in [x, 1]\} - C$ for each $x \in \mathrm{X}$. Thus, $(\min_w)_{x\in X}(\max_w)_{y\in X} f(x, y) \subset \max_w f(x, \mathrm{X}) - C$ for each $x \in \mathrm{X}$.

Finally, from Figure 1, we can check that $(\min_w)_{x\in X}(\max_w)_{y\in X} f(x, y) = \{(x, x) : x \in [0, 1]\} \subset (1, 1) - C = \max_{x\in X} f(x, x) - C$, that is, Corollary 3.8 holds.

Acknowledgments

The author gratefully acknowledges the referee for his/her ardent corrections and valuable suggestions, and is thankful to Professor Junyi Fu and Professor Xunhua Gong for their help. This work was supported by the Young Foundation of Wuyi University.

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