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Research Article

The C^1 Solutions of the Series-Like Iterative Equation with Variable Coefficients

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By constructing a structure operator quite different from that ofZhang and Baker (2000) and using the Schauder fixed point theory, the existence and uniqueness of the C^1 solutions of the series-like iterative equations with variable coefficients are discussed.

1. Introduction

An important form of iterative equations is the polynomial-like iterative equation

$$\lambda_1 f(x) + \lambda_2 f^2(x) + \cdots + \lambda_n f^n(x) = F(x), \quad x \in I := [a,b], \tag{1.1}$$

where F is a given function, f is an unknown function, $\lambda_i \in \mathrm{R}^1$ $(i = 1,2,\ldots,n)$, and f^k $(k = 1,2,\ldots,n)$ is the kth iterate of f, that is, $f^0(x) = x$, $f^k(x) = f \circ f^{k-1}(x)$. The case of all constant $\lambda_i's$ was considered in [1–10]. In 2000, W. N. Zhang and J. A. Baker first discussed the continuous solutions of such an iterative equation with variable coefficients $\lambda_i = \lambda_i(x)$ which are all continuous in interval $[a,b]$. In 2001, J. G. Si and X. P. Wang furthermore gave the continuously differentiable solution of such equation in the same conditions as in [11]. In this paper, we continue the works of [11, 12], and consider the series-like iterative equation with variable coefficients

$$\sum_{i=1}^{\infty} \lambda_i(x) f^i(x) = F(x), \quad x \in I := [a,b], \tag{1.2}$$

where $\lambda_i(x) : I \to [0,1]$ are given continuous functions and $\sum_{i=1}^{\infty}\lambda_i(x) = 1$, $\lambda_1(x) \geq c > 0$ $(\forall x \in I)$, $\max_{x\in I}\lambda_i(x) = c_i$. We improve the methods given by the authors in [11, 12], and the conditions of [11, 12] are weakened by constructing a new structure operator.

2. Preliminaries

Let $C^0(I,\mathrm{R}) = \{f : I \to \mathrm{R},\ f \text{ is continuous}\}$, clearly $(C^0(I,\mathrm{R}), \|\cdot\|_{c^0})$ is a Banach space, where $\|f\|_{c^0} = \max_{x\in I}|f(x)|$, for f in $C^0(I,\mathrm{R})$.

Let $C^1(I,\mathrm{R}) = \{f : I \to \mathrm{R}, f \text{ is continuous and continuously differentiable}\}$, then $C^1(I,\mathrm{R})$ is a Banach space with the norm $\|\cdot\|_{c^1}$, where $\|f\|_{c^1} = \|f\|_{c^0} + \|f'\|_{c^0}$, for f in $C^1(I,\mathrm{R})$.

Being a closed subset, $C^1(I,I)$ defined by

$$C^1(I,I) = \left\{ f \in C^1(I,R),\ f(I) \subseteq I,\ \forall x \in I \right\} \tag{2.1}$$

is a complete space.

The following lemmas are useful, and the methods of proof are similar to those of paper [4], but the conditions are weaker than those of [4].

Lemma 2.1. *Suppose that $\varphi \in C^1(I,I)$ and*

$$\left|\varphi'(x)\right| \leq M, \quad \forall x \in I, \tag{2.2}$$

$$\left|\varphi'(x_1) - \varphi'(x_2)\right| \leq M'|x_1 - x_2|, \quad \forall x_1, x_2 \in I, \tag{2.3}$$

where M and M' are positive constants.Then

$$\left|(\varphi^n(x_1))' - (\varphi^n(x_2))'\right| \leq M'\left(\sum_{i=n-1}^{2n-2} M^i\right)|x_1 - x_2|, \tag{2.4}$$

for any $x_1,\ x_2$ in I, where $(\varphi^n)'$ denotes $d\varphi^n/dx$.

Lemma 2.2. *Suppose that $\varphi_1, \varphi_2 \in C^1(I,I)$ satisfy* (2.2)*.Then*

$$\|\varphi_1^n - \varphi_2^n\|_{c^0} \leq \left(\sum_{i=1}^{n} M^{i-1}\right)\|\varphi_1 - \varphi_2\|_{c^0}. \tag{2.5}$$

Lemma 2.3. *Suppose that $\varphi_1, \varphi_2 \in C^1(I,I)$ satisfy* (2.2) *and* (2.3)*.Then*

$$\begin{aligned}\left\|\left(\varphi_1^{k+1}\right)' - \left(\varphi_2^{k+1}\right)'\right\|_{c^0} &\leq (k+1)M^k\|\varphi_1' - \varphi_2'\|_{c^0} \\ &\quad + Q(k+1)M'\left(\sum_{i=1}^{k}(k-i+1)M^{k+i-1}\right)\|\varphi_1 - \varphi_2\|_{c^0},\end{aligned} \tag{2.6}$$

for $k = 0,1,2,\ldots$, where $Q(s) = 0$ as $s = 1$ and $Q(s) = 1$ as $s = 2,3,\ldots$.

3. Main Results

For given constants $M_1 > 0$ and $M_2 > 0$, let

$$\mathcal{A}(M_1, M_2) = \Big\{\varphi \in C^1(I, I) : \left|\varphi'(x)\right| \le M_1,\ \forall x \in I,\\ \left|\varphi'(x_1) - \varphi'(x_2)\right| \le M_2|x_1 - x_2|,\ \forall x_1, x_2 \in I\Big\}. \tag{3.1}$$

Theorem 3.1 (existence). *Given positive constants* $M_1,\ M_2$ *and* $F \in \mathcal{A}(M_1, M_2)$*, if there exists constants* $N_1 \ge 1$ *and* $N_2 > 0$*, such that*

(P$_1$) $c - \sum_{i=2}^{\infty} c_i N_1^{i-1} \ge M_1/N_1$,

(P$_2$) $c - \sum_{i=2}^{\infty} c_i (\sum_{j=i-1}^{2i-2} N_1^j) \ge M_2/N_2$,

then (1.2) *has a solution* f *in* $\mathcal{A}(N_1, N_2)$.

Proof. For convenience, let $d = \max\{|a|, |b|\}$.

Define $K : \mathcal{A}(N_1, N_2) \to C^1(I, I)$ such that $K : f \to K_f$, where

$$K_f(t) = \sum_{i=1}^{\infty} \lambda_i(x) f^i(t), \quad \forall x, t \in I. \tag{3.2}$$

Since $f \in \mathcal{A}(N_1, N_2)$, it is easy to see that $|f^i(t)| \le d$ for all $t \in I$, and $|\lambda_i(x) f^i(t)| \le d|\lambda_i(x)|$ for all $x, t \in I$. It follows from $\sum_{i=1}^{\infty} \lambda_i(x) = 1$ that $\sum_{i=1}^{\infty} \lambda_i(x) f^i(t)$ is uniformly convergent. Then $K_f(t)$ is continuous for $t \in I$. Also we have

$$a = \sum_{i=1}^{\infty} \lambda_i(x) a \le \sum_{i=1}^{\infty} \lambda_i(x) f^i(t) \le \sum_{i=1}^{\infty} \lambda_i(x) b = b, \tag{3.3}$$

thus $K_f \in C^0(I, I)$.

For any $f \in \mathcal{A}(N_1, N_2)$, we have

$$\left|\frac{d}{dt}\Big(\lambda_i(x)\Big(f^i(t)\Big)\Big)\right| = \lambda_i(x)\left|f'\Big(f^{i-1}(t)\Big)\Big(f^{i-1}(t)\Big)'\right| \le c_i N_1^i. \tag{3.4}$$

By condition (P$_1$), we see that $\sum_{i=1}^{\infty} c_i N_1^i$ is convergent, therefore $\sum_{i=1}^{\infty} c_i (f^i(t))'$ is uniformly convergent for $t \in I$, this implies that $K_f(t)$ is continuously differentiable for $t \in I$. Moreover

$$\left|\frac{d}{dt} K_f(t)\right| \le \sum_{i=1}^{\infty} \lambda_i(x) \left|\Big(f^i(t)\Big)'\right| \le \sum_{i=1}^{\infty} c_i N_1^i := \mu_1. \tag{3.5}$$

By Lemma 2.1,

$$\begin{aligned}\left|\frac{d}{dt}(K_f(t_1))-\frac{d}{dt}(K_f(t_2))\right| &\le \sum_{i=1}^{\infty}\lambda_i(x)\left|\left(f^i(t_1)\right)'-\left(f^i(t_2)\right)'\right| \\ &\le \sum_{i=1}^{\infty}c_i\left(N_2\sum_{j=i-1}^{2i-2}N_1^j\right)|t_1-t_2| := \mu_2|t_1-t_2|.\end{aligned} \tag{3.6}$$

Thus $K_f \in \mathcal{A}(\mu_1,\mu_2)$.

Define $T:\mathcal{A}(N_1,N_2)\to C^1(I,I)$ as follows:

$$Tf(t)=\frac{1}{\lambda_1(x)}F(t)-\frac{1}{\lambda_1(x)}K_f(t)+f(t),\quad \forall t,x\in I, \tag{3.7}$$

where $f\in\mathcal{A}(N_1,N_2)$. Because K_f, F, and f are continuously differentiable for all $t\in I$, Tf is continuously differentiable for all $t\in I$. By conditions (P_1) and (P_2), for any t_1, t_2 in I, we have

$$\begin{aligned}\left|\frac{d}{dt}(Tf(t))\right| &\le \frac{1}{\lambda_1(x)}\left|F'(t)\right|+\frac{1}{\lambda_1(x)}\sum_{i=2}^{\infty}\lambda_i(x)\left|\left(f^i(t)\right)'\right| \le \frac{1}{c}M_1+\frac{1}{c}\sum_{i=2}^{\infty}c_iN_1^i \\ &\le \frac{1}{c}M_1+\frac{1}{c}(cN_1-M_1)=N_1.\end{aligned} \tag{3.8}$$

We furthermore have

$$\begin{aligned}\left|\frac{d}{dt}(Tf(t_1))-\frac{d}{dt}(Tf(t_2))\right| &\le \frac{1}{\lambda_1(x)}\left|F'(t_1)-F'(t_2)\right|+\frac{1}{\lambda_1(x)}\sum_{i=2}^{\infty}c_i\left|\left(f^i(t_1)\right)'-\left(f^i(t_2)\right)'\right| \\ &\le \frac{1}{c}M_2|t_1-t_2|+\frac{1}{c}\sum_{i=2}^{\infty}c_iN_2\left(\sum_{j=i-1}^{2i-2}N_1^j\right)|t_1-t_2| \\ &\le N_2|x_1-x_2|.\end{aligned} \tag{3.9}$$

Thus $T:\mathcal{A}(N_1,N_2)\to\mathcal{A}(N_1,N_2)$ is a self-diffeomorphism.

Now we prove the continuity of T under the norm $\|\cdot\|_{c^1}$. For arbitrary $f_1,f_2\in\mathcal{A}(N_1,N_2)$,

$$
\begin{aligned}
\|Tf_1 - Tf_2\|_{c^0} &= \max_{t\in I}\left|-\frac{1}{\lambda_1(x)}K_{f_1}(t) + f_1(t) + \frac{1}{\lambda_1(x)}K_{f_2}(t) - f_2(t)\right| \\
&\le \frac{1}{c}\max_{t\in I}\left|\sum_{i=2}^{\infty}\lambda_i(x)f_1^i(t) - \sum_{i=2}^{\infty}\lambda_i(x)f_2^i(t)\right| \\
&\le \frac{1}{c}\sum_{i=2}^{\infty}c_i\left\|f_1^i - f_2^i\right\|_{c^0} \\
&\le \frac{1}{c}\sum_{i=2}^{\infty}c_i\left(\sum_{k=1}^{i}N_1^{k-1}\right)\|f_1 - f_2\|_{c^0}, \\
\left\|\frac{d}{dt}(Tf_1) - \frac{d}{dt}(Tf_2)\right\|_{c^0} &= \max_{t\in I}\left|-\frac{1}{\lambda_1(x)}\left(K_{f_1}(t)\right)' + \left(f_1(t)\right)' + \frac{1}{\lambda_1(x)}\left(K_{f_2}(t)\right)' - \left(f_2(t)\right)'\right| \\
&\le \frac{1}{c}\max_{t\in I}\left|\sum_{i=2}^{\infty}\lambda_i(x)\left(f_1^i(t)\right)' - \sum_{i=2}^{\infty}\lambda_i(x)\left(f_2^i(t)\right)'\right| \\
&\le \frac{1}{c}\sum_{i=2}^{\infty}c_i\left\|\left(f_1^i\right)' - \left(f_2^i\right)'\right\|_{c^0} \\
&\le \frac{1}{c}\sum_{i=2}^{\infty}c_i\left[iN_1^{i-1}\|f_1' - f_2'\|_{c^0} + Q(i)N_2\left(\sum_{k=1}^{i-1}(i-k)N_1^{i+k-2}\right)\|f_1 - f_2\|_{c^0}\right].
\end{aligned}
\tag{3.10}
$$

Let

$$
\begin{aligned}
E_1 &= \frac{1}{c}\sum_{i=2}^{\infty}c_i\left(\sum_{k=1}^{i}N_1^{k-1} + Q(i)N_2\sum_{k=1}^{i-1}(i-k)N_1^{i+k-2}\right), \\
E_2 &= \frac{1}{c}\sum_{i=2}^{\infty}c_i iN_1^{i-1}, \qquad E = \max\{E_1, E_2\}.
\end{aligned}
\tag{3.11}
$$

Then we have

$$
\begin{aligned}
\|Tf_1 - Tf_2\|_{c^1} &= \|Tf_1 - Tf_2\|_{c^0} + \left\|(Tf_1)' - (Tf_2)'\right\|_{c^0} \le E_1\|f_1 - f_2\|_{c^0} + E_2\|f_1' - f_2'\|_{c^0} \\
&\le E\|f_1 - f_2\|_{c^0} + E\|f_1' - f_2'\|_{c^0} = E\|f_1 - f_2\|_{c^1},
\end{aligned}
\tag{3.12}
$$

which gives continuity of T.

It is easy to show that $\mathcal{A}(N_1, N_2)$ is a compact convex subset of $C^1(I, I)$. By Schauder's fixed point theorem, we assert that there is a mapping $f \in \mathcal{A}(N_1, N_2)$ such that

$$
f(t) = Tf(t) = \frac{1}{\lambda_1(x)}F(t) - \frac{1}{\lambda_1(x)}K_f(t) + f(t), \quad \forall t \in I. \tag{3.13}
$$

Let $t = x$, we have $f(x)$ as a solution of (1.2) in $\mathcal{A}(N_1, N_2)$. This completes the proof. □

Theorem 3.2 (Uniqueness). *Suppose that* (P_1) *and* (P_2) *are satisfied, also one supposes that*

(P_3) $E < 1$,

then for arbitrary function F *in* $\mathcal{A}(M_1, M_2)$, (1.2) *has a unique solution* $f \in \mathcal{A}(N_1, N_2)$.

Proof. The existence of (1.2) in $\mathcal{A}(N_1, N_2)$ is given by Theorem 3.1, from the proof of Theorem 3.1, we see that $\mathcal{A}(N_1, N_2)$ is a closed subset of $C^1(I, I)$, by (3.12) and (P_3), we see that $T : \mathcal{A}(N_1, N_2) \to \mathcal{A}(N_1, N_2)$ is a contraction. Therefore T has a unique fixed point $f(x)$ in $\mathcal{A}(N_1, N_2)$, that is, (1.2) has a unique solution in $\mathcal{A}(N_1, N_2)$, this proves the theorem. □

4. Example

Consider the equation

$$\sum_{i=1}^{\infty} \lambda_i(x) f^i(x) = \frac{1}{4}x^2, \quad x \in I := [-1, 1], \tag{4.1}$$

where $\lambda_1(x) = 33/36 + (1/36)\cos^2(\pi x/2)$, $\lambda_2(x) = 1/36 + (1/36)\sin^2(\pi x/2)$, $\lambda_3(x) = 1/36$, $\lambda_4(x) = \lambda_5(x) = \cdots = 0$. It is easy to see that $0 \le \lambda_i(x) \le 1$, $\sum_{i=1}^{\infty}\lambda_i(x) = 1$, $c = 33/36$, $c_2 = 2/36$, $c_3 = 1/36$, $c_4 = c_5 = \cdots = 0$.

For any x, y in $[-1, 1]$,

$$|F'(x)| = |0.5x| \le 0.5, \qquad |F'(x) - F'(y)| \le |0.5x| + |0.5y| \le 1, \tag{4.2}$$

thus $F \in \mathcal{A}(0.5, 1)$. By condition ($P_1$), we can choose $N_1 = 1.1$, and by condition (P_1), we can choose $N_2 = 1.5$. Then by Theorem 3.1, there is a continuously differentiable solution of (4.1) in $\mathcal{A}(1.1, 1.5)$.

Remark 4.1. Here $F(x)$ is not monotone for $x \in [-1, 1]$, hence it cannot be concluded by [11, 12].

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