Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2008, Article ID 672301, 13 pages
doi:10.1155/2008/672301

Research Article

Weak Convergence Theorems of Three Iterative Methods for Strictly Pseudocontractive Mappings of Browder-Petryshyn Type

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Received 25 September 2007; Revised 29 January 2008; Accepted 28 February 2008

Recommended by Huang Nan-Jing

Let E be a real q-uniformly smooth Banach space which is also uniformly convex (e.g., L_p or l_p spaces $(1 < p < \infty)$), and K a nonempty closed convex subset of E. By constructing nonexpansive mappings, we elicit the weak convergence of Mann's algorithm for a κ-strictly pseudocontractive mapping of Browder-Petryshyn type on K in condition thet the control sequence $\{\alpha_n\}$ is chosen so that (i) $\mu \leq \alpha_n < 1, n \geq 0$; (ii) $\sum_{n=0}^{\infty}(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] = \infty$, where $\mu \in [\max\{0, 1 - (q\kappa/C_q)^{1/(q-1)}\}, 1)$. Moreover, we consider to find a common fixed point of a finite family of strictly pseudocontractive mappings and consider the parallel and cyclic algorithms for solving this problem. We will prove the weak convergence of these algorithms.

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1. Introduction

Let E be a real Banach space and let J_q $(q > 1)$ denote the generalized duality mapping from E into 2^{E^*} given by $J_q(x) = \{f \in E^* : \langle x, f\rangle = \|x\|^q \text{ and } \|f\| = \|x\|^{q-1}\}$, where E^* denotes the dual space of E and $\langle\cdot,\cdot\rangle$ denotes the generalized duality pairing. In particular, J_2 is called the normalized duality mapping and it is usually denoted by J. If E^* is strictly convex then J_q is single-valued. In the sequel, we will denote the single-valued generalized duality mapping by j_q and $F(T) = \{x \in E : Tx = x\}$.

Definition 1.1. A mapping T with domain $D(T)$ and range $R(T)$ in E is called *strictly pseudocontractive* of Browder-Petryshyn type [1], if for all $x, y \in D(T)$, there exists $\kappa \in [0, 1)$ and $j_q(x-y) \in J_q(x-y)$ such that

$$\langle Tx - Ty, j_q(x-y)\rangle \leq \|x-y\|^q - \kappa\|x - y - (Tx - Ty)\|^q. \tag{1.1}$$

(If (1.1) holds, we also say that T is κ-strictly pseudocontractive.)

Remark 1.2. If I denotes the identity operator, then (1.1) can be written in the form

$$\langle (I-T)x-(I-T)y, j_q(x-y)\rangle \geq \kappa \|(I-T)x-(I-T)y\|^q. \tag{1.2}$$

In Hilbert spaces, (1.1) (and hence (1.2)) is equivalent to the inequality

$$\|Tx-Ty\|^2 \leq \|x-y\|^2 + k\|x-y-(Tx-Ty)\|^2, \quad k=(1-2\kappa)<1, \tag{1.3}$$

and we can assume also that $k \geq 0$, so that $k \in [0,1)$. Note that the class of strictly pseudocontractive mappings strictly includes the class of nonexpansive mappings which are mappings T on $D(T)$ such that $\|Tx-Ty\| \leq \|x-y\|$ for all $x, y \in D(T)$. That is, T is nonexpansive if and only if T is 0-strictly pseudocontractive.

The class of strictly pseudocontractive mappings has been studied by several authors (see, e.g., [1–7]). However their iterative methods are far less developed though Browder and Petryshyn [1] initiated their work in 1967. As a matter of fact, strictly pseudocontractive mappings have more powerful applications in solving inverse problems (see Scherzer [8]). Therefore it is interesting to develop the theory of iterative methods for strictly pseudocontractive mappings.

Browder and Petryshyn proved the following theorem.

Theorem BP (see [1]). *Let H be a real Hilbert space and K a nonempty closed convex and bounded subset of H. Let $T: K \to K$ be a κ-strictly pseudocontractive map. Then for any fixed $\gamma \in (1-\kappa, 1)$, the sequence $\{x_n\}$ generated from an arbitrary $x_1 \in K$ by*

$$x_{n+1} = \gamma x_n + (1-\gamma)Tx_n, \quad n \geq 1 \tag{1.4}$$

converges weakly to a fixed point of T.

Recently Marino and Xu [9] have extended Browder and Petryshyn's above-mentioned result by proving that the sequence $\{x_n\}$ generated by the following Mann's algorithm [10]:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)Tx_n, \quad n \geq 0. \tag{1.5}$$

Theorem MX (see [9]). *Let K be a closed convex subset of a Hilbert space H. Let $T: K \to K$ be a κ-strictly pseudocontractive mapping for some $0 \leq \kappa < 1$ and $F(T) \neq \emptyset$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm* (1.5). *Assume that the control sequence $\{\alpha_n\}_{n=0}^{\infty}$ is chosen so that $\kappa < \alpha_n < 1$ for all n and*

$$\sum_{n=0}^{\infty} (\alpha_n - \kappa)(1-\alpha_n) = \infty. \tag{1.6}$$

Then $\{x_n\}$ converges weakly to a fixed point of T.

Meanwhile, Marino and Xu raised the open question: whether Theorem MX can be extended to Banach spaces which are uniformly convex and have a Frechet differentiable norm. As a partial affirmative answer, Osilike and Udomene [2] proved the following theorem.

Theorem OU. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and let $T : K \to K$ be a κ-strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the conditions:*

(i*) $0 \leq \alpha_n \leq 1,\ n \geq 0$;

(ii*) $0 < a \leq \alpha_n \leq b < (q\kappa/C_q)^{1/(q-1)},\ n \geq 0$ *and for some constants* $a, b \in (0,1)$.

Then, the sequence $\{x_n\}$ is generated by the Mann's algorithm:

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \tag{1.7}$$

converges weakly to a fixed point of T.

We would like to point out that Osilike's and Udomene's condition (*ii**) excludes the natural choice $1 - 1/n$ for α_n. This is overcome by our paper. We prove that if α_n satisfies the conditions

$$\begin{gathered} \mu \leq \alpha_n < 1; \\ \sum_{n=0}^{\infty} (1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right] = \infty; \end{gathered} \tag{1.8}$$

where $\mu \in [\max\{0, 1-(q\kappa/C_q)^{1/(q-1)}\}, 1)$, then the iterative sequence (1.5) converges weakly to a fixed point of T.

Moreover, we are concerned with the problem of finding a point x such that

$$x \in \bigcap_{i=1}^{N} F(T_i), \tag{1.9}$$

where $N \geq 1$ is a positive integer and $\{T_i\}_{i=1}^N$ are N strictly pseudocontractive mappings defined on a closed convex subset K of a real Banach space E which is q-uniformly smooth and uniformly convex. Assume that $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i = 1$. We will show that the sequence $\{x_n\}$ generated by the following parallel algorithm:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i T_i x_n, \quad n \geq 0 \tag{1.10}$$

will converge weakly to a solution to the problem (1.9).

We will consider a more general situation by allowing the weights $\{\lambda_i\}_{i=1}^N$ in (1.10) to depend on n, the number of steps of the iteration. That is we consider the algorithm which generates a sequence $\{x_n\}$ in the following way:

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n, \quad n \geq 0. \tag{1.11}$$

Under appropriate assumptions on the sequences of the weights $\{\lambda_i^{(n)}\}_{i=1}^N$ we will also prove the weak convergence, to a solution of the problem (1.9), of the algorithm (1.11).

Another approach to the problem (1.9) is the cyclic algorithm [11]. (For convenience, we relabel the mappings $\{T_i\}_{i=1}^N$ as $\{T_i\}_{i=0}^{N-1}$.) This means that beginning with an $x_0 \in K$, we define the sequence $\{x_n\}$ cyclically by

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) T_{[n]} x_n, \quad n \geq 0, \tag{1.12}$$

where $T_{[n]} = T_i$, with $i = n(\text{mod} N)$, $0 \leq i \leq N-1$. We will show that this cyclic algorithm (1.12) is also weakly convergent if the sequence $\{\alpha_n\}$ of parameters is appropriately chosen.

We will use the notations:

(1) $\rightharpoonup$ for weak convergence;

(2) $\omega_w(x_n) = \{x : \exists x_{n_j} \rightharpoonup x\}$ denotes the weak ω-limit set of $\{x_n\}$.

2. Preliminaries

Let E be a real Banach space. The *modulus of smoothness* of E is the function $\rho_E : [0,\infty) \to [0,\infty)$ defined by

$$\rho_E(\tau) = \sup\left\{\frac{1}{2}(\|x+y\| + \|x-y\|) - 1 : \|x\| \leq 1, \ \|y\| \leq \tau\right\}. \tag{2.1}$$

E is *uniformly smooth* if and only if $\lim_{\tau\to 0}(\rho_E(\tau)/\tau) = 0$.

Let $q > 1$. E is said to be *q-uniformly smooth* (or to have a modulus of smoothness of power type $q > 1$) if there exists a constant $c > 0$ such that $\rho_E(\tau) \leq c\tau^q$. Hilbert spaces, L_p (or l_p) spaces $(1 < p < \infty)$, and the Sobolev spaces, W_m^p $(1 < p < \infty)$ are q-uniformly smooth. Hilbert spaces are 2 uniformly smooth, while

$$L_p \ (\text{or } l_p) \text{ or } W_m^p \text{ is } \begin{cases} p\text{-uniformly smooth if } 1 < p \leq 2, \\ 2\text{-uniformly smooth if } p \geq 2. \end{cases} \tag{2.2}$$

Theorem HKX (see [12, page 1130]). *Let $q > 1$ and let E be a real q-uniformly smooth Banach space. Then there exists a constant $C_q > 0$ such that for all $x, y \in E$,*

$$\|x+y\|^q \leq \|x\|^q + q\langle y, j_q(x)\rangle + C_q\|y\|^q . \tag{2.3}$$

E is said to have a Frechet differentiable norm if for all $x \in U = \{x \in E : \|x\| = 1\}$

$$\lim_{t\to 0}\frac{\|x+ty\| - \|x\|}{t} \tag{2.4}$$

exists and is attained uniformly in $y \in U$. In this case there exists an increasing function $b : [0,\infty) \to [0,\infty)$ with $\lim_{t\to 0} b(t) = 0$ such that for all $x, h \in E$,

$$\frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle \leq \frac{1}{2}\|x+h\|^2 \leq \frac{1}{2}\|x\|^2 + \langle h, j(x)\rangle + b(\|h\|). \tag{2.5}$$

It is well known (see, e.g., [13, page 107]) that q-uniformly smooth Banach space has a Frechet differentiable norm.

Lemma 2.1 (see [2]). *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let K be a nonempty closed convex subset of E and $T : K \to K$ a strictly pseudocontractive mapping of Browder-Petryshyn type. Then $(I - T)$ is demiclosed at zero, that is, $\{x_n\} \subset D(T)$ such that $\{x_n\}$ converges weakly to $x \in D(T)$ and $\{(I-T)x_n\}$ converges strongly to 0, then $Tx = x$.*

Lemma 2.2 (see [14, 15]). *Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$, $\{\delta_n\}_{n=1}^{\infty}$ be nonnegative sequences satisfying the following inequality*

$$a_{n+1} \le (1+\delta_n)a_n + b_n, \quad \forall n \ge 1. \tag{2.6}$$

If $\sum_{n=1}^{\infty}\delta_n < \infty$ and $\sum_{n=1}^{\infty} b_n < \infty$, then $\lim_{n\to\infty} a_n$ exists.

Lemma 2.3. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let T be a self-mapping on K with $F(T) \neq \varnothing$. Let $\{x_n\}_{n=0}^{\infty}$ be the sequence satisfying the following conditions:*

(a) $\lim_{n\to\infty}\|x_n - p\|$ *exists for every* $p \in F(T)$;

(b) $\lim_{n\to\infty}\|x_n - Tx_n\| = 0$;

(c) $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ *exists for all* $t \in [0,1]$ *and for all* $p_1, p_2 \in F(T)$.

Then, the sequence $\{x_n\}$ converges weakly to a fixed point of T.

Proof. Since $\lim_{n\to\infty}\|x_n - p\|$ exists, then $\{x_n\}$ is bounded. By (b) and Lemma 2.1, we have $\omega_{\mathcal{W}}(x_n) \subset F(T)$. Assume that $p_1, p_2 \in \omega_{\mathcal{W}}(x_n)$ and that $\{x_{n_i}\}$ and $\{x_{m_j}\}$ be subsequences of $\{x_n\}$ such that $x_{n_i} \rightharpoonup p_1$ and $x_{m_j} \rightharpoonup p_2$, respectively. Since E is a real q-uniformly smooth Banach space which is also uniformly convex, then E has a Frechet differentiable norm. Set $x = p_1 - p_2$, $h = t(x_n - p_1)$ in (2.5), we obtain

$$\begin{aligned} &\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle \\ &\quad \le \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \le \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(t\|x_n - p_1\|), \end{aligned} \tag{2.7}$$

where b is increasing. Since $\|x_n - p_1\| \le M$, for all $n \ge 0$, for some $M > 0$, then

$$\begin{aligned} &\frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle \\ &\quad \le \frac{1}{2}\|tx_n + (1-t)p_1 - p_2\|^2 \le \frac{1}{2}\|p_1 - p_2\|^2 + t\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM). \end{aligned} \tag{2.8}$$

Therefore,

$$\begin{aligned} &\frac{1}{2}\|p_1 - p_2\|^2 + t\limsup_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle \\ &\quad \le \frac{1}{2}\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|^2 \le \frac{1}{2}\|p_1 - p_2\|^2 + t\liminf_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM). \end{aligned} \tag{2.9}$$

Hence $\limsup_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle \le \liminf_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle + b(tM)/t$. Since $\lim_{t\to 0^+} b(tM)/t = 0$, then $\lim_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle$ exists. Since $\lim_{n\to\infty}\langle x_n - p_1, j(p_1 - p_2)\rangle = \langle p - p_1, j(p_1 - p_2)\rangle$, for all $p \in \omega_{\mathcal{W}}(x_n)$. Set $p = p_2$. We have $\langle p_2 - p_1, j(p_1 - p_2)\rangle = \|p_2 - p_1\|^2 = 0$, that is, $p_2 = p_1$. Hence $\omega_{\mathcal{W}}(x_n)$ is singleton, so that $\{x_n\}$ converges weakly to a fixed point of T. □

3. Mann's algorithm

Theorem 3.1. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let $T : K \to K$ be a κ-strictly pseudocontractive mapping with $F(T) \neq \emptyset$. Let $\{\alpha_n\}$ be a real sequence satisfying the condition* (1.8). *Given $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by Mann's algorithm* (1.5). *Then the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let $\beta_n = (\alpha_n - \mu)/(1-\mu)$. Since $\alpha_n \in (\mu, 1)$, then $\beta_n \in (0,1)$. We compute

$$\begin{aligned} x_{n+1} &= \alpha_n x_n + (1-\alpha_n)Tx_n = [\mu + (1-\mu)\beta_n]x_n + (1-\mu)(1-\beta_n)Tx_n \\ &= \beta_n x_n + (1-\beta_n)[\mu x_n + (1-\mu)Tx_n] = \beta_n x_n + (1-\beta_n)S_\mu x_n, \end{aligned} \tag{3.1}$$

where $S_\mu = \mu I + (1-\mu)T$. We will show that S_μ is a nonexpansive mapping and that $F(S_\mu) = F(T)$. Indeed, it follows from (1.2) and (2.3) that

$$\begin{aligned} \|S_\mu x - S_\mu y\|^q &= \|\mu x + (1-\mu)Tx - [\mu y + (1-\mu)Ty]\|^q = \|x - y - (1-\mu)[x - y - (Tx - Ty)]\|^q \\ &\le \|x-y\|^q - q(1-\mu)\langle (I-T)x - (I-T)y, j_q(x-y)\rangle + C_q(1-\mu)^q\|x-y-(Tx-Ty)\|^q \\ &\le \|x-y\|^q - q\kappa(1-\mu)\|x-y-(Tx-Ty)\|^q + C_q(1-\mu)^q\|x-y-(Tx-Ty)\|^q \\ &= \|x-y\|^q - (1-\mu)[q\kappa - C_q(1-\mu)^{q-1}]\|x-y-(Tx-Ty)\|^q. \end{aligned} \tag{3.2}$$

When $1-(q\kappa/C_q)^{1/(q-1)} \le \mu < 1$, we have $\|S_\mu x - S_\mu y\|^q \le \|x-y\|^q$, that is, S_μ is nonexpansive. On the other hand, for all $x \in F(S_\mu)$, $x = S_\mu x = \mu x + (1-\mu)Tx$. Then $x = Tx$, that is, $x \in F(T)$.

Now we show that $\|x_n - S_\mu x_n\|$ is decreasing. By (3.1), we have

$$\begin{aligned} \|x_{n+1} - S_\mu x_{n+1}\| &= \|\beta_n x_n + (1-\beta_n)S_\mu x_n - S_\mu x_{n+1}\| \\ &= \|\beta_n(x_n - S_\mu x_n) + \beta_n(S_\mu x_n - S_\mu x_{n+1}) + (1-\beta_n)(S_\mu x_n - S_\mu x_{n+1})\| \\ &\le \beta_n\|x_n - S_\mu x_n\| + \|S_\mu x_n - S_\mu x_{n+1}\| \le \beta_n\|x_n - S_\mu x_n\| + \|x_n - x_{n+1}\| \\ &= \beta_n\|x_n - S_\mu x_n\| + (1-\beta_n)\|x_n - S_\mu x_n\| = \|x_n - S_\mu x_n\|, \\ \|x_n - Tx_n\| &= \frac{1}{1-\alpha_n}\|x_{n+1} - x_n\| = \frac{1-\beta_n}{1-\alpha_n}\|x_n - S_\mu x_n\| = \frac{1}{1-\mu}\|x_n - S_\mu x_n\|. \end{aligned} \tag{3.3}$$

It follows from (3.3) that

$$\|x_n - Tx_n\| = \frac{1}{1-\mu}\|x_n - S_\mu x_n\| \le \frac{1}{1-\mu}\|x_{n-1} - S_\mu x_{n-1}\| = \|x_{n-1} - Tx_{n-1}\|. \tag{3.4}$$

Hence $\lim_{n\to\infty}\|x_n - Tx_n\|$ exists.

Pick a $p \in F(T)$. We then show that the real sequence $\{\|x_n - p\|\}_{n=0}^{\infty}$ is decreasing, hence $\lim_{n\to\infty}\|x_n - p\|$ exists. To see this, using (1.2) and (2.3), we obtain

$$\begin{aligned} \|x_{n+1} - p\|^q &= \|x_n - p - (1-\alpha_n)[x_n - p - (Tx_n - p)]\|^q \\ &\le \|x_n - p\|^q - q(1-\alpha_n)\langle x_n - p - (Tx_n - p), j_q(x_n - p)\rangle + C_q(1-\alpha_n)^q\|x_n - p - (Tx_n - p)\|^q \\ &\le \|x_n - p\|^q - q\kappa(1-\alpha_n)\|x_n - p - (Tx_n - p)\|^q + C_q(1-\alpha_n)^q\|x_n - p - (Tx_n - p)\|^q \\ &= \|x_n - p\|^q - (1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}]\|x_n - p - (Tx_n - p)\|^q. \end{aligned} \tag{3.5}$$

Then

$$(1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right]\left\|x_n - p - (Tx_n - p)\right\|^q \le \left\|x_n - p\right\|^q - \left\|x_{n+1} - p\right\|^q. \tag{3.6}$$

Since $\mu \le \alpha_n < 1$ for all n, where $\mu \in [\max\{0, 1-(q\kappa/C_q)^{1/(q-1)}\}, 1)$, we get $(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] \ge 0$. Therefore, (3.6) implies the sequence $\{\|x_n - p\|\}$ is decreasing (and hence $\lim_{n\to\infty}\|x_n - p\|$ exists). It follows from (3.6) that

$$\sum_{n=0}^{\infty}(1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right]\left\|x_n - p - (Tx_n - p)\right\|^q < \left\|x_0 - p\right\|^q < \infty. \tag{3.7}$$

Since $\sum_{n=0}^{\infty}(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] = \infty$, then (3.7) implies that

$$\liminf_{n\to\infty}\left\|x_n - Tx_n\right\| = 0. \tag{3.8}$$

Thus

$$\lim_{n\to\infty}\left\|x_n - Tx_n\right\| = 0. \tag{3.9}$$

Then we prove that for all $p_1, p_2 \in F(T)$, $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$. Let $a_n(t) = \|tx_n + (1-t)p_1 - p_2\|$. It is obvious that $\lim_{n\to\infty} a_n(0) = \|p_1 - p_2\|$ and $\lim_{n\to\infty} a_n(1) = \lim_{n\to\infty}\|x_n - p_2\|$ exist. So we only need to consider the case of $t \in (0,1)$. Define $T_n : K \to K$ by

$$T_n x = \alpha_n x + (1-\alpha_n)Tx, \quad x \in K. \tag{3.10}$$

Then for all $x, y \in K$,

$$\begin{aligned}\left\|T_n x - T_n y\right\|^q &\le \|x-y\|^q - q(1-\alpha_n)\langle (I-T)x - (I-T)y, j_q(x-y)\rangle + C_q(1-\alpha_n)^q\left\|x-y-(Tx-Ty)\right\|^q \\ &\le \|x-y\|^q - (1-\alpha_n)\left[q\kappa - C_q(1-\alpha_n)^{q-1}\right]\left\|x-y-(Tx-Ty)\right\|^q.\end{aligned} \tag{3.11}$$

By the choice of α_n, we have $(1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}] \ge 0$, so it follows that $\|T_n x - T_n y\| \le \|x - y\|$. Set $S_{n,m} = T_{n+m-1}T_{n+m-2}\cdots T_n$, $m \ge 1$. We have

$$\begin{gathered}\left\|S_{n,m}x - S_{n,m}y\right\| \le \|x-y\| \quad \forall x, y \in K, \\ S_{n,m}x_n = x_{n+m}, \qquad S_{n,m}p = p \quad \forall p \in F(T).\end{gathered} \tag{3.12}$$

Set $b_{n,m} = \|S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\|$. Let δ denote the *modulus of convexity* of E. If $\|x_n - p_1\| = 0$ for some n_0, then $x_n = p_1$ for any $n \ge n_0$ so that $\lim_{n\to\infty}\|x_n - p_1\| = 0$, in fact $\{x_n\}$ converges strongly to $p_1 \in F(T)$. Thus we may assume $\|x_n - p_1\| > 0$ for any $n \ge 0$. It is well known (see, e.g., [16, page 108]) that

$$\left\|tx + (1-t)y\right\| \le 1 - 2\min\{t, (1-t)\}\delta(\|x-y\|) \le 1 - 2t(1-t)\delta(\|x-y\|) \tag{3.13}$$

for all $t \in [0,1]$ and for all $x, y \in E$ such that $\|x\| \leq 1$, $\|y\| \leq 1$. Set

$$w_{n,m} = \frac{S_{n,m}p_1 - S_{n,m}(tx_n + (1-t)p_1)}{t\|x_n - p_1\|},$$
$$z_{n,m} = \frac{S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}x_n}{(1-t)\|x_n - p_1\|}. \tag{3.14}$$

Then $\|w_{n,m}\| \leq 1$ and $\|z_{n,m}\| \leq 1$ so that it follows from (3.13) that

$$2t(1-t)\delta(\|w_{n,m} - z_{n,m}\|) \leq 1 - \|tw_{n,m} + (1-t)z_{n,m}\|. \tag{3.15}$$

Observe that

$$\|w_{n,m} - z_{n,m}\| = \frac{b_{n,m}}{t(1-t)\|x_n - p_1\|},$$
$$\|tw_{n,m} + (1-t)z_{n,m}\| = \frac{\|S_{n,m}x_n - S_{n,m}p_1\|}{\|x_n - p_1\|}, \tag{3.16}$$

it follows from (3.15) that

$$2t(1-t)\|x_n - p_1\|\delta\left(\frac{b_{n,m}}{t(1-t)\|x_n - p_1\|}\right) \leq \|x_n - p_1\| - \|S_{n,m}x_n - S_{n,m}p_1\| = \|x_n - p_1\| - \|x_{n+m} - p_1\|. \tag{3.17}$$

Since E is uniformly convex, then $\delta(s)/s$ is nondecreasing, and since $\|x_n - p\|$ is decreasing, hence it follows from (3.17) that

$$\frac{\|x_0 - p_1\|}{2}\delta\left(\frac{4}{\|x_0 - p_1\|}b_{n,m}\right) \leq \|x_n - p_1\| - \|x_{n+m} - p_1\| \quad \left(\text{since } t(1-t) \leq \frac{1}{4}\ \forall t \in [0,1]\right). \tag{3.18}$$

Since $\delta(0) = 0$ and $\lim_{n\to\infty}\|x_n - p\|$ exists, then the continuity of yields $\lim_{n\to\infty} b_{n,m} = 0$ uniformly for all m. Observe that

$$\begin{aligned}
a_{n+m}(t) &\leq \|tx_{n+m} + (1-t)p_1 - p_2 + (S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1)\| \\
&\quad + \|S_{n,m}(tx_n + (1-t)p_1) - tS_{n,m}x_n - (1-t)S_{n,m}p_1\| \\
&= \|S_{n,m}(tx_n + (1-t)p_1) - S_{n,m}p_2\| + b_{n,m} \leq \|tx_n + (1-t)p_1 - p_2\| + b_{n,m} = a_n(t) + b_{n,m}.
\end{aligned} \tag{3.19}$$

Hence $\limsup_{n\to\infty} a_n(t) \leq \liminf_{n\to\infty} a_n(t)$, this ensures that $\lim_{n\to\infty} a_n(t)$ exists for all $t \in (0,1)$.

Now apply Lemma 2.3 to conclude that $\{x_n\}$ converges weakly to a fixed point of T. □

Remark 3.2. In particular, set $q = 2$, $C_q = 1$, our result reduces to Theorem MX. Moreover, if T is nonexpansive, then $\kappa = 0$ and our Theorem 3.1. reduces to Reich's theorem [17].

4. Parallel algorithm

The following proposition lists some useful properties for strictly pseudocontractive mappings.

Proposition 4.1. *Let K be a closed convex subset of a Banach space E. Given an integer $N \geq 1$, assume, for each $1 \leq i \leq N$, $T_i : K \to K$ is a κ_i-strictly pseudocontractive mapping for some $0 \leq \kappa_i < 1$. Assume $\{\lambda_i\}_{i=1}^N$ is a positive sequence such that $\sum_{i=1}^N \lambda_i = 1$. Then.*

(i) *$\sum_{i=1}^N \lambda_i T_i$ is a κ-strictly pseudocontractive mapping, with $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$.*

(ii) *Suppose that $\{T_i\}_{i=1}^N$ has a common fixed point. Then*

$$F\left(\sum_{i=1}^N \lambda_i T_i\right) = \bigcap_{i=1}^N F(T_i). \tag{4.1}$$

Proof. To prove (i), we only need to consider the case of $N = 2$ (the general case can be proved by induction). Set $G = (1-\lambda)T_1 + \lambda T_2$, where $\lambda \in (0,1)$ and for $i = 1,2$, T_i is a κ_i-strictly pseudocontractive mapping. Set $\kappa = \min\{\kappa_1, \kappa_2\}$;

$$\begin{aligned}
&\langle Gx - Gy, j_q(x-y)\rangle \\
&\quad\leq (1-\lambda)\langle T_1x - T_1y, j_q(x-y)\rangle + \lambda\langle T_2x - T_2y, j_q(x-y)\rangle \\
&\quad\leq (1-\lambda)\left[\|x-y\|^q - \kappa_1\|(I-T_1)x-(I-T_1)y\|^q\right] + \lambda\left[\|x-y\|^q - \kappa_2\|(I-T_2)x-(I-T_2)y\|^q\right] \\
&\quad\leq \|x-y\|^q - \kappa\left[(1-\lambda)\|(I-T_1)x-(I-T_1)y\|^q + \lambda\|(I-T_2)x-(I-T_2)y\|^q\right] \\
&\quad\leq \|x-y\|^q - \kappa\|(I-G)x-(I-G)y\|^q.
\end{aligned} \tag{4.2}$$

Hence G is a κ-strictly pseudocontractive mapping.

To prove (ii), again we can assume $N = 2$. It suffices to prove that $F(G) \subset F(T_1) \cap F(T_2)$, where $G = (1-\lambda)T_1 + \lambda T_2$, with $\lambda \in (0,1)$. Let $x \in F(G)$ and take $z \in F(T_1) \cap F(T_2)$ to deduce that

$$\begin{aligned}
\|x-z\|^q &= (1-\lambda)\langle T_1x - z, j_q(x-z)\rangle + \lambda\langle T_2x - z, j_q(x-z)\rangle \\
&\leq (1-\lambda)\left[\|x-z\|^q - \kappa\|(I-T_1)x-(I-T_1)z\|^q\right] + \lambda\left[\|x-z\|^q - \kappa\|(I-T_2)x-(I-T_2)z\|^q\right] \\
&= \|x-z\|^q - \kappa\left[(1-\lambda)\|(I-T_1)x\|^q + \lambda\|(I-T_2)x\|^q\right].
\end{aligned} \tag{4.3}$$

Since $\kappa > 0$, we get $(1-\lambda)\|(I-T_1)x\|^q + \lambda\|(I-T_2)x\|^q \leq 0$. This together with $0 < \lambda < 1$ implies that $T_1x = x$ and $T_2x = x$. Thus $x \in F(T_1) \cap F(T_2)$. □

Theorem 4.2. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : K \to K$ be a κ_i-strictly pseudocontractive mapping for some $0 \leq \kappa_i < 1$. Let $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^N F(T_i)$ is nonempty. Assume also $\{\lambda_i\}_{i=1}^N$ is a finite sequence of positive numbers such that $\sum_{i=1}^N \lambda_i = 1$. Given $x_0 \in K$, let $\{x_n\}_{n=0}^\infty$ be the sequence generated by Mann's algorithm* (1.10)*:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)\sum_{i=1}^N \lambda_i T_i x_n, \quad n \geq 0. \tag{4.4}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the conditions (1.8)*. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{N}$.*

Proof. Put

$$A = \sum_{i=1}^{N} \lambda_i T_i. \tag{4.5}$$

Then by Proposition 4.1, A is a κ-strictly pseudocontractive mapping and $F(A) = \bigcap_{i=1}^{N} F(T_i)$.

We can rewrite the algorithm (1.10) as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) A x_n, \quad n \geq 0. \tag{4.6}$$

Now apply Theorem 3.1 to conclude that sequence $\{x_n\}$ converges weakly to a fixed point of A. □

Theorem 4.3. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let $N \geq 1$ be an integer. Let, for each $1 \leq i \leq N$, $T_i : K \to K$ be a κ_i-strictly pseudocontractive mapping for some $0 \leq \kappa_i < 1$. Let $\kappa = \min\{\kappa_i : 1 \leq i \leq N\}$. Assume the common fixed point set $\bigcap_{i=1}^{N} F(T_i)$ is nonempty. Assume also for each n, $\{\lambda_i^{(n)}\}_{i=1}^{N}$ is a finite sequence of positive numbers such that $\sum_{i=1}^{N} \lambda_i^{(n)} = 1$ for all n and $\inf_{n\geq 1} \lambda_i^{(n)} > 0$ for all $1 \leq i \leq N$. Given $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the algorithm* (1.11)*:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) \sum_{i=1}^{N} \lambda_i^{(n)} T_i x_n, \quad n \geq 0. \tag{4.7}$$

Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the condition (1.8)*. Assume also that*

$$\sum_{n=0}^{\infty} \left(\sum_{i=1}^{N} \left| \lambda_i^{(n+1)} - \lambda_i^{(n)} \right| \right) < \infty. \tag{4.8}$$

Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{N}$.

Proof. Write, for each $n \geq 1$,

$$A_n = \sum_{i=1}^{N} \lambda_i^{(n)} T_i. \tag{4.9}$$

By Proposition 4.1, each A_n is a κ-strictly pseudocontractive mapping with $F(A_n) = \bigcap_{i=1}^{N} F(T_i)$, and the algorithm (1.11) can be rewritten as

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n) A_n x_n, \quad n \geq 0. \tag{4.10}$$

As Theorem 3.1, if set $\beta_n = (\alpha_n - \mu)/(1-\mu)$, then $\{x_n\}_{n=0}^{\infty}$ can also be generated by the following algorithm:

$$x_{n+1} = \beta_n x_n + (1-\beta_n) S_{\mu,n} x_n, \tag{4.11}$$

where $S_{\mu,n} = \mu I + (1-\mu) A_n$ and $S_{\mu,n}$ is a nonexpansive mapping with $F(S_{\mu,n}) = F(A_n)$. Similarly, we can prove that $\lim_{n\to\infty} \|x_n - p\|$ exists for every $p \in \bigcap_{i=1}^{N} F(T_i)$, and that

$$\liminf_{n\to\infty} \left\| x_n - A_n x_n \right\| = 0. \tag{4.12}$$

Since we can write $A_{n+1}x_{n+1} = A_n x_{n+1} + y_n$, where $y_n = \sum_{i=1}^{N}(\lambda_i^{(n+1)} - \lambda_i^{(n)})T_i x_{n+1}$, then by (4.11) we obtain

$$\begin{aligned}
\|x_{n+1} - S_{\mu,n+1}x_{n+1}\| &= \|\beta_n x_n + (1-\beta_n)S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1}\| \\
&= \|\beta_n(x_n - S_{\mu,n}x_n) + \beta_n(S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1}) + (1-\beta_n)(S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1})\| \\
&\le \beta_n\|x_n - S_{\mu,n}x_n\| + \|S_{\mu,n}x_n - S_{\mu,n+1}x_{n+1}\| \\
&\le \beta_n\|x_n - S_{\mu,n}x_n\| + \|S_{\mu,n}x_n - S_{\mu,n}x_{n+1}\| + \|S_{\mu,n}x_{n+1} - S_{\mu,n+1}x_{n+1}\| \\
&\le \beta_n\|x_n - S_{\mu,n}x_n\| + \|x_n - x_{n+1}\| + (1-\mu)\|A_n x_{n+1} - A_{n+1}x_{n+1}\| \\
&\le \beta_n\|x_n - S_{\mu,n}x_n\| + (1-\beta_n)\|x_n - S_{\mu,n}x_n\| + (1-\mu)\|y_n\| \\
&= \|x_n - S_{\mu,n}x_n\| + (1-\mu)\|y_n\|.
\end{aligned} \tag{4.13}$$

Assumption (4.8) implies that

$$\sum_{n=0}^{\infty}\|y_n\| < \infty. \tag{4.14}$$

Using Lemma 2.2, we conclude that $\lim_{n\to\infty}\|x_n - S_{\mu,n}x_n\|$ exists. Then $\lim_{n\to\infty}\|x_n - A_n x_n\|$ exists. Thus, by (4.12) we have $\lim_{n\to\infty}\|x_n - A_n x_n\| = 0$.

If we define $T_n : K \to K$ by

$$T_n x = \alpha_n x + (1-\alpha_n)A_n x, \quad x \in K. \tag{4.15}$$

According to the corresponding deductive process of Theorem 3.1, we can prove that $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$ and for all $p_1, p_2 \in F(A_n)$.

Consequently, $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=1}^{N}$ by Lemma 2.3. □

5. Cyclic algorithm

Theorem 5.1. *Let E be a real q-uniformly smooth Banach space which is also uniformly convex and let K be a nonempty closed convex subset of E. Let $N \ge 1$ be an integer. Let, for each $0 \le i \le N-1$, $T_i : K \to K$ be a κ_i-strictly pseudocontractive mapping for some $0 \le \kappa_i < 1$. Let $\kappa = \min\{\kappa_i : 0 \le i \le N-1\}$. Assume the common fixed point set $\bigcap_{i=0}^{N-1}F(T_i)$ is nonempty. Given $x_0 \in K$, let $\{x_n\}_{n=0}^{\infty}$ be the sequence generated by the cyclic algorithm* (1.12)*:*

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T_{[n]}x_n, \quad n \ge 0, \tag{5.1}$$

where $T_{[n]} = T_i$, with $i = n(\bmod N)$, $0 \le i \le N-1$. Let $\{\alpha_n\}_{n=0}^{\infty}$ be a real sequence satisfying the condition

$$\mu \le \alpha_n < 1 - \varepsilon \tag{5.2}$$

for all n and some $\varepsilon \in (0, 1-\mu)$, where $\mu \in [\max\{0, 1-(q\kappa/C_q)^{1/(q-1)}\}, 1)$. Then $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=0}^{N-1}$.

Proof. Pick a $p \in F = \bigcap_{i=0}^{N-1} F(T_i)$. We first show that the real sequence $\{\|x_n - p\|\}_{n=0}^{\infty}$ is decreasing, hence $\lim_{n\to\infty}\|x_n - p\|$ exists. To see this, using (1.2) and (2.3), we obtain

$$\begin{aligned}\|x_{n+1}-p\|^q &= \|x_n - p - (1-\alpha_n)[x_n - p - (T_{[n]}x_n - p)]\|^q \\ &\le \|x_n-p\|^q - q(1-\alpha_n)\langle x_n-p-(T_{[n]}x_n-p), j_q(x_n-p)\rangle + C_q(1-\alpha_n)^q\|x_n-p-(T_{[n]}x_n-p)\|^q \\ &\le \|x_n-p\|^q - q\kappa(1-\alpha_n)\|x_n-p-(T_{[n]}x_n-p)\|^q C_q(1-\alpha_n)^q\|x_n-p-(T_{[n]}x_n-p)\|^q \\ &= \|x_n-p\|^q - (1-\alpha_n)[q\kappa - C_q(1-\alpha_n)^{q-1}]\|x_n - p - (T_{[n]}x_n - p)\|^q.\end{aligned} \tag{5.3}$$

Since $\mu \le \alpha_n < 1-\varepsilon$, we get by (5.3)

$$\varepsilon[q\kappa - C_q(1-\mu)^{q-1}]\|x_n - p - (T_{[n]}x_n - p)\|^q \le \|x_n - p\|^q - \|x_{n+1} - p\|^q. \tag{5.4}$$

It follows that the sequence $\{\|x_n - p\|\}$ is decreasing (and hence $\lim_{n\to\infty}\|x_n - p\|$ exists) and that $\lim_{n\to\infty}\|x_n - T_{[n]}x_n\| = 0$. This implies that

$$\lim_{n\to\infty}\|x_{n+1} - x_n\| = \lim_{n\to\infty}(1-\alpha_n)\|x_n - T_{[n]}x_n\| = 0. \tag{5.5}$$

Claim: $\omega_{\mathcal{W}}(x_n) \subset F$.

Indeed, assume $x^* \in \omega_{\mathcal{W}}(x_n)$ and $x_{n_i} \rightharpoonup x^*$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. We may further assume $n_i = l(\mathrm{mod} N)$ for all i. Since by (5.5), we also have $x_{n_{i+j}} \rightharpoonup x^*$ for all $j \ge 0$, we deduce that

$$\|x_{n_{i+j}} - T_{[l+j]}x_{n_{i+j}}\| = \|x_{n_{i+j}} - T_{[n_i+j]}x_{n_{i+j}}\| \longrightarrow 0. \tag{5.6}$$

Then Lemma 2.1 implies that $x^* \in F(T_{[l+j]})$ for all j. This ensures that $x^* \in F$.

If we define $T_n : K \to K$ by

$$T_n x = \alpha_n x + (1-\alpha_n)T_{[n]}x, \quad x \in K. \tag{5.7}$$

According to the corresponding deductive process of Theorem 3.1, we can prove that $\lim_{n\to\infty}\|tx_n + (1-t)p_1 - p_2\|$ exists for all $t \in [0,1]$ and for all $p_1, p_2 \in F$.

Consequently, we conclude that $\{x_n\}$ converges weakly to a common fixed point of $\{T_i\}_{i=0}^{N-1}$ by using Lemma 2.3. This completes the proof. □

Acknowledgement

The author would like to thank Professor Haiyun Zhou for his useful suggestions, and the referees for their valuable comments. This paper is sponsored by the North China Electric Power University Youth Foundation (no. 200611004).

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