

Research Article

T-Stability of Picard Iteration in Metric Spaces

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We establish a general result for the stability of Picard's iteration. Several theorems in the literature are obtained as special cases.

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Let (X, d) be a complete metric space and T a self-map of X . Let $x_{n+1} = f(T, x_n)$ be some iteration procedure. Suppose that $F(T)$, the fixed point set of T , is nonempty and that x_n converges to a point $q \in F(T)$. Let $\{y_n\} \subset X$ and define $\epsilon_n = d(y_{n+1}, f(T, y_n))$. If $\lim \epsilon_n = 0$ implies that $\lim y_n = q$, then the iteration procedure $x_{n+1} = f(T, x_n)$ is said to be T -stable. Without loss of generality, we may assume that $\{y_n\}$ is bounded, for if $\{y_n\}$ is not bounded, then it cannot possibly converge. If these conditions hold for $x_{n+1} = Tx_n$, that is, Picard's iteration, then we will say that Picard's iteration is T -stable.

We will obtain sufficient conditions that Picard's iteration is T -stable for an arbitrary self-map, and then demonstrate that a number of contractive conditions are Picard T -stable.

We will need the following lemma from [1].

Lemma 1. *Let $\{x_n\}$, $\{\epsilon_n\}$ be nonnegative sequences satisfying $x_{n+1} \leq hx_n + \epsilon_n$, for all $n \in \mathbb{N}$, $0 \leq h < 1$, $\lim \epsilon_n = 0$. Then, $\lim x_n = 0$.*

Theorem 1. *Let (X, d) be a nonempty complete metric space and T a self-map of X with $F(T) \neq \emptyset$. If there exist numbers $L \geq 0$, $0 \leq h < 1$, such that*

$$d(Tx, q) \leq Ld(x, Tx) + hd(x, q) \tag{1}$$

for each $x \in X$, $q \in F(T)$, and, in addition,

$$\lim d(y_n, Ty_n) = 0, \quad (2)$$

then Picard's iteration is T -stable.

Proof. First, we show that the fixed point q of T is unique. Suppose p is another fixed point of T , then

$$d(p, q) = d(Tp, q) \leq Ld(p, Tp) + hd(p, q) = hd(p, q). \quad (3)$$

Since $0 \leq h < 1$, so $d(p, q) = 0$, that is, $p = q$.

Let $\{y_n\} \subset X$, $\epsilon_n = d(y_{n+1}, Ty_n)$, and $\lim \epsilon_n = 0$. We need to show that $\lim y_n = q$.

Using (1), (2), and Lemma 1,

$$d(y_{n+1}, q) \leq d(y_{n+1}, Ty_n) + d(Ty_n, q) \leq \epsilon_n + Ld(y_n, Ty_n) + hd(y_n, q), \quad (4)$$

and $\lim y_n = q$. \square

Corollary 1. Let (X, d) be a nonempty complete metric space and T a self-map of X satisfying the following: there exists $0 \leq h < 1$, such that, for each $x, y \in X$,

$$d(Tx, Ty) \leq h \max \{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \quad (5)$$

Then, Picard's iteration is T -stable.

Proof. From [2, Theorem 11], T has a unique fixed point q . Also, T satisfies (1). It remains to show that (2) is satisfied.

Define p_n to be the diameter of the orbit of y_n ; that is, $p_n = \delta(O(y_n, Ty_n, \dots))$. First, we show that p_n is bounded:

$$\begin{aligned} d(Ty_n, q) &\leq h \max \{d(y_n, q), d(y_n, Ty_n), d(y_n, Tq), d(q, Ty_n), d(q, Tq)\} \\ &\leq h \max \{d(y_n, q), d(y_n, Ty_n), d(y_n, q), d(q, Ty_n), 0\} \\ &= h \max \{d(y_n, q), d(y_n, Ty_n), d(y_n, q), d(q, Ty_n)\}. \end{aligned} \quad (6)$$

Hence, $d(Ty_n, q) \leq hd(y_n, q)$ or $d(Ty_n, q) \leq hd(y_n, Ty_n)$ or $d(Ty_n, q) \leq hd(q, Ty_n)$.

If $d(Ty_n, q) \leq hd(y_n, q)$, it is clear that

$$d(Ty_n, q) \leq hd(y_n, q) \leq \frac{h}{1-h} d(y_n, q). \quad (7)$$

If $d(Ty_n, q) \leq hd(q, Ty_n)$, then

$$d(Ty_n, q) = 0 \leq \frac{h}{1-h} d(y_n, q). \quad (8)$$

If $d(Ty_n, q) \leq hd(y_n, Ty_n)$, then

$$d(y_n, Ty_n) \leq d(Ty_n, q) + d(y_n, q) \leq hd(y_n, Ty_n) + d(y_n, q). \quad (9)$$

Hence, $d(Ty_n, q) \leq (h/(1-h))d(y_n, q)$. Now it is easy to see that $\{Ty_n\}$ is bounded and so is $\{p_n\}$, since $\{y_n\}$ is bounded.

For any $i, j \geq n$, using (5),

$$d(Ty_i, Ty_j) \leq h \max \{d(y_i, y_j), d(y_i, Ty_i), d(y_j, Ty_j), d(y_i, Ty_j), d(y_j, Ty_i)\} \leq hp_n. \quad (10)$$

Thus,

$$d(y_i, Ty_j) \leq d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_j) \leq \epsilon_{i-1} + hp_{n-1}. \quad (11)$$

But

$$d(y_i, y_j) \leq d(y_i, Ty_{i-1}) + d(Ty_{i-1}, Ty_{j-1}) + d(Ty_{j-1}, y_j) \leq \epsilon_{i-1} + hp_{n-1} + \epsilon_{i-1}, \quad (12)$$

which implies that

$$p_n \leq 2\epsilon_{i-1} + hp_{n-1}, \quad (13)$$

and $\lim p_n = 0$ by Lemma 1. Since $d(y_n, Ty_n) \leq p_n$, $\lim d(y_n, Ty_n) = 0$.

The conclusion now follows from Theorem 1. \square

Corollary 2 (see [3, Theorem 1]). *Let (X, d) be a nonempty complete metric space and T a self-map of X satisfying*

$$d(Tx, Ty) \leq Ld(x, Tx) + ad(x, y) \quad (14)$$

for all $x, y \in X$, where $L \geq 0$, $0 \leq a < 1$. Suppose that T has a fixed point p . Then, T is Picard T -stable.

Proof. Since T satisfies (14) for all $x, y \in X$, then T satisfies inequality (1) of our paper. Let $\{y_n\} \subset X$ and define $\epsilon_n = d(y_{n+1}, y_n)$. From the proof of Theorem 1 of [3], $\lim d(y_n, Ty_n) = 0$. Therefore, by our theorem (Theorem 1), T is Picard T -stable. \square

Definition (5) of this paper is actually Definition (24) of [2]. Therefore, many contractive conditions are special cases of (5), and, for each of these, Picard's iteration is T -stable. For example, Theorems 1 and 2 of [4] and Theorem 1 of [5] are special cases of Corollary 1.

We will not examine the analogues of Theorem 1 for Mann, Ishikawa, Kirk, or any other iteration scheme since, if one obtains convergence to a fixed point for a map using Picard's iteration, there is no point in considering any other more complicated iteration procedure.

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