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*Research Article*

# Strong Convergence to Common Fixed Points of Countable Relatively Quasi-Nonexpansive Mappings

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We prove that a sequence generated by the monotone CQ-method converges strongly to a common fixed point of a countable family of relatively quasi-nonexpansive mappings in a uniformly convex and uniformly smooth Banach space. Our result is applicable to a wide class of mappings.



## 1. Introduction

Let $E$ be a real Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $T : C \to E$ be a mapping. Recall that $T$ is nonexpansive if

$$\|Tx - Ty\| \le \|x - y\| \quad \forall x, y \in C. \tag{1.1}$$

We denote by $\mathrm{F}(T)$ the set of fixed points of $T$, that is, $\mathrm{F}(T) = \{x \in C : x = Tx\}$. A mapping $T$ is said to be quasi-nonexpansive if $\mathrm{F}(T) \neq \emptyset$ and

$$\|Tx - y\| \le \|x - y\| \quad \forall x \in C,\ y \in \mathrm{F}(T). \tag{1.2}$$

It is easy to see that if $T$ is nonexpansive with $\mathrm{F}(T) \neq \emptyset$, then it is quasi-nonexpansive. There are many methods for approximating fixed points of a quasi-nonexpansive mapping. In 1953, Mann [1] introduced the iteration as follows: a sequence $\{x_n\}$ is defined by

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n) T x_n, \tag{1.3}$$

where the initial guess element $x_0 \in C$ is arbitrary and $\{\alpha_n\}$ is a real sequence in $[0,1]$. Mann iteration has been extensively investigated for nonexpansive mappings. One of the fundamental convergence results was proved by Reich [2]. In an infinite-dimensional Hilbert space, Mann iteration can yield only weak convergence (see [3, 4]). Attempts to modify the Mann iteration method (1.3) so that strong convergence is guaranteed have recently been made. Nakajo and Takahashi [5] proposed the following modification of Mann iteration method (1.3) for a nonexpansive mapping $T$ from $C$ into itself in a Hilbert space:

$$\begin{aligned}
&x_0 \in C \text{ is arbitrary,}\\
&y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\
&C_n = \{z \in C : \|y_n - z\| \le \|x_n - z\|\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0,1,2,\ldots,
\end{aligned} \tag{1.4}$$

where $P_K$ denotes the metric projection from a Hilbert space $H$ onto a closed convex subset $K$ of $H$ and prove that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$. A projection onto intersection of two halfspaces is computed by solving a linear system of two equations with two unknowns (see [6, Section 3]).

Recently, Su and Qin [7] modified iteration (1.4), so-called the monotone CQ method for nonexpansive mapping, as follows:

$$\begin{aligned}
&x_0 \in C \text{ is arbitrary,}\\
&y_n = \alpha_n x_n + (1-\alpha_n)Tx_n,\\
&C_0 = \{z \in C : \|y_0 - z\| \le \|x_0 - z\|\},\\
&Q_0 = C,\\
&C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \le \|x_n - z\|\},\\
&Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n\rangle \ge 0\},\\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0,1,2,\ldots,
\end{aligned} \tag{1.5}$$

and prove that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x_0$.

We now recall some definitions concerning relatively quasi-nonexpansive mappings and what have been proved until now. Let $E$ be a real smooth Banach space with norm $\|\cdot\|$ and let $E^*$ be the dual of $E$. Denote by $\langle\cdot,\cdot\rangle$ the pairing between $E$ and $E^*$. The normalized duality mapping $J$ from $E$ to $E^*$ is defined by

$$Jx = \{f \in E^* : \langle x, f\rangle = \|x\|^2 = \|f\|^2\}, \quad \text{where } x \in E. \tag{1.6}$$

The reader is directed to [8] (and its review [9]), where the properties on the duality mapping and several related topics are presented. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x,y) = \|x\|^2 - 2\langle x, Jy\rangle + \|y\|^2 \quad \forall x, y \in E. \tag{1.7}$$

Let $T$ be a mapping from $C$ into $E$. A point $p$ in $C$ is said to be an *asymptotic fixed point* of $T$ [10] if $C$ contains a sequence $\{x_n\}$ which converges weakly to $p$ and $\lim_{n\to\infty}(x_n - Tx_n) = 0$. The set of asymptotic fixed points of $T$ is denoted by $\widehat{\mathrm{F}}(T)$. We say that the mapping $T$ is *relatively nonexpansive* if the following conditions are satisfied:

(R1) $\mathrm{F}(T) \neq \varnothing$;

(R2) $\phi(p, Tx) \leq \phi(p, x)$ for each $x \in C,\ p \in \mathrm{F}(T)$;

(R3) $\mathrm{F}(T) = \widehat{\mathrm{F}}(T)$.

If $T$ satisfies (R1) and (R2), then $T$ is called *relatively quasi-nonexpansive*.

Several articles have appeared providing method for approximating fixed points of relatively quasi-nonexpansive mappings [11–16]. Matsushita and Takahashi [12] introduced the following iteration: a sequence $\{x_n\}$ defined by

$$x_{n+1} = \prod_C J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T x_n\big), \tag{1.8}$$

where the initial guess element $x_0 \in C$ is arbitrary, $\{\alpha_n\}$ is a real sequence in $[0,1]$, $T$ is a relatively nonexpansive mapping, and $\Pi_C$ denotes the generalized projection from $E$ onto a closed convex subset $C$ of $E$. They prove that the sequence $\{x_n\}$ converges weakly to a fixed point of $T$. Moreover, Matsushita and Takahashi [13] proposed the following modification of iteration (1.8):

$$\begin{aligned}
&x_0 \in C \text{ is arbitrary},\\
&y_n = J^{-1}\big(\alpha_n J x_n + (1-\alpha_n) J T x_n\big),\\
&C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\},\\
&Q_n = \{z \in C : \langle x_n - z, x_0 - x_n\rangle \geq 0\},\\
&x_{n+1} = \prod_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{1.9}$$

and prove that the sequence $\{x_n\}$ converges strongly to $\Pi_{\mathrm{F}(T)} x_0$.

Recently, Kohsaka and Takahashi [11] extended iteration (1.8) to obtain a weak convergence theorem for common fixed points of a finite family of relatively nonexpansive mapping $\{T_i\}_{i=1}^m$ by the following iteration:

$$x_{n+1} = \prod_C J^{-1}\left(\sum_{i=1}^m w_{n,i}\big(\alpha_{n,i} J x_n + (1-\alpha_{n,i}) J T_i x_n\big)\right), \quad n = 1, 2, \ldots, \tag{1.10}$$

where $\alpha_{n,i} \subset [0,1]$ and $w_{n,i} \subset [0,1]$ with $\sum_{i=1}^m w_{n,i} = 1$ for all $n \in \mathbb{N}$.

Employing the ideas of Su and Qin [7], and of Aoyama et al. [17], we modify iterations (1.5), (1.8)–(1.10) to obtain strong convergence theorems for common fixed points of countable relatively quasi-nonexpansive mappings in a Banach space. Consequently, we obtain strong convergence theorems for quasi-nonexpansive mappings in a Hilbert space without using demiclosedness principle. Moreover, we introduce a new certain condition for an infinite family of mappings which is inspired by Aoyama et al. [17], and we also show how to generate a corresponding sequence of mappings satisfying our condition.

## 2. Preliminaries

Throughout the paper, let $E$ be a real Banach space. We say that $E$ is *strictly convex* if the following implication holds for $x, y \in E$:

$$\|x\| = \|y\| = 1, \quad x \neq y \text{ imply } \left\| \frac{x+y}{2} \right\| < 1. \tag{2.1}$$

It is also said to be *uniformly convex* if for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|x\| = \|y\| = 1, \quad \|x - y\| \geq \varepsilon \text{ imply } \left\| \frac{x+y}{2} \right\| \leq 1 - \delta. \tag{2.2}$$

It is known that if $E$ is uniformly convex Banach space, then $E$ is reflexive and strictly convex. A Banach space $E$ is said to be *smooth* if

$$\lim_{t \to 0} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for each $x, y \in S(E) := \{x \in E : \|x\| = 1\}$. In this case, the norm of $E$ is said to be *Gâteaux differentiable*. The space $E$ is said to have *uniformly Gâteaux differentiable norm* if for each $y \in S(E)$, the limit (2.3) is attained uniformly for $x \in S(E)$. The norm of $E$ is said to be *Fréchet differentiable* if for each $x \in S(E)$, the limit (2.3) is attained uniformly for $y \in S(E)$. The norm of $E$ is said to be *uniformly Fréchet differentiable* (and $E$ is said to be *uniformly smooth*) if the limit (2.3) is attained uniformly for $x, y \in S(E)$.

We also know the following properties (see, e.g., [18] for details).

(a) $E$ ($E^*$, resp.) is uniformly convex if and only if $E^*$ ($E$, resp.) is uniformly smooth.

(b) $J(x) \neq \emptyset$ for each $x \in E$.

(c) If $E$ is reflexive, then $J$ is a mapping of $E$ onto $E^*$.

(d) If $E$ is strictly convex, then $J(x) \cap J(y) = \emptyset$ for all $x \neq y$.

(e) If $E$ is smooth, then $J$ is single valued.

(f) If $E$ has a Fréchet differentiable norm, then $J$ is norm to norm continuous.

(g) If $E$ is uniformly smooth, then $J$ is uniformly norm to norm continuous on each bounded subset of $E$.

(h) If $E$ is a Hilbert space, then $J$ is the identity operator.

Let $E$ be a smooth Banach space. The function $\phi : E \times E \to \mathbb{R}$ is defined by

$$\phi(x, y) = \|x\|^2 - 2\langle x, Jy \rangle + \|y\|^2 \quad \forall x, y \in E. \tag{2.4}$$

It is obvious from the definition of the function $\phi$ that

$$(\|x\| - \|y\|)^2 \leq \phi(x, y) \leq (\|x\| + \|y\|)^2 \quad \forall x, y \in E. \tag{2.5}$$

Moreover, we know the following results.

**Lemma 2.1** (see [13, Remark 2.1]). *Let $E$ be a strictly convex and smooth Banach space, then $\phi(x,y)=0$ if and only if $x=y$.*

**Lemma 2.2** (see [11, Lemma 2.5]). *Let $E$ be a uniformly convex and smooth Banach space and let $r>0$. Then there exists a continuous, strictly increasing, and convex function $g:[0,2r]\to[0,\infty)$ such that $g(0)=0$ and*

$$g(\|x-y\|)\le\phi(x,y) \tag{2.6}$$

*for all $x,y\in B_r=\{z\in E:\|z\|\le r\}$.*

Let $C$ be a nonempty closed convex subset of $E$. Suppose that $E$ is reflexive, strictly convex, and smooth. It is known that [19] for any $x\in E$, there exists a unique point $x^*\in C$ such that

$$\phi(x^*,x)=\min_{y\in C}\phi(y,x). \tag{2.7}$$

Following Alber [20], we denote such an $x^*$ by $\Pi_C x$. The mapping $\Pi_C$ is called the *generalized projection* from $E$ onto $C$. It is easy to see that in a Hilbert space, the mapping $\Pi_C$ coincides with the metric projection $P_C$. Concerning the generalized projection, the following are well known.

**Lemma 2.3** (see [19, Proposition 4]). *Let $C$ be a nonempty closed convex subset of a smooth Banach space $E$ and let $x\in E$. Then*

$$x^*=\prod_C x\Longleftrightarrow\langle x^*-y,Jx-Jx^*\rangle\ge 0\quad\text{for each } y\in C. \tag{2.8}$$

**Lemma 2.4** (see [19, Proposition 5]). *Let $E$ be a reflexive, strictly convex, and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $x\in E$. Then*

$$\phi\left(y,\prod_C x\right)+\phi\left(\prod_C x,x\right)\le\phi(y,x)\quad\text{for each } y\in C. \tag{2.9}$$

Dealing with the generalized projection from $E$ onto the fixed point set of a relatively quasi-nonexpansive mapping, we get the following result.

**Lemma 2.5.** *Let $E$ be a strictly convex and smooth Banach space, let $C$ be a nonempty closed convex subset of $E$, and let $T$ be a relatively quasi-nonexpansive mapping from $C$ into $E$. Then $\mathrm{F}(T)$ is closed and convex.*

*Proof.* The proof of [13, Proposition 2.4] does not invoke condition (R3) at all. So the conclusion holds for relatively quasi-nonexpansive mappings as well. □

Let $C$ be a subset of a Banach space $E$ and let $\{T_n\}$ be a family of mappings from $C$ into $E$. For a subset $B$ of $C$, we say that

(i) $(\{T_n\},B)$ satisfies condition AKTT if

$$\sum_{n=1}^{\infty}\sup\left\{\left\|T_{n+1}z-T_nz\right\|:z\in B\right\}<\infty; \tag{2.10}$$

(ii) $(\{T_n\}, B)$ satisfies condition *AKTT if

$$\sum_{n=1}^{\infty} \sup \left\{ \left\| JT_{n+1}z - JT_n z \right\| : z \in B \right\} < \infty. \tag{2.11}$$

Aoyama et al. [17, Lemma 3.2] prove the following result which is very useful in our main result.

**Lemma 2.6.** *Let $C$ be a nonempty subset of a Banach space $E$ and let $\{T_n\}$ be a sequence of mappings from $C$ into $E$. Let $B$ be a subset of $C$ with $(\{T_n\}, B)$ satisfying condition AKTT, then there exists a mapping $\widetilde{T} : B \to E$ such that*

$$\widetilde{T}x = \lim_{n \to \infty} T_n x \quad \forall x \in B \tag{2.12}$$

*and* $\lim_{n \to \infty} \sup\{\|\widetilde{T}z - T_n z\| : z \in B\} = 0$.

Inspired by the preceding lemma, we have the following result.

**Lemma 2.7.** *Let $E$ be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let $C$ be a nonempty subset of $E$, and let $\{T_n\}$ be a sequence of mappings from $C$ into $E$. Let $B$ be a subset of $C$ with $(\{T_n\}, B)$ satisfying condition *AKTT, then there exists a mapping $\widehat{T} : B \to E$ such that*

$$\widehat{T}x = \lim_{n \to \infty} T_n x \quad \forall x \in B \tag{2.13}$$

*and* $\lim_{n \to \infty} \sup\{\|J\widehat{T}z - JT_n z\| : z \in B\} = 0$.

*Proof.* For $x \in B$, we show that $\{JT_n x\}$ is a Cauchy sequence in $E^*$. Let $\varepsilon > 0$. By the condition *AKTT of $(\{T_n\}, B)$, there exists $l_0 \in \mathbb{N}$ such that

$$\sum_{n=l_0}^{\infty} \sup\left\{ \left\| JT_{n+1}z - JT_n z \right\| : z \in B \right\} < \varepsilon. \tag{2.14}$$

In particular, if $k > l \geq l_0$, then

$$\begin{aligned} \left\| JT_k x - JT_l x \right\| &\leq \sum_{n=l}^{k-1} \sup \left\{ \left\| JT_{n+1}z - JT_n z \right\| : z \in B \right\} \\ &\leq \sum_{n=l_0}^{\infty} \sup\left\{ \left\| JT_{n+1}z - JT_n z \right\| : z \in B \right\} < \varepsilon. \end{aligned} \tag{2.15}$$

Hence, $\{JT_n x\}$ is a Cauchy sequence in $E^*$. It follows then that $\lim_{n \to \infty} JT_n x$ exists for all $x \in B$. Moreover, it is noted that the convergence is uniform on $B$. Since $E$ is reflexive and strictly convex, $J$ is bijective and we can define a mapping $\widehat{T}$ from $B$ into $E$ such that

$$\widehat{T}x = J^{-1}\left( \lim_{n \to \infty} JT_n x \right) \quad \forall x \in B. \tag{2.16}$$

Since $E$ has a Fréchet differentiable norm, $J$ is norm-to-norm continuous and hence

$$\widehat{T}x = J^{-1}J\left(\lim_{n\to\infty} T_n x\right) = \lim_{n\to\infty} T_n x \quad \forall x \in B. \tag{2.17}$$

This completes the proof. □

Combining Lemmas 2.6 and 2.7, we obtain a crucial tool for our main result.

**Lemma 2.8.** *Let $E$ be a reflexive and strictly convex Banach space whose norm is Fréchet differentiable, let $C$ be a nonempty subset of $E$, and let $\{T_n\}$ be a sequence of mappings from $C$ into $E$. Suppose that for each bounded subset $B$ of $C$, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Then there exists a mapping $T : C \to E$ such that*

$$Tx = \lim_{n\to\infty} T_n x \quad \forall x \in C. \tag{2.18}$$

*Proof.* To see that $T$ is well defined, we suppose that $(\{T_n\}, \{x\})$ satisfies condition AKTT and condition *AKTT. Then, by Lemmas 2.6 and 2.7, there exist $\widetilde{T}$ and $\widehat{T}$ such that $\widetilde{T}x = \lim_{n\to\infty} T_n x = \widehat{T}x$. □

**Lemma 2.9** (see [11, Lemma 3.2]). *Let $E$ be a reflexive, strictly convex, and smooth Banach space, let $z \in E$, and let $\{t_i\}_{i=1}^m \subset (0,1)$ with $\sum_{i=1}^m t_i = 1$. If $\{x_i\}_{i=1}^m$ is a finite sequence in $E$ such that*

$$\phi\left(z, J^{-1}\left(\sum_{i=1}^m t_i J x_i\right)\right) = \sum_{i=1}^m t_i \phi(z, x_i), \tag{2.19}$$

*then $x_1 = x_2 \cdots = x_m$.*

**Lemma 2.10.** *Let $E$ be a strictly convex Banach space and let $\{t_n\} \subset (0,1)$ with $\sum_{n=1}^\infty t_n = 1$. If $\{x_n\}$ is a sequence in $E$ such that $\sum_{n=1}^\infty t_n x_n$ and $\sum_{n=1}^\infty t_n \|x_n\|^2$ converge, and*

$$\left\|\sum_{n=1}^\infty t_n x_n\right\|^2 = \sum_{n=1}^\infty t_n \|x_n\|^2, \tag{2.20}$$

*then $\{x_n\}$ is a constant sequence.*

*Proof.* Suppose that $x_i \neq x_j$ for some $i, j \in \mathbb{N}$. Then, by the strict convexity of $E$,

$$\left\|\frac{t_i}{t_i+t_j}x_i + \frac{t_j}{t_i+t_j}x_j\right\|^2 < \frac{t_i}{t_i+t_j}\|x_i\|^2 + \frac{t_j}{t_i+t_j}\|x_j\|^2. \tag{2.21}$$

It follows that

$$\begin{aligned}
\left\|\sum_{n=1}^\infty t_n x_n\right\|^2 &= \left\|(t_i+t_j)\left(\frac{t_i}{t_i+t_j}x_i + \frac{t_j}{t_i+t_j}x_j\right) + \sum_{n\neq i,j} t_n x_n\right\|^2 \\
&\le (t_i+t_j)\left\|\frac{t_i}{t_i+t_j}x_i + \frac{t_j}{t_i+t_j}x_j\right\|^2 + \sum_{n\neq i,j} t_n\|x_n\|^2 \\
&< (t_i+t_j)\left(\frac{t_i}{t_i+t_j}\|x_i\|^2 + \frac{t_j}{t_i+t_j}\|x_j\|^2\right) + \sum_{n\neq i,j} t_n\|x_n\|^2 \\
&= \sum_{n=1}^\infty t_n\|x_n\|^2.
\end{aligned} \tag{2.22}$$

This is a contradiction. □

## 3. Main results

In this section, we establish strong convergence theorem for finding common fixed points of a countable family of relatively quasi-nonexpansive mappings in a Banach space.

This theorem generalizes a recent theorem by Su et al. [21, Theorem 3.1]. It is noted that relative quasi-nonexpansiveness considered in the paper and hemirelative nonexpansiveness of [21] are the same. We do prefer the former name because in a Hilbert space setting, relatively quasi-nonexpansive mappings are just quasi-nonexpansive.

Recall that an operator $T$ in a Banach space is *closed* if $x_n \to x$ and $Tx_n \to y$, then $Tx = y$.

**Theorem 3.1.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{T_n\}$ be a sequence of relatively quasi-nonexpansive mappings from $C$ into $E$ such that $\bigcap_{n=0}^{\infty} \mathrm{F}(T_n)$ is nonempty and let $\{x_n\}$ be a sequence in $C$ defined as follows:*

$$\begin{aligned} &x_0 \in C, \quad C_{-1} = Q_{-1} = C, \\ &y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JT_n x_n), \\ &C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \le \phi(z, x_n)\}, \\ &Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ &x_{n+1} = \prod_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots, \end{aligned} \tag{3.1}$$

*where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that for each bounded subset $B$ of $C$, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. Let $T$ be the mapping from $C$ into $E$ defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $T$ is closed and $\mathrm{F}(T) = \bigcap_{n=0}^{\infty} \mathrm{F}(T_n)$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{F}(T)} x_0$.*

*Proof.* We first note that each $C_n$ and $Q_n$ are closed and convex. This follows since $\phi(z, y_n) \le \phi(z, x_n)$ is equivalent to

$$2\langle z, Jx_n - Jy_n \rangle \le \|x_n\|^2 - \|y_n\|^2. \tag{3.2}$$

It is clear that $\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) \subset C = C_{-1} \cap Q_{-1}$. Next, we show that

$$\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) \subset C_n \cap Q_n \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.3}$$

Suppose that $\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) \subset C_{k-1} \cap Q_{k-1}$ for some $k \in \mathbb{N} \cup \{0\}$. Let $p \in \bigcap_{n=0}^{\infty} \mathrm{F}(T_n)$. Then

$$\begin{aligned} \phi(p, y_k) &= \phi(p, J^{-1}(\alpha_k Jx_k + (1-\alpha_k)JT_k x_k)) \\ &= \|p\|^2 - 2\langle p, \alpha_k Jx_k + (1-\alpha_k)JT_k x_k \rangle + \|\alpha_k Jx_k + (1-\alpha_k)JT_k x_k\|^2 \\ &\le \|p\|^2 - 2\alpha_k \langle p, Jx_k \rangle - 2(1-\alpha_k)\langle p, JT_k x_k \rangle + \alpha_k \|x_k\|^2 + (1-\alpha_k)\|T_k x_k\|^2 \\ &= \alpha_k (\|p\|^2 - 2\langle p, Jx_k \rangle + \|x_k\|^2) + (1-\alpha_k)\left(\|p\|^2 - 2\langle p, JT_k x_k \rangle + \|T_k x_k\|^2\right) \\ &= \alpha_k \phi(p, x_k) + (1-\alpha_k)\phi(p, T_k x_k) \\ &\le \alpha_k \phi(p, x_k) + (1-\alpha_k)\phi(p, x_k) \\ &= \phi(p, x_k). \end{aligned} \tag{3.4}$$

This implies that $\bigcap_{n=0}^{\infty} F(T_n) \subset C_k$. From $x_k = \Pi_{C_{k-1} \cap Q_{k-1}} x_0$ and by Lemma 2.3, we have

$$\langle x_k - z, Jx_0 - Jx_k \rangle \geq 0 \quad \text{for each } z \in C_{k-1} \cap Q_{k-1}. \tag{3.5}$$

In particular,

$$\langle x_k - p, Jx_0 - Jx_k \rangle \geq 0 \quad \text{for every } p \in \bigcap_{n=0}^{\infty} F(T_n) \tag{3.6}$$

and hence $\bigcap_{n=0}^{\infty} F(T_n) \subset Q_k$. It follows that

$$\bigcap_{n=0}^{\infty} F(T_n) \subset C_k \cap Q_k. \tag{3.7}$$

By induction, (3.3) holds. This implies that $\{x_n\}$ is well defined. It follows from the definition of $Q_n$ and Lemma 2.3 that $x_n = \Pi_{Q_n} x_0$. Since $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in Q_n$, we have

$$\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.8}$$

Therefore, $\phi(x_n, x_0)$ is nondecreasing. Using $x_n = \Pi_{Q_n} x_0$ and Lemma 2.4, we have

$$\phi(x_n, x_0) = \phi(\Pi_{Q_n} x_0, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0) \tag{3.9}$$

for all $p \in \bigcap_{n=0}^{\infty} F(T_n)$ for all $n \in \mathbb{N} \cup \{0\}$. Therefore, $\phi(x_n, x_0)$ is bounded. So

$$\lim_{n \to \infty} \phi(x_n, x_0) \text{ exists.} \tag{3.10}$$

In particular, by (2.5), the sequence $\{(\|x_n\| - \|x_0\|)^2\}$ is bounded. This implies that $\{x_n\}$ is bounded. Noticing again that $x_n = \Pi_{Q_n} x_0$, and for any positive integer $k$, we have $x_{n+k} \in Q_{n+k-1} \subset Q_n$. By Lemma 2.4,

$$\phi(x_{n+k}, x_n) = \phi\left( x_{n+k}, \prod_{Q_n} x_0 \right) \leq \phi(x_{n+k}, x_0) - \phi\left( \prod_{Q_n} x_0, x_0 \right) = \phi(x_{n+k}, x_0) - \phi(x_n, x_0). \tag{3.11}$$

Using Lemma 2.2, we have, for $m, n$ with $m > n$,

$$g(\|x_m - x_n\|) \leq \phi(x_m, x_n) \leq \phi(x_m, x_0) - \phi(x_n, x_0), \tag{3.12}$$

where $g : [0, \infty) \to [0, \infty)$ is a continuous, strictly increasing, and convex function with $g(0) = 0$. Then the properties of the function $g$ yield that $\{x_n\}$ is a Cauchy sequence in $C$, so there exists $w \in C$ such that $x_n \to w$. In view of $x_{n+1} = \Pi_{C_n \cap Q_n} x_0 \in C_n$ and the definition of $C_n$, we also have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.13}$$

It follows that

$$\lim_{n\to\infty}\phi(x_{n+1},y_n)=\lim_{n\to\infty}\phi(x_{n+1},x_n)=0. \tag{3.14}$$

By using Lemma 2.2, we obtain

$$\lim_{n\to\infty}\|x_{n+1}-y_n\|=\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{3.15}$$

Since $J$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|Jx_{n+1}-Jy_n\|=\lim_{n\to\infty}\|Jx_{n+1}-Jx_n\|=0. \tag{3.16}$$

On the other hand, we have, for each $n\in\mathbb{N}\cup\{0\}$,

$$\begin{aligned}\|Jx_{n+1}-Jy_n\|&=\|Jx_{n+1}-(\alpha_nJx_n+(1-\alpha_n)JT_nx_n)\|\\&=\|(1-\alpha_n)(Jx_{n+1}-JT_nx_n)-\alpha_n(Jx_n-Jx_{n+1})\|\\&\geq(1-\alpha_n)\|Jx_{n+1}-JT_nx_n\|-\alpha_n\|Jx_n-Jx_{n+1}\|,\end{aligned} \tag{3.17}$$

and hence

$$\|Jx_{n+1}-JT_nx_n\|\leq\frac{1}{1-\alpha_n}\|Jx_{n+1}-Jy_n\|+\frac{\alpha_n}{1-\alpha_n}\|Jx_n-Jx_{n+1}\|. \tag{3.18}$$

From (3.16) and $\limsup_{n\to\infty}\alpha_n<1$, we obtain

$$\lim_{n\to\infty}\|Jx_{n+1}-JT_nx_n\|=0. \tag{3.19}$$

Since $J^{-1}$ is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty}\|x_{n+1}-T_nx_n\|=\lim_{n\to\infty}\|J^{-1}(Jx_{n+1})-J^{-1}(JT_nx_n)\|=0. \tag{3.20}$$

It follows from (3.15) that

$$\|x_n-T_nx_n\|\leq\|x_n-x_{n+1}\|+\|x_{n+1}-T_nx_n\|\longrightarrow 0 \tag{3.21}$$

and so

$$\lim_{n\to\infty}\|Jx_n-JT_nx_n\|=0. \tag{3.22}$$

*Case 1.* $(\{T_n\},\{x_n\})$ satisfies condition AKTT. We apply Lemma 2.6 to get

$$\begin{aligned}\|x_n-Tx_n\|&\leq\|x_n-T_nx_n\|+\|T_nx_n-Tx_n\|\\&\leq\|x_n-T_nx_n\|+\sup\{\|T_nz-Tz\|:z\in\{x_n\}\}\longrightarrow 0.\end{aligned} \tag{3.23}$$

*Case 2.* $(\{T_n\}, \{x_n\})$ satisfies condition *AKTT. It follows from Lemma 2.7 that

$$\begin{aligned}\|Jx_n - JTx_n\| &\leq \|Jx_n - JT_nx_n\| + \|JT_nx_n - JTx_n\| \\ &\leq \|Jx_n - JT_nx_n\| + \sup\{\|JT_nz - JTz\| : z \in \{x_n\}\} \longrightarrow 0.\end{aligned} \tag{3.24}$$

Hence,

$$\lim_{n\to\infty} \|x_n - Tx_n\| = \lim_{n\to\infty} \|J^{-1}(Jx_n) - J^{-1}(JTx_n)\| = 0. \tag{3.25}$$

From both cases, we obtain

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0. \tag{3.26}$$

Since $T$ is closed and $x_n \to w$, we have $w \in \mathrm{F}(T)$. Furthermore, by (3.9),

$$\phi(w, x_0) = \lim_{n\to\infty} \phi(x_n, x_0) \leq \phi(p, x_0) \quad \forall p \in \mathrm{F}(T). \tag{3.27}$$

Hence, $w = \Pi_{\mathrm{F}(T)}x_0$. □

**Corollary 3.2** (see [21, Theorem 3.1]). *Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $T$ be a closed relatively quasi-nonexpansive mapping from $C$ into $E$ such that $\mathrm{F}(T)$ is nonempty and let $\{x_n\}$ be a sequence in $C$ defined as follows:*

$$\begin{aligned}&x_0 \in C, \quad C_{-1} = Q_{-1} = C, \\ &y_n = J^{-1}(\alpha_n Jx_n + (1-\alpha_n)JTx_n), \\ &C_n = \{z \in C_{n-1} \cap Q_{n-1} : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ &Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ &x_{n+1} = \prod_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,\end{aligned} \tag{3.28}$$

*where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty}\alpha_n < 1$. Then $\{x_n\}$ converges strongly to $\Pi_{\mathrm{F}(T)}x_0$.*

*Remark 3.3.* If, in Theorem 3.1, $T_n$ is continuous for each $n \in \mathbb{N}$, then the mapping $T$ is continuous and closed.

In our main theorem, we assume that for each bounded subset $B$ of $C$, the ordered pair $(\{T_n\}, B)$ satisfies either condition AKTT or condition *AKTT. As in [17], we can generate a sequence $\{T_n\}$ of relatively quasi-nonexpansive mappings satisfying such an assumption by using convex combination of a given sequence $\{S_k\}$ of relatively quasi-nonexpansive mappings with a nonempty common fixed point set.

Let $\{\beta_n^k\}$ be a family of positive real numbers with indices $n,k \in \mathbb{N} \cup \{0\}$ with $k \leq n$ such that

(i) $\sum_{k=0}^{n} \beta_n^k = 1$ for every $n \in \mathbb{N} \cup \{0\}$;

(ii) $\lim_{n\to\infty} \beta_n^k = \beta^k > 0$ for every $k \in \mathbb{N} \cup \{0\}$; and

(iii) $\sum_{n=0}^{\infty} \sum_{k=0}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. For a sequence $\{S_k\}_{k=1}^{\infty}$ of continuous relatively quasi-nonexpansive mappings with a common fixed point and $S_0$ is the identity mapping, we define a sequence $\{T_n\}$ of mappings from $C$ into $E$ by

$$T_n x = J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k x\right) \tag{3.29}$$

for $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. We note that

$$\bigcap_{k=0}^{\infty} \mathrm{F}(S_k) \subset \bigcap_{k=0}^{n} \mathrm{F}(S_k) \subset \mathrm{F}(T_n) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.30}$$

For $n \in \mathbb{N} \cup \{0\}$, let $p \in \bigcap_{k=0}^{n} \mathrm{F}(S_k)$. Then

$$\begin{aligned}
\phi(p, T_n x) &= \phi\left(p, J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k x\right)\right) \\
&= \|p\|^2 - 2\left\langle p, \sum_{k=0}^{n} \beta_n^k J S_k x\right\rangle + \left\|\sum_{k=0}^{n} \beta_n^k J S_k x\right\|^2 \\
&\leq \|p\|^2 - 2\sum_{k=0}^{n} \beta_n^k \langle p, J S_k x\rangle + \sum_{k=0}^{n} \beta_n^k \|S_k x\|^2 \\
&= \sum_{k=0}^{n} \beta_n^k \phi(p, S_k x) \\
&\leq \phi(p, x)
\end{aligned} \tag{3.31}$$

for all $x \in C$. Then, for all $z \in \mathrm{F}(T_n)$ and fix $q \in \bigcap_{k=0}^{\infty} \mathrm{F}(S_k)$,

$$\phi(q, z) = \phi(q, T_n z) = \phi\left(q, J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k z\right)\right) \leq \sum_{k=0}^{n} \beta_n^k \phi(q, S_k z) \leq \phi(q, z), \tag{3.32}$$

that is,

$$\phi\left(q, J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k z\right)\right) = \sum_{k=0}^{n} \beta_n^k \phi(q, S_k z) = \phi(q, z). \tag{3.33}$$

By Lemma 2.9, we have $z = S_0 z = S_1 z = \cdots = S_n z$. So

$$\mathrm{F}(T_n) \subset \bigcap_{k=0}^{n} \mathrm{F}(S_k) \quad \forall n \in \mathbb{N} \cup \{0\}. \tag{3.34}$$

This implies that

$$\mathrm{F}(T_n) = \bigcap_{k=0}^{n} \mathrm{F}(S_k) \quad \forall n \in \mathbb{N} \cup \{0\}, \tag{3.35}$$

and so

$$\bigcap_{n=0}^{\infty} \mathrm{F}(T_n) = \bigcap_{k=0}^{\infty} \mathrm{F}(S_k) \neq \varnothing. \tag{3.36}$$

Then, by (3.31), we have that $\{T_n\}$ is a sequence of relatively quasi-nonexpansive mappings. Let $B$ be a bounded subset of $C$ and let $p \in \bigcap_{k=0}^{\infty} \mathrm{F}(S_k)$. By (2.5), we have

$$\left(\|S_k x\| - \|p\|\right)^2 \leq \phi(p, S_k x) \leq \phi(p, x) \leq \left(\|x\| + \|p\|\right)^2, \tag{3.37}$$

and hence

$$\|S_k x\| \leq 2\|p\| + \sup\{\|z\| : z \in B\} \tag{3.38}$$

for all $x \in B$ and $k \in \mathbb{N} \cup \{0\}$. Let $M = \sup\{\|S_k x\| : x \in B,\ k \in \mathbb{N} \cup \{0\}\}$. For $x \in B$ and $n \in \mathbb{N} \cup \{0\}$, we have

$$\begin{aligned}
\|JT_{n+1}x - JT_n x\| &= \left\| \sum_{k=0}^{n+1} \beta_{n+1}^k J S_k x - \sum_{k=0}^{n} \beta_n^k J S_k x \right\| \\
&\leq \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| \|J S_k x\| + \beta_{n+1}^{n+1} \|J S_k x\| \\
&= \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| \|S_k x\| + \left(1 - \sum_{k=0}^{n} \beta_{n+1}^k\right) \|S_k x\| \\
&\leq \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right| M + \left(\sum_{k=0}^{n} \beta_n^k - \sum_{k=0}^{n} \beta_{n+1}^k\right) M \\
&\leq 2M \sum_{k=0}^{n} \left|\beta_{n+1}^k - \beta_n^k\right|.
\end{aligned} \tag{3.39}$$

Therefore,

$$\sup\{\|JT_{n+1}x - JT_nx\| : x \in B\} \le 2M\sum_{k=0}^{n}|\beta_{n+1}^k - \beta_n^k|. \tag{3.40}$$

It follows from (iii) that

$$\sum_{n=0}^{\infty}\sup\{\|JT_{n+1}x - JT_nx\| : x \in B\} \le 2M\sum_{n=0}^{\infty}\sum_{k=0}^{n}|\beta_{n+1}^k - \beta_n^k| < \infty. \tag{3.41}$$

By Lemma 2.7, we can define a mapping $T$ by

$$Tx = \lim_{n\to\infty}T_nx, \quad \forall x \in C. \tag{3.42}$$

Using the same argument presented in the proof of [17, pages 2357-2358], we have

$$\lim_{n\to\infty}\sum_{k=0}^{n}|\beta_n^k - \beta^k| = 0, \qquad \sum_{k=0}^{\infty}\beta^k = 1. \tag{3.43}$$

For each $x \in C$, the series $\sum_{k=0}^{\infty}\beta^k JS_kx$ converges absolutely and

$$\begin{aligned}
\left\|JTx - \sum_{k=0}^{\infty}\beta^k JS_kx\right\| &= \lim_{n\to\infty}\left\|JT_nx - \sum_{k=0}^{\infty}\beta^k JS_kx\right\| \\
&= \lim_{n\to\infty}\left\|\sum_{k=0}^{n}\beta_n^k JS_kx - \sum_{k=0}^{\infty}\beta^k JS_kx\right\| \\
&\le \lim_{n\to\infty}\left(\sum_{k=0}^{n}|\beta_n^k - \beta^k|\|JS_kx\| + \sum_{k=n+1}^{\infty}\beta^k\|JS_kx\|\right) \\
&\le \lim_{n\to\infty}\sum_{k=0}^{n}|\beta_n^k - \beta^k|\|S_kx\| + \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\beta^k\|S_kx\| \\
&\le \lim_{n\to\infty}\sum_{k=0}^{n}|\beta_n^k - \beta^k|M + \lim_{n\to\infty}\sum_{k=n+1}^{\infty}\beta^kM = 0.
\end{aligned} \tag{3.44}$$

This implies that

$$Tx = J^{-1}\left(\sum_{k=0}^{\infty}\beta^k JS_kx\right) \quad \forall x \in C. \tag{3.45}$$

It is obvious that

$$\bigcap_{k=0}^{\infty}F(S_k) \subset F(T). \tag{3.46}$$

Let $z \in \mathrm{F}(T)$ and fix $p \in \bigcap_{k=0}^{\infty} \mathrm{F}(S_k)$. Then

$$\begin{aligned}
\phi(p,z) = \phi(p,Tz) &= \phi\left(p, J^{-1}\left(\sum_{k=0}^{\infty}\beta^k JS_k z\right)\right) \\
&= \lim_{n\to\infty}\phi\left(p, J^{-1}\left(\sum_{k=0}^{n}\beta^k JS_k z\right)\right) \\
&= \lim_{n\to\infty}\left(\|p\|^2 - 2\left\langle p, \sum_{k=0}^{n}\beta^k JS_k z\right\rangle + \left\|\sum_{k=0}^{n}\beta^k JS_k z\right\|^2\right) \\
&\le \lim_{n\to\infty}\left(\|p\|^2 - 2\left\langle p, \sum_{k=0}^{n}\beta^k JS_k z\right\rangle + \sum_{k=0}^{n}\beta^k \left\|JS_k z\right\|^2\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=0}^{\infty}\beta^k\|p\|^2 - 2\sum_{k=0}^{n}\beta^k\langle p, JS_k z\rangle + \sum_{k=0}^{n}\beta^k\left\|S_k z\right\|^2\right) \\
&= \lim_{n\to\infty}\left(\sum_{k=0}^{n}\beta^k\phi(p,S_k z) + \sum_{k=n+1}^{\infty}\beta^k\|p\|^2\right) \\
&= \lim_{n\to\infty}\sum_{k=0}^{n}\beta^k\phi(p,S_k z) \\
&= \sum_{k=0}^{\infty}\beta^k\phi(p,S_k z) \\
&\le \sum_{k=0}^{\infty}\beta^k\phi(p,z) \\
&= \phi(p,z).
\end{aligned} \tag{3.47}$$

It follows that

$$\left\|\sum_{k=0}^{\infty}\beta^k JS_k z\right\|^2 = \sum_{k=0}^{\infty}\beta^k\left\|JS_k z\right\|^2. \tag{3.48}$$

By the strict convexity of $E^*$ and Lemma 2.10,

$$JS_k z = JS_0 z = Jz \quad \forall k \in \mathbb{N}. \tag{3.49}$$

Since $J$ is one to one,

$$S_k z = S_0 z = z \quad \forall k \in \mathbb{N}. \tag{3.50}$$

So $z \in \bigcap_{k=0}^{\infty}\mathrm{F}(S_k)$. Therefore,

$$\mathrm{F}(T) \subset \bigcap_{k=0}^{\infty}\mathrm{F}(S_k). \tag{3.51}$$

This together with (3.36) and (3.46) gives

$$F(T) = \bigcap_{n=0}^{\infty} F(T_n) = \bigcap_{k=0}^{\infty} F(S_k). \tag{3.52}$$

Hence, we obtain that $\{T_n\}$ satisfies all the conditions of our main theorem. Now, we have the following result.

**Theorem 3.4.** *Let $E$ be a uniformly convex and uniformly smooth Banach space and let $C$ be a nonempty closed convex subset of $E$. Let $\{\beta_n^k\}$ be a family of positive real numbers with indices $n$, $k \in \mathbb{N} \cup \{0\}$ with $k \leq n$ such that*

(i) $\sum_{k=0}^{n} \beta_n^k = 1$ *for every* $n \in \mathbb{N} \cup \{0\}$;

(ii) $\lim_{n\to\infty} \beta_n^k = \beta^k > 0$ *for every* $k \in \mathbb{N} \cup \{0\}$;

(iii) $\sum_{n=0}^{\infty} \sum_{k=0}^{n} |\beta_{n+1}^k - \beta_n^k| < \infty$.

*Let $\{S_k\}$ be a sequence of continuous relatively quasi-nonexpansive mappings with a common fixed point and let $S_0$ be the identity operator, one defines a sequence $\{T_n\}$ of relatively quasi-nonexpansive mappings from $C$ into $E$ by*

$$T_n x = J^{-1}\left(\sum_{k=0}^{n} \beta_n^k J S_k x\right) \tag{3.53}$$

*for all $x \in C$ and $n \in \mathbb{N} \cup \{0\}$. Then the sequence $\{x_n\}$ in $C$ defined by* (3.1) *converges strongly to* $\Pi_{\bigcap_{k=0}^{\infty} F(S_k)} x_0$.

## 4. Deduced theorems

In Hilbert spaces, relatively quasi-nonexpansive mappings and quasi-nonexpansive mappings are the same. We obtain the following result.

**Theorem 4.1.** *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $\{T_n\}$ be a sequence of quasi-nonexpansive mappings from $C$ into $E$ such that $\bigcap_{n=0}^{\infty} F(T_n)$ is nonempty and let $\{x_n\}$ be a sequence in $C$ defined as follows:*

$$\begin{aligned}
&x_0 \in C, \quad C_{-1} = Q_{-1} = C, \\
&y_n = \alpha_n x_n + (1 - \alpha_n) T_n x_n, \\
&C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \leq \|x_n - z\|\}, \\
&Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{4.1}$$

*where $\{\alpha_n\}$ is a sequence in $[0,1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. Suppose that for each bounded subset $B$ of $C$, the ordered pair $(\{T_n\}, B)$ satisfies condition AKTT. Let $T$ be the mapping from $C$ into $E$ defined by $Tz = \lim_{n\to\infty} T_n z$ for all $z \in C$ and suppose that $T$ is closed and $F(T) = \bigcap_{n=0}^{\infty} F(T_n)$. Then $\{x_n\}$ converges strongly to $P_{F(T)} x_0$.*

*Proof.* Since $J$ is an identity operator, we have

$$\phi(x, y) = \|x - y\|^2, \tag{4.2}$$

for every $x, y \in H$. Therefore,

$$\|T_n x - p\| \leq \|x - p\| \Longleftrightarrow \phi(p, T_n x) \leq \phi(p, x) \tag{4.3}$$

for every $x \in C$ and $p \in \mathrm{F}(T_n)$. Hence, $T_n$ is quasi-nonexpansive if and only if $T_n$ is relatively quasi-nonexpansive. Then, by Theorem 3.1, we obtain the result. □

**Corollary 4.2** (see [22, Theorem 2.1]). *Let $C$ be a nonempty closed convex subset of a Hilbert space $H$. Let $T$ be a closed quasi-nonexpansive mapping from $C$ into $E$ such that $\mathrm{F}(T)$ is nonempty and let $\{x_n\}$ be a sequence in $C$ defined as follows:*

$$\begin{aligned}
&x_0 \in C, \quad C_{-1} = Q_{-1} = C, \\
&y_n = \alpha_n x_n + (1 - \alpha_n) T x_n, \\
&C_n = \{z \in C_{n-1} \cap Q_{n-1} : \|y_n - z\| \leq \|x_n - z\|\}, \\
&Q_n = \{z \in C_{n-1} \cap Q_{n-1} : \langle x_n - z, x_0 - x_n \rangle \geq 0\}, \\
&x_{n+1} = P_{C_n \cap Q_n} x_0, \quad n = 0, 1, 2, \ldots,
\end{aligned} \tag{4.4}$$

*where $\{\alpha_n\}$ is a sequence in $[0, 1)$ with $\limsup_{n\to\infty} \alpha_n < 1$. Then $\{x_n\}$ converges strongly to $P_{\mathrm{F}(T)} x_0$.*

We give an example of a countable family of quasi-nonexpansive mappings which are not nonexpansive but satisfy all the requirements of our main theorem.

*Example 4.3.* Let $E = \mathbb{R}$ with the usual norm. For $n \in \mathbb{N}$, we define a mapping $T_n$ on $\mathbb{R}$ by

$$T_n x = \begin{cases} 0 & \text{if } x \leq \dfrac{1}{n^2}, \\[2mm] \dfrac{1}{n^2} & \text{if } x > \dfrac{1}{n^2}, \end{cases} \tag{4.5}$$

for all $x \in \mathbb{R}$. Then $\bigcap_{n=1}^{\infty} \mathrm{F}(T_n) = \mathrm{F}(T_n) = \{0\}$ and

$$|T_n x - 0| \leq |x - 0| \quad \forall x \in \mathbb{R}. \tag{4.6}$$

So $\{T_n\}$ is a sequence of quasi-nonexpansive mappings. Let $z \in \mathbb{R}$, then

$$|T_{n+1} z - T_n z| = \begin{cases} 0 & \text{if } z \leq \dfrac{1}{(n+1)^2}, \\[2mm] \dfrac{1}{n^2} & \text{if } \dfrac{1}{(n+1)^2} < z \leq \dfrac{1}{n^2}, \\[2mm] \dfrac{1}{n^2} - \dfrac{1}{(n+1)^2} & \text{if } z > \dfrac{1}{n^2}, \end{cases} \tag{4.7}$$

for all $n \in \mathbb{N}$. It follows that

$$\sum_{n=1}^{\infty} \sup\{ |T_{n+1}z - T_n z| : z \in \mathbb{R}\} = \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \tag{4.8}$$

We now define a mapping $T$ on $\mathbb{R}$ by

$$Tx = \lim_{n\to\infty} T_n x = 0 \quad \forall x \in \mathbb{R}. \tag{4.9}$$

Hence, the sequence $\{T_n\}$ satisfies all conditions in our main result. We also note that each $T_n$ is neither nonexpansive nor relatively nonexpansive. Actually, $T_n$ above fails to have the condition (R3). Let $\{x_m\}$ be a sequence define by $x_m = 1/n^2 + 1/m$. Then

$$x_m \longrightarrow \frac{1}{n^2}, \quad x_m - T_n x_m = \frac{1}{m} \longrightarrow 0 \text{ as } m \longrightarrow \infty. \tag{4.10}$$

This implies that $1/n^2 \in \widehat{\mathrm{F}}(T_n)$ and $1/n^2 \notin \mathrm{F}(T_n)$.

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