Hindawi Publishing Corporation
Fixed Point Theory and Applications
Volume 2007, Article ID 76971, 9 pages
doi:10.1155/2007/76971

Research Article

Strong Convergence Theorems for a Finite Family of Nonexpansive Mappings

Meijuan Shang, Yongfu Su, and Xiaolong Qin

Received 23 May 2007; Accepted 2 August 2007

Recommended by J. R. L. Webb

We modified the classic Mann iterative process to have strong convergence theorem for a finite family of nonexpansive mappings in the framework of Hilbert spaces. Our results improve and extend the results announced by many others.

1. Introduction and preliminaries

Let H be a real Hilbert space, C a nonempty closed convex subset of H, and $T : C \to C$ a mapping. Recall that T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. A point $x \in C$ is called a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of fixed points of T, that is, $F(T) = \{x \in C : Tx = x\}$. Recall that a self-mapping $f : C \to C$ is a contraction on C, if there exists a constant $\alpha \in (0,1)$ such that $\|f(x) - f(y)\| \leq \alpha\|x - y\|$ for all $x, y \in C$. We use Π_C to denote the collection of all contractions on C, that is, $\Pi_C = \{f \mid f : C \to C \text{ a contraction}\}$. An operator A is strongly positive if there exists a constant $\overline{\gamma} > 0$ with the property

$$\langle Ax, x\rangle \geq \overline{\gamma}\|x\|^2 \quad \forall x \in H. \tag{1.1}$$

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems (see, e.g., [1, 2] and the references therein). A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in C} \frac{1}{2}\langle Ax, x\rangle - \langle x, b\rangle, \tag{1.2}$$

where C is the fixed point set of a nonexpansive mapping S, and b is a given point in H. In [2], it is proved that the sequence $\{x_n\}$ defined by the iterative method below, with the initial guess $x_0 \in H$ chosen arbitrarily,

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n b, \quad n \geq 0, \tag{1.3}$$

converges strongly to the unique solution of the minimization problem (1.2) provided the sequence $\{\alpha_n\}$ satisfies certain conditions. Recently, Marino and Xu [1] introduced a new iterative scheme by the viscosity approximation method

$$x_{n+1} = (I - \alpha_n A)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0. \tag{1.4}$$

They proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality $\langle (A - \gamma f)x^*, x - x^* \rangle \geq 0$, $x \in C$, which is the optimality condition for the minimization problem

$$\min_{x \in C} \frac{1}{2} \langle Ax, x \rangle - h(x), \tag{1.5}$$

where C is the fixed point set of a nonexpansive mapping S, and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$.)

Mann's iteration process [3] is often used to approximate a fixed point of a nonexpansive mapping, which is defined as

$$x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Tx_n, \quad n \geq 0, \tag{1.6}$$

where the initial guess x_0 is taken in C arbitrarily and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ is in the interval $[0,1]$. But Mann's iteration process has only weak convergence, in general. For example, Reich [4] shows that if E is a uniformly convex Banach space and has a Frehet differentiable norm and if the sequence $\{\alpha_n\}$ is such that $\sum \alpha_n(1 - \alpha_n) = \infty$, then the sequence $\{x_n\}$ generated by process (1.6) converges weakly to a point in $F(T)$. Therefore, many authors try to modify Mann's iteration process to have strong convergence.

Kim and Xu [5] introduced the following iteration process:

$$\begin{aligned} x_0 &= x \in C \text{ arbitrarily chosen}, \\ y_n &= \beta_n x_n + (1 - \beta_n)Tx_n, \\ x_{n+1} &= \alpha_n u + (1 - \alpha_n)y_n. \end{aligned} \tag{1.7}$$

They proved that the sequence $\{x_n\}$ defined by (1.7) converges strongly to a fixed point of T provided the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy appropriate conditions.

Recently, Yao et al. [6] also modified Mann's iterative scheme (1.7) and got a strong convergence theorem. They improved the results of Kim and Xu [5] to some extent.

In this paper, we study the mapping W_n defined by

$$\begin{aligned}
U_{n0} &= I, \\
U_{n1} &= \gamma_{n1} T_1 U_{n0} + (1-\gamma_{n1}) I, \\
U_{n2} &= \gamma_{n2} T_2 U_{n1} + (1-\gamma_{n2}) I, \\
&\vdots \\
U_{n,N-1} &= \gamma_{n,N-1} T_{N-1} U_{n,N-2} + (1-\gamma_{n,N-1}) I, \\
W_n = U_{nN} &= \gamma_{nN} T_N U_{n,N-1} + (1-\gamma_{nN}) I,
\end{aligned} \tag{1.8}$$

where $\{\gamma_{n1}\},\{\gamma_{n2}\},\dots,\{\gamma_{nN}\} \in (0,1]$. Such a mapping W_n is called the W_n-mapping generated by $T_1,T_2,\dots,T_N$ and $\{\gamma_{n1}\},\{\gamma_{n2}\},\dots,\{\gamma_{nN}\}$. Nonexpansivity of each T_i ensures the nonexpansivity of W_n. It follows from [7, Lemma 3.1] that $F(W_n) = \cap_{i=1}^N F(T_i)$.

Very recently, Xu [2] studied the following iterative scheme:

$$x_{n+1} = \alpha_n u + (I - \alpha_n A) T_{n+1} x_n, \quad n \geq 0, \tag{1.9}$$

where A is a linear bounded operator, $T_n = T_{n \bmod N}$ and the mod function takes values in $\{1,2,\dots,N\}$. He proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the minimization problem (1.2) provided T_n satisfy

$$F(T_N \cdots T_2 T_1) = F(T_1 T_N \cdots T_3 T_2) = \cdots = F(T_{N-1} \cdots T_1 T_n), \tag{1.10}$$

and $\{\alpha_n\} \in (0,1)$ satisfying the following control conditions:

(C1) $\lim_{n\to\infty} \alpha_n = 0$;

(C2) $\sum_{n=1}^{\infty} \alpha_n = \infty$;

(C3) $\sum_{n=1}^{\infty} |\alpha_n - \alpha_{n+N}| < \infty$ or $\lim_{n\to\infty} \alpha_n/\alpha_{n+N} = 0$.

Remark 1.1. There are many nonexpansive mappings, which do not satisfy (1.10). For example, if $X = [0,1]$ and T_1, T_2 are defined by $T_1 x = x/2 + 1/2$ and $T_2 x = x/4$, then $F(T_1 T_2) = \{4/7\}$, whereas $F(T_2 T_1) = \{1/7\}$.

In this paper, motivated by Kim and Xu [5], Marino and Xu [1], Xu [2], and Yao et al. [6], we introduce a composite iteration scheme as follows:

$$\begin{aligned}
x_0 &= x \in C \text{ arbitrarily chosen}, \\
y_n &= \beta_n x_n + (1-\beta_n) W_n x_n, \\
x_{n+1} &= \alpha_n \gamma f(x_n) + (I - \alpha_n A) y_n,
\end{aligned} \tag{1.11}$$

where $f \in \Pi_C$ is a contraction, and A is a linear bounded operator. We prove, under certain appropriate assumptions on the sequences $\{\alpha_n\}$ and $\{\beta_n\}$, that $\{x_n\}$ defined by (1.11) converges to a common fixed point of the finite family of nonexpansive mappings, which solves some variation inequality and is also the optimality condition for the minimization problem (1.5).

Now, we consider some special cases of iterative scheme (1.11). When $A = I$ and $\gamma = 1$ in (1.11), we have that (1.11) collapses to

$$x_0 = x \in C \text{ arbitrarily chosen,}$$
$$y_n = \beta_n x_n + (1-\beta_n) W_n x_n, \tag{1.12}$$
$$x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n) y_n.$$

When $A = I$ and $\gamma = 1$ in (1.11), $N = 1$ and $\{\gamma_{n1}\} = 1$ in (1.8), we have that (1.11) collapses to the iterative scheme which was considered by Yao et al. [6]. When $A = I$ and $\gamma = 1$ in (1.11), $N = 1$ and $\{\gamma_{n1}\} = 1$ in (1.8), and $f(y) = u \in C$ for all $y \in C$ in (1.11), we have that (1.11) reduces to (1.7), which was considered by Kim and Xu [5].

In order to prove our main results, we need the following lemmas.

Lemma 1.2. *In a Hilbert space H, there holds the inequality*

$$\|x+y\|^2 \le \|x\|^2 + 2\langle y,(x+y)\rangle, \quad x,y \in H. \tag{1.13}$$

Lemma 1.3 (Suzuki [8]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a Banach space X and let β_n be a sequence in $[0,1]$ with $0 < \liminf_{n\to\infty}\beta_n \le \limsup_{n\to\infty}\beta_n < 1$. Suppose $x_{n+1} = (1-\beta_n)y_n + \beta_n x_n$ for all integers $n \ge 0$ and*

$$\limsup_{n\to\infty}\left(\|y_{n+1}-y_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{1.14}$$

Then $\lim_{n\to\infty}\|y_n - x_n\| = 0$.

Lemma 1.4 (Xu [2]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \le (1-\gamma_n)\alpha_n + \delta_n, \tag{1.15}$$

where γ_n is a sequence in $(0,1)$ and $\{\delta_n\}$ is a sequence such that

(i) $\sum_{n=1}^{\infty}\gamma_n = \infty$;

(ii) $\limsup_{n\to\infty}\delta_n/\gamma_n \le 0$ *or* $\sum_{n=1}^{\infty}|\delta_n| < \infty$.

Then $\lim_{n\to\infty}\alpha_n = 0$.

Lemma 1.5 (Marino and Xu [1]). *Assume that A is a strongly positive linear bounded operator on a Hilbert space H with coefficient $\overline{\gamma} > 0$ and $0 < \rho \le \|A\|^{-1}$, then $\|I - \rho A\| \le 1 - \rho\overline{\gamma}$.*

Lemma 1.6 (Marino and Xu [1]). *Let H be a Hilbert space. Let A be a strongly positive linear bounded selfadjoint operator with coefficient $\overline{\gamma} > 0$. Assume that $0 < \gamma < \overline{\gamma}/\alpha$. Let $T : C \to C$ be a nonexpansive mapping with a fixed point $x_t \in C$ of the contraction $C \ni x \mapsto t\gamma f(x) + (1-tA)Tx$. Then $\{x_t\}$ converges strongly as $t \to 0$ to a fixed point $\overline{x}$ of T, which solves the variational inequality*

$$\langle (A-\gamma f)\overline{x}, z - \overline{x}\rangle \le 0, \quad z \in F(T). \tag{1.16}$$

2. Main results

Theorem 2.1. *Let C be a closed convex subset of a Hilbert space H. Let A be a strongly positive linear bounded operator with coefficient $\overline{\gamma} > 0$ and W_n is defined by (1.8). Assume that*

$0<\gamma<\overline{\gamma}/\alpha$ and $F=\cap_{i=1}^{N}F(T_i)\neq\emptyset$. Given a map $f\in\Pi_C$, the initial guess $x_0\in C$ is chosen arbitrarily and given sequences $\{\alpha_n\}_{n=0}^{\infty}$ and $\{\beta_n\}_{n=0}^{\infty}$ in $(0,1)$, the following conditions are satisfied:

(C1) $\sum_{n=0}^{\infty}\alpha_n=\infty$;

(C2) $\lim_{n\to\infty}\alpha_n=0$;

(C3) $0<\liminf_{n\to\infty}\beta_n\leq\limsup_{n\to\infty}\beta_n<1$;

(C4) $\lim_{n\to\infty}|\gamma_{n,i}-\gamma_{n-1,i}|=0$, *for all* $i=1,2,\ldots,N$.

Let $\{x_n\}_{n=1}^{\infty}$ be the composite process defined by (1.11). Then $\{x_n\}_{n=1}^{\infty}$ converges strongly to $q\in F$, which also solves the following variational inequality:

$$\langle\gamma f(q)-Aq,p-q\rangle\leq 0,\quad p\in F. \tag{2.1}$$

Proof. First, we observe that $\{x_n\}_{n=0}^{\infty}$ is bounded. Indeed, take a point $p\in F$ and notice that

$$||y_n-p||\leq\beta_n||x_n-p||+(1-\beta_n)||W_nx_n-p||\leq||x_n-p||. \tag{2.2}$$

It follows that

$$\begin{aligned}||x_{n+1}-p||&=||\alpha_n(\gamma f(x_n)-Ap)+(I-\alpha_nA)(y_n-p)||\\&\leq[1-\alpha_n(\overline{\gamma}-\gamma\alpha)]||x_n-p||+\alpha_n||\gamma f(p)-Ap||.\end{aligned} \tag{2.3}$$

By simple inductions, we have $\|x_n-p\|\leq\max\{\|x_0-p\|,\|Ap-\gamma f(p)\|/(\overline{\gamma}-\gamma\alpha)\}$, which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$ and $\{z_n\}$.

Next, we claim that

$$||x_{n+1}-x_n||\longrightarrow 0. \tag{2.4}$$

Put $l_n=(x_{n+1}-\beta_nx_n)/(1-\beta_n)$. Now, we compute $l_{n+1}-l_n$, that is,

$$x_{n+1}=(1-\beta_n)l_n+\beta_nx_n,\quad n\geq 0. \tag{2.5}$$

Observing that

$$\begin{aligned}l_{n+1}-l_n&=\frac{\alpha_{n+1}\gamma f(x_{n+1})+(I-\alpha_{n+1}A)y_{n+1}-\beta_{n+1}x_{n+1}}{1-\beta_{n+1}}\\&\quad-\frac{\alpha_n\gamma f(x_n)+(I-\alpha_nA)y_n-\beta_nx_n}{1-\beta_n}\\&=\frac{\alpha_{n+1}(\gamma f(x_{n+1})-Ay_{n+1})}{1-\beta_{n+1}}-\frac{\alpha_n(\gamma f(x_n)-Ay_n)}{1-\beta_n}\\&\quad+W_{n+1}x_{n+1}-W_nx_n,\end{aligned} \tag{2.6}$$

we have

$$\begin{aligned}||l_{n+1}-l_n|| \leq{}& \frac{\alpha_{n+1}}{1-\beta_{n+1}}||\gamma f(x_{n+1})-Ay_{n+1}||+\frac{\alpha_n}{1-\beta_n}||Ay_n-\gamma f(x_n)|| \\ &+||x_{n+1}-x_n||+||W_{n+1}x_n-W_nx_n||.\end{aligned} \tag{2.7}$$

Next, we will use M to denote the possible different constants appearing in the following reasoning. It follows from the definition of W_n that

$$\begin{aligned}&||W_{n+1}x_n-W_nx_n|| \\ &\quad=||\gamma_{n+1,N}T_NU_{n+1,N-1}x_n+(1-\gamma_{n+1,N})x_n-\gamma_{n,N}T_NU_{n,N-1}x_n-(1-\gamma_{n,N})x_n|| \\ &\quad\leq|\gamma_{n+1,N}-\gamma_{n,N}|\,||x_n||+||\gamma_{n+1,N}T_NU_{n+1,N-1}x_n-\gamma_{n,N}T_NU_{n,N-1}x_n|| \\ &\quad\leq|\gamma_{n+1,N}-\gamma_{n,N}|\,||x_n||+||\gamma_{n+1,N}(T_NU_{n+1,N-1}x_n-T_NU_{n,N-1}x_n)|| \\ &\qquad+|\gamma_{n+1,N}-\gamma_{n,N}|\,||T_NU_{n,N-1}x_n|| \\ &\quad\leq 2M|\gamma_{n+1,N}-\gamma_{n,N}|+\gamma_{n+1,N}||U_{n+1,N-1}x_n-U_{n,N-1}x_n||.\end{aligned} \tag{2.8}$$

Next, we consider

$$\begin{aligned}&||U_{n+1,N-1}x_n-U_{n,N-1}x_n|| \\ &\quad=||\gamma_{n+1,N-1}T_{N-1}U_{n+1,N-2}x_n+(1-\gamma_{n+1,N-1})x_n \\ &\qquad-\gamma_{n,N-1}T_{N-1}U_{n,N-2}x_n-(1-\gamma_{n,N-1})x_n|| \\ &\quad\leq|\gamma_{n+1,N-1}-\gamma_{n,N-1}|\,||x_n||+\gamma_{n+1,N-1}||T_{N-1}U_{n+1,N-2}y_n-T_{N-1}U_{n,N-2}x_n|| \\ &\qquad+|\gamma_{n+1,N-1}-\gamma_{n,N-1}|\,||T_{N-1}U_{n,N-2}x_n|| \\ &\quad\leq 2M|\gamma_{n+1,N-1}-\gamma_{n,N-1}|+||U_{n+1,N-2}x_n-U_{n,N-2}x_n||.\end{aligned} \tag{2.9}$$

It follows that

$$\begin{aligned}&||U_{n+1,N-1}x_n-U_{n,N-1}x_n|| \\ &\quad\leq 2M|\gamma_{n+1,N-1}-\gamma_{n,N-1}|+2M|\gamma_{n+1,N-2}-\gamma_{n,N-2}|+||U_{n+1,N-3}x_n-U_{n,N-3}x_n|| \\ &\quad\leq 2M\sum_{i=2}^{N-1}|\gamma_{n+1,i}-\gamma_{n,i}|+||U_{n+1,1}x_n-U_{n,1}x_n|| \\ &\quad\leq 2M\sum_{i=1}^{N-1}|\gamma_{n+1,i}-\gamma_{n,i}|.\end{aligned} \tag{2.10}$$

Substituting (2.10) into (2.8) yields that

$$\begin{aligned} \|W_{n+1}x_n - W_nx_n\| &\le 2M\,|\,\gamma_{n+1,N}-\gamma_{n,N}\,| + 2\gamma_{n+1,N}M\sum_{i=1}^{N-1}|\,\gamma_{n+1,i}-\gamma_{n,i}\,| \\ &\le 2M\sum_{i=1}^{N}|\,\gamma_{n+1,i}-\gamma_{n,i}\,|. \end{aligned} \tag{2.11}$$

It follows that

$$\begin{aligned} &\|l_{n+1}-l_n\| - \|x_n - x_{n-1}\| \\ &\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}) - Ay_{n+1}\| + \frac{\alpha_n}{1-\beta_n}\|Ay_n - \gamma f(x_n)\| + 2M\sum_{i=1}^{N}|\,\gamma_{n+1,i}-\gamma_{n,i}\,|. \end{aligned} \tag{2.12}$$

Observing conditions (C1), (C4) and takeing the limits as $n \to \infty$, we get

$$\limsup_{n\to\infty}\left(\|l_{n+1}-l_n\| - \|x_{n+1}-x_n\|\right) \le 0. \tag{2.13}$$

We can obtain $\lim_{n\to\infty}\|l_n - x_n\| = 0$ easily by Lemma 1.3. Since $x_{n+1}-x_n = (1-\beta_n)(l_n - x_n)$, we have that (2.4) holds. Observing that $x_{n+1}-y_n = \alpha_n(\gamma f(x_n) - Ay_n)$, we can easily get $\lim_{n\to\infty}\|y_n - x_{n+1}\| = 0$, which implies that

$$\|y_n - x_n\| \le \|x_n - x_{n+1}\| + \|x_{n+1}-y_n\|, \tag{2.14}$$

that is,

$$\lim_{n\to\infty}\|y_n - x_n\| = 0. \tag{2.15}$$

On the other hand, we have

$$\|W_nx_n - x_n\| \le \|x_n - y_n\| + \|y_n - W_nx_n\| \le \|x_n - y_n\| + \beta_n\|x_n - W_nx_n\|, \tag{2.16}$$

which implies $(1-\beta_n)\|W_nx_n - x_n\| \le \|x_n - y_n\|$. From condition (C3) and (2.15), we obtain

$$\|W_nx_n - x_n\| \longrightarrow 0. \tag{2.17}$$

Next, we claim that

$$\limsup_{n\to\infty}\langle \gamma f(q) - Aq, x_n - q\rangle \le 0, \tag{2.18}$$

where $q = \lim_{t\to 0}x_t$ with x_t being the fixed point of the contraction $x \mapsto t\gamma f(x) + (I - tA)W_nx$. Then, x_t solves the fixed point equation $x_t = t\gamma f(x_t) + (I-tA)W_nx_t$. Thus, we have $\|x_t - x_n\| = \|(I - tA)(W_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)\|$. It follows from Lemma 1.2 that

$$\begin{aligned} ||x_t - x_n||^2 &= ||(I - tA)(W_n x_t - x_n) + t(\gamma f(x_t) - Ax_n)||^2 \\ &\le (1 - \overline{\gamma}t)^2 ||W_n x_t - x_n||^2 + 2t\langle \gamma f(x_t) - Ax_n, x_t - x_n\rangle \\ &\le (1 - 2\overline{\gamma}t + (\overline{\gamma}t)^2)||x_t - x_n||^2 + f_n(t) \\ &\quad + 2t\langle \gamma f(x_t) - Ax_t, x_t - x_n\rangle + 2t\langle Ax_t - Ax_n, x_t - x_n\rangle, \end{aligned} \tag{2.19}$$

where

$$f_n(t) = (2||x_t - x_n|| + ||x_n - W_n x_n||)||x_n - W_n x_n|| \longrightarrow 0, \quad \text{as } n \longrightarrow 0. \tag{2.20}$$

It follows that

$$\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \le \frac{\overline{\gamma}t}{2}\langle Ax_t - Ax_n, x_t - x_n\rangle + \frac{1}{2t}f_n(t). \tag{2.21}$$

Let $n \to \infty$ in (2.21) and note that (2.20) yields

$$\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \le \frac{t}{2}M, \tag{2.22}$$

where $M > 0$ is a constant such that $M \ge \overline{\gamma}\langle Ax_t - Ax_n, x_t - x_n\rangle$ for all $t \in (0,1)$ and $n \ge 1$. Taking $t \to 0$ from (2.22), we have $\limsup_{t\to 0}\limsup_{n\to\infty}\langle Ax_t - \gamma f(x_t), x_t - x_n\rangle \le 0$. Since H is a Hilbert space, the order of $\limsup_{t\to 0}$ and $\limsup_{n\to\infty}$ is exchangeable, and hence (2.18) holds. Now from Lemma 1.2, we have

$$\begin{aligned} ||x_{n+1} - q||^2 &= ||(I - \alpha_n A)(y_n - q) + \alpha_n(\gamma f(x_n) - Aq)||^2 \\ &\le ||(I - \alpha_n A)(y_n - q)||^2 + 2\alpha_n\langle \gamma f(x_n) - Aq, x_{n+1} - q\rangle \\ &\le (1 - \alpha_n\overline{\gamma})^2||x_n - q||^2 + \alpha_n\gamma\alpha\left(||x_n - q||^2 + ||x_{n+1} - q||^2\right) \\ &\quad + 2\alpha_n\langle \gamma f(q) - Aq, x_{n+1} - q\rangle, \end{aligned} \tag{2.23}$$

which implies that

$$\begin{aligned} ||x_{n+1} - q||^2 &\le \frac{(1 - \alpha_n\overline{\gamma})^2 + \alpha_n\gamma\alpha}{1 - \alpha_n\gamma\alpha}||x_n - q||^2 + \frac{2\alpha_n}{1 - \alpha_n\gamma\alpha}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle \\ &\le \left[1 - \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}\right]||x_n - q||^2 \\ &\quad + \frac{2\alpha_n(\overline{\gamma} - \alpha\gamma)}{1 - \alpha_n\gamma\alpha}\left[\frac{1}{\overline{\gamma} - \alpha\gamma}\langle \gamma f(q) - Aq, x_{n+1} - q\rangle + \frac{\alpha_n\overline{\gamma}^2}{2(\overline{\gamma} - \alpha\gamma)}M\right]. \end{aligned} \tag{2.24}$$

Put $l_n = 2\alpha_n(\overline{\gamma} - \alpha_n\gamma)/(1 - \alpha_n\alpha\gamma)$ and $t_n = 1/(\overline{\gamma} - \alpha\gamma)\langle \gamma f(q) - Aq, x_{n+1} - q\rangle + \alpha_n\overline{\gamma}^2/(2(\overline{\gamma} - \alpha\gamma))M$, that is,

$$||x_{n+1} - q||^2 \le (1 - l_n)||x_n - q|| + l_n t_n. \tag{2.25}$$

It follows from conditions (C1), (C2), and (2.22) that $\lim_{n\to\infty} l_n = 0$, $\sum_{n=1}^{\infty} l_n = \infty$, and $\limsup_{n\to\infty} t_n \le 0$. Apply Lemma 1.4 to (2.25) to conclude that $x_n \to q$. This completes the proof. □

Remark 2.2. Our results relax the condition of Kim and Xu [1] imposed on control sequences and also improve the results of Yao et al. [6] from one single mapping to a finite family of nonexpansive mappings, respectively.

References

[1] G. Marino and H.-K. Xu, "A general iterative method for nonexpansive mappings in Hilbert spaces," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 43–52, 2006.

[2] H. K. Xu, "An iterative approach to quadratic optimization," *Journal of Optimization Theory and Applications*, vol. 116, no. 3, pp. 659–678, 2003.

[3] W. R. Mann, "Mean value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, pp. 506–510, 1953.

[4] S. Reich, "Weak convergence theorems for nonexpansive mappings in Banach spaces," *Journal of Mathematical Analysis and Applications*, vol. 67, no. 2, pp. 274–276, 1979.

[5] T.-H. Kim and H.-K. Xu, "Strong convergence of modified Mann iterations," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 61, no. 1-2, pp. 51–60, 2005.

[6] Y. Yao, R. Chen, and J.-C. Yao, "Strong convergence and certain control conditions for modified Mann iteration," to appear in *Nonlinear Analysis: Theory, Methods & Applications* .

[7] W. Takahashi and K. Shimoji, "Convergence theorems for nonexpansive mappings and feasibility problems," *Mathematical and Computer Modelling*, vol. 32, no. 11–13, pp. 1463–1471, 2000.

[8] T. Suzuki, "Strong convergence of Krasnoselskii and Mann's type sequences for one-parameter nonexpansive semigroups without Bochner integrals," *Journal of Mathematical Analysis and Applications*, vol. 305, no. 1, pp. 227–239, 2005.

Meijuan Shang: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China; Department of Mathematics, Shijiazhuang University, Shijiazhuang 050035, China
Email address: meijuanshang@yahoo.com.cn

Yongfu Su: Department of Mathematics, Tianjin Polytechnic University, Tianjin 300160, China
Email address: suyongfu@tjpu.edu.cn

Xiaolong Qin: Department of Mathematics, Gyeongsang National University, Chinju 660-701, Korea
Email address: qxlxajh@163.com