ON THE ORBITS OF *G*-CLOSURE POINTS OF ULTIMATELY NONEXPANSIVE MAPPINGS

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Let X be a closed subset of a Banach space and G an ultimately nonexpansive commutative semigroup of continuous selfmappings. If the G-closure of X is nonempty, then the closure of the orbit of any G-closure point is a commutative topological group.

1. Introduction

Let (X,d) be a metric space. A mapping $f : X \to X$ is called *nonexpansive* if for every $x,y \in X$, we have $d(f(x),f(y)) \leq d(x,y)$. Edelstein introduced in [2] the concept of f-closure points for nonexpansive mappings and proved that a nonexpansive mapping of $\mathbb{E}^n$ admits a fixed point if it has a nonempty set of f-closure points (points which are cluster points of $\{f^n(x)\}$ for some $x \in X$).

When G is a family of mappings $g : X \to X$ forming a semigroup under composition, the notion of G-closure points of X was introduced in [5] to generalize the concept of f-closure point. A G-closure point x of X is a cluster point of an orbit $G(z)$ for some $z \in X$. The study of f-closure points sets (called ω-limit sets in [1, 7]), orbits, and G-closure points (e.g., [3, 4, 6]) has since been of great interest in the fixed points theorems for various contractive-type mappings. In [7], Roehrig and Sine showed that when C is a closed set in a Banach space B and $f : C \to C$ a nonexpansive mapping, suppose for some $x \in C$, the ω-limit set S (i.e., the set of f-closure points) of x is nonempty, then there exists a binary operation in the set S under which it is a monothetic topological group in the topology induced by the metric of B. It is the purpose of this paper to show that when G is a commutative ultimately nonexpansive semigroup of mappings (a concept introduced by Edelstein and the author in [3, 4]) of a closed subset X of a Banach space into itself and if there is a G-closure point $z \in X$, then there exists a binary operation in the closure of the orbit of z such that it is a commutative topological group.

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2. Definitions and notations

Definition 2.1. Let (X,d) be a metric space and $G: X \to X$ a semigroup of mappings. For any $x \in X$, the set $G(x) = \{g(x) : g \in G\}$ is called the *orbit* of x under G.

Definition 2.2. A semigroup of selfmappings G of a metric space (X,d) is called *asymptotically nonexpansive* if for all $x, y \in X$ there exists $g \in G$ such that for all $f \in G$, $d(fg(x), fg(y)) \leq d(x,y)$.

Definition 2.3. A semigroup G of continuous selfmappings of a metric space (X,d) is called *ultimately nonexpansive* if for every pair of points $x, y \in X$ and for every $\alpha > 0$ there is $g \in G$ such that for all $f \in G$, $d(fg(x), fg(y)) \leq (1+\alpha)d(x,y)$. (When $\alpha = 0$, G is asymptotically nonexpansive.)

Definition 2.4. Let $f : (X,d) \to (X,d)$. Then the *ω-limit set of x* (denoted by $\omega(x)$ in [1, 7]) or the *f-closure of x* (denoted by X^f in [2]) is the set

$$\left\{ y \in X : y = \lim_{n \in N_1} f^n(x) \right\}, \tag{2.1}$$

where N_1 is a strictly increasing sequence in $\mathbb{Z}^+$.

Definition 2.5. Let G be a family of mappings of (X,d) into itself. The *G-closure of X* consists of all points $x \in X$ such that for some $z \in X$, any $\varepsilon > 0$, and any $f \in G$, there is a $g \in G$ such that $d(fg(z), x) < \varepsilon$. The G-closure of X is denoted by X^G.

Definition 2.6. A point x of (X,d) is called *G-recurrent* (or *recurrent under G*) if for any $\varepsilon > 0$ and any $f \in G$, there is a $g \in G$ such that $d(fg(x), x) < \varepsilon$.

3. Preliminaries

In the following, G is a family of ultimately nonexpansive commutative semigroups of continuous mappings of a metric space (X,d) into itself.

Proposition 3.1. *If $X^G \neq \emptyset$ and $z \in X^G$, then for all $f \in G$, for all $\varepsilon > 0$, there exists $g \in G$ with $d(fg(z), z) < \varepsilon$.*

Proof. See [3, Proposition 1(a)]. □

Proposition 3.2. *If $z \in X^G$, then $G|_{G(z)}$ is a family of asymptotically nonexpansive mappings.*

Proof. See [3, Proposition 2(a)]. □

Proposition 3.3. *If $z \in X^G$, then $G|_{G(z)}$ is a family of isometries.*

Proof. By Proposition 3.2, $G|_{G(z)}$ is a family of asymptotically nonexpansive mappings. By a result of Holmes and Narayanaswami (see [5, Proposition 2]), $G|_{G(z)}$ is a family of isometries. □

Corollary 3.4. *If $z \in X^G$, then $G|_{\overline{G(z)}}$ is a family of isometries.*

Proof. Obvious. □

Proposition 3.5. *When (X,d) is complete and $z \in X^G$, then for each $f \in G$, $f(\overline{G(z)}) = \overline{G(z)}$. That is, each f is an onto mapping when restricted to $\overline{G(z)}$.*

Proof. For each $f \in G$, clearly $f\overline{G(z)} \subseteq \overline{fG(z)} \subseteq \overline{G(z)}$ since f is continuous. It suffices to show that $\overline{G(z)} \subseteq f\overline{G(z)}$. Let $p \in \overline{G(z)}$. Then for all $\varepsilon = 1/n$, there exists $g_n \in G$ such that $d(g_n(z), p) < 1/2n$.

Since $z \in X^G$, for the above f and g_n, there exists t_n corresponding to fg_n such that $d(fg_nt_n(z), z) < 1/2n$. By Proposition 3.3, each member in G is an isometry on $G(z)$. Hence $d(fg_ng_nt_n(z), p) \le d(fg_ng_nt_n(z), g_n(z)) + d(g_n(z), p) < 1/2n + 1/2n = 1/n$. Let $h_n = g_ng_nt_n$. Then for each $\varepsilon = 1/n$, there exists $h_n \in G$ such that $d(fh_n(z), p) < 1/n$. Now $\{fh_n(z)\}$ converges to p implies that $\{h_n(z)\}$ is a Cauchy sequence since f is an isometry. Since X is complete $\{h_n(z)\}$, converges to a point $q \in \overline{G(z)}$.

Clearly $f(q) = f(\lim_{n\to\infty} h_n(z)) = \lim_{n\to\infty} fh_n(z) = p$, showing that $\overline{G(z)} \subseteq f\overline{G(z)}$. □

Proposition 3.6. *For each $f \in G$, $f|_{\overline{G(z)}}$ is a homeomorphism.*

Proof. By the corollary to Propositions 3.3 and 3.5, each f is an isometry of $\overline{G(z)}$ onto itself. Hence, each f is a homeomorphism. □

4. Main result

Theorem 4.1. *Let X be a closed subset of a Banach space and let $G : X \to X$ be a commutative semigroup (under composition) of ultimately nonexpansive mappings. If $X^G \neq \varnothing$ and z is any arbitrary member in X^G, then a binary operation can be introduced in $\overline{G(z)}$ such that $\overline{G(z)}$ is a commutative topological group in the topology induced by the metric of X.*

Proof. By Proposition 3.6, each $f \in G$ is an isometry and therefore a homeomorphism of $\overline{G(z)}$ onto itself. Hence, the inverse of each $f \in G$ exists. Let f^{-1} denote the inverse of f. By Proposition 3.1, since $z \in X^G$, for each $\varepsilon = 1/n$, for the above f, there exists $f_n \in G$ such that $d(ff_n(z), z) < 1/n$. Denote $g_n = ff_n$. We have $\lim_{n\to\infty} g_n(z) = z$. Let $p, q \in \overline{G(z)}$. Then there exist $h_n \in G$ and $t_n \in G$ such that $\lim_{n\to\infty} h_n(z) = p$ and $\lim_{n\to\infty} t_n(z) = q$. Denote $h_n^* = h_ng_n^{-1}$ and $t_n^* = t_ng_n^{-1}$. Then $h_n = h_n^*g_n$ and $t_n = t_n^*g_n$.

Define $q \circ p = \lim_{n\to\infty} t_n^*g_nh_n^*(z)$. This limit exists since each member of G is an isometry. It is also unique. Clearly $q \circ p \in \overline{G(z)}$. The following results are immediate:

(1) the operation $\circ$ is associative,

(2) z is the identity of $\overline{G(z)}$ (since $z \circ p = \lim_{n\to\infty} g_n^*g_nh_n^*(z) = \lim_{n\to\infty} h_n(z) = p$),

(3) $q \circ p = p \circ q$ since G is commutative.

If $p = \lim_{n\to\infty} h_n(z) = \lim_{n\to\infty} h_n^*g_n(z)$, define $p^{-1} = \lim_{n\to\infty} g_n(h_n^*)^{-1}(z)$. This limit exists as each member of G is an isometry; clearly $p^{-1} \circ p = \lim_{n\to\infty} (h_n^*)^{-1}g_nh_n^*(z) = z = p \circ p^{-1}$. Hence $\overline{G(z)}$ is a commutative group.

Next, let $p_i \to p$ and $q_i \to q$, where $p_i, q_i, p, q \in \overline{G(z)}$. Then there exist $h_{i,n}$ and $t_{i,n}$ such that $\lim_{n\to\infty} h_{i,n}(z) = p_i$ and $\lim_{n\to\infty} t_{i,n}(z) = q_i$. Denote $h_{i,n}^* = h_{i,n}g_n^{-1}$ and $t_{i,n}^* = t_{i,n}g_n^{-1}$.

Then $(t_{i,n}^*)^{-1} = g_n t_{i,n}^{-1}$. Since $(t_n^*)^{-1} = g_n t_n^{-1}$, $q^{-1} = \lim_{n\to\infty} g_n (t_n^*)^{-1}(z)$, and $q_i^{-1} = \lim_{n\to\infty} g_n (t_{i,n}^*)^{-1}(z)$, we have

$$\begin{aligned}
\|p_i \circ q_i^{-1} - p\circ q^{-1}\| &\le \|q_i^{-1}\circ p_i - q^{-1}\circ p_i\| + \|p_i\circ q^{-1} - p\circ q^{-1}\| \\
&= \left\|\lim_{n\to\infty}(t_{i,n}^*)^{-1} g_n h_{i,n}^*(z) - \lim_{n\to\infty}(t_n^*)^{-1} g_n h_{i,n}^*(z)\right\| \\
&\quad + \left\|\lim_{n\to\infty} h_{i,n}^* g_n (t_n^*)^{-1}(z) - \lim_{n\to\infty} h_n^* g_n (t_n^*)^{-1}(z)\right\| \\
&= \left\|\lim_{n\to\infty} g_n (t_{i,n}^*)^{-1}(z) - \lim_{n\to\infty} g_n (t_n^*)^{-1}(z)\right\| \\
&\quad + \left\|\lim_{n\to\infty} h_{i,n}^*(z) - \lim_{n\to\infty} h_n^*(z)\right\| \\
&= \left\|\lim_{n\to\infty} g_n (t_{i,n}^*)^{-1}(z) - \lim_{n\to\infty} g_n (t_n^*)^{-1}(z)\right\| \\
&\quad + \left\|\lim_{n\to\infty} h_{i,n} g_n^{-1}(z) - \lim_{n\to\infty} h_n g_n^{-1}(z)\right\| \\
&= \|q_i^{-1} - q^{-1}\| + \left\|\lim_{n\to\infty} h_{i,n}(z) - \lim_{n\to\infty} h_n(z)\right\| \\
&= \|q_i^{-1} - q^{-1}\| + \|p_i - p\|,
\end{aligned} \tag{4.1}$$

since all mappings are isometries.

As $i\to\infty$, $\|q_i^{-1} - q^{-1}\|$ and $\|p_i - p\|$ become arbitrarily small, so $\|p_i\circ q_i^{-1} - p\circ q^{-1}\|$ approaches zero. Hence the operation $\circ$ is continuous in both variables and $\overline{G(z)}$ is a topological group. □

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