Hindawi Publishing Corporation
International Journal of Differential Equations
Volume 2010, Article ID 432759, 11 pages
doi:10.1155/2010/432759

Research Article

On the Existence of Nodal Solutions for a Nonlinear Elliptic Problem on Symmetric Riemannian Manifolds

Anna Maria Micheletti[1] and Angela Pistoia[2]

[1] *Dipartimento di Matematica Applicata "U.Dini", Università di Pisa, via F. Buonarroti 1/c, 56100 Pisa, Italy*
[2] *Dipartimento di Metodi e Modelli Matematici, Università di Roma "La Sapienza", via Antonio Scarpa 16, 00161 Roma, Italy*

Correspondence should be addressed to Angela Pistoia, pistoia@dmmm.uniroma1.it

Received 1 October 2009; Accepted 7 December 2009

Academic Editor: Thomas Bartsch

Given that (M, g) is a smooth compact and symmetric Riemannian n-manifold, $n \geq 2$, we prove a multiplicity result for antisymmetric sign changing solutions of the problem $-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u$ in M. Here $p > 2$ if $n = 2$ and $2 < p < 2^* = 2n/(n-2)$ if $n \geq 3$.

1. Introduction

Let (M, g) be a smooth compact connected Riemannian manifold without boundary of dimension $n \geq 2$. Let us consider the problem

$$-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u \quad \text{in } M, \qquad u \in \mathrm{H}^1_g(M), \tag{1.1}$$

where $p > 2$ if $n = 2$, $2 < p < 2n/(n-2)$ if $n \geq 3$ and ε is a positive parameter. Here $\mathrm{H}^1_g(M)$ is the completion of $C^\infty(M)$ with respect to

$$\|u\|_g^2 := \int_M \left|\nabla_g u\right|^2 d\mu_g + \int_M u^2 d\mu_g. \tag{1.2}$$

It is well known that any critical point of the energy functional $J_\varepsilon : \mathrm{H}^1_g(M) \to \mathbb{R}$ constrained to the Nehari manifold $\mathcal{N}_\varepsilon$ is a solution to (1.1). Here

$$J_\varepsilon(u) := \frac{1}{\varepsilon^n}\int_M \left(\frac{1}{2}\varepsilon^2\left|\nabla_g u\right|^2 + \frac{1}{2u^2} - \frac{1}{p}|u|^p\right)d\mu_g, \tag{1.3}$$

$$\mathcal{N}_\varepsilon := \left\{u \in \mathrm{H}^1_g(M)\setminus\{0\} \ : J'_\varepsilon(u)[u] = 0\right\}. \tag{1.4}$$

In [1] the authors show that the least energy solution of (1.1), that is, the minimum of J_ε on $\mathcal{N}_\varepsilon$ is a positive solution with a spike layer, whose peak converges to the maximum point of the scalar curvature $\mathcal{S}_g$ of (M, g) as ε goes to zero. Successively, in [2] (see also [3, 4]) the authors point out that the topology of the manifold M influences the multiplicity of positive solutions of (1.1), that is, (1.1) has at least $\mathrm{cat}(M)$ nontrivial solutions provided that ε is small enough. Here $\mathrm{cat}(M)$ denotes the Lusternik-Schnirelman category of M. Recently, in [5–7] it has been proved that the existence of positive solutions is strongly related to the geometry of M, that is stable critical points of the scalar curvature $\mathcal{S}_g$ generate positive solutions with one or more peaks as ε goes to zero.

As far as it concerns the existence of sign changing solutions to (1.1), a few results are known. The first result has been obtained in [7] where it has been constructed solutions with one positive peak and one negative peak, which approach, as ε goes to zero, the minimum point and the maximum point of $\mathcal{S}_g$, provided the scalar curvature is not constant. In [8] the authors assume the following:

(S) *the manifold M is a regular submanifold of $\mathbb{R}^N$ invariant with respect to τ, where $\tau : \mathbb{R}^N \to \mathbb{R}^N$ is an orthogonal linear transformation such that $\tau \neq I$ and $\tau^2 = I$, I being the identity of $\mathbb{R}^N$.*

They prove problem (1.1) has at least $G_\tau\!-\!\mathrm{cat}(M - M_\tau)$ pairs of sign changing solutions which change sign exactly once. Here $G_\tau\!-\!\mathrm{cat}(M - M_\tau)$ denotes the G_τ-equivariant Lusternik-Schnirelman category for the group $G_\tau := \{I, \tau\}$ and $M_\tau := \{x \in M : \tau x = x\}$.

In this paper we assume M satisfies (S) in the particular case $\tau = -I$. We look for solutions of the problem

$$\begin{gathered}-\varepsilon^2\Delta_g u + u = |u|^{p-2}u \quad \text{in } M,\\ u \in \mathrm{H}^1_g(M),\\ u(-x) = -u(x).\end{gathered} \tag{1.5}$$

We evaluate the number of solutions of problem (1.5) using Morse theory. Our main result reads as following.

Theorem 1.1. *Assume that for ε small enough all the solutions to problem* (1.5) *with energy close to $2m_\infty$ are nondegenerate. Then there are at least $P_1(M/G)$ pairs $(u, -u)$ of nontrivial solutions to* (1.5) *which change sign exactly once, where*

$$m_\infty := \inf_{\int_{\mathbb{R}^n}|\nabla u|^2 + u^2 = \int_{\mathbb{R}^n}|u|^p}\int_{\mathbb{R}^n}\left(\frac{1}{2}|\nabla u|^2 + \frac{1}{2}u^2 - \frac{1}{p}|u|^p\right)dx. \tag{1.6}$$

Here $G = \{I, -I\}$ and $P_1(M/G)$ is Poincaré polynomial $P_t(M/g)$ when $t = 1$.

Concerning the assumptions of nondegeneracy of all the critical points with energy close to $2m_\infty$, we think that it is true "generically" in some sense with respect to (ε, g) where ε is a positive parameter and g is a Riemannian metric.

We point out that problem (1.1) has been widely studied when the manifold M is replaced by an open bounded and smooth domain in $\mathbb{R}^N$ with Dirichlet or Neumann boundary condition. In particular, it has been studied the effect of the domain topology or the domain geometry on the number of solutions. See, for example, [9–19] for the Dirichlet problem and [20–32] for the Neumann problem,

The paper is organized as follows. In Section 2 we set the problem and we recall some known results; in Section 3 we give the proof of Theorem 1.1; in Section 4 we prove the technical Lemma 4.5, which is crucial for the proof of Theorem 1.1.

2. Setting of the Problem

First of all, we will recall some topological notions which are used in the paper.

Definition 2.1 (Poincaré polynomial). If (X, Y) is a couple of the topological spaces, the Poincaré polynomial $P_t(X, Y)$ is defined as the following power series in t:

$$P_t(X, Y) := \sum_k \dim H_k(X, Y) t^k, \tag{2.1}$$

where $H_k(X, Y)$ is the kth homology group with coefficients in some fields. Moreover, we set

$$P_t(X) := P_t(X, \emptyset) = \sum_k \dim H_k(X) t^k. \tag{2.2}$$

If X is a compact manifold, we have that $\dim H_k(X) < +\infty$ and in this case $P_t(X)$ is a polynomial and not a formal series.

Definition 2.2 (Morse index). Let J be a $C2$-functional on a Banach space X and $u \in X$ an isolated critical point of J with $J(u) = c$. If $J^c := \{v \in X : J(v) \le c\}$ then the (polynomial) Morse index $i_t(u)$ of u is the following series:

$$i_t(u) := \sum_k \dim H_k(J^c, J^c \setminus \{u\}) t^k, \tag{2.3}$$

where $H_k(J^c, J^c \setminus \{u\})$ is the kth homology group of the couple $(J^c, J^c \setminus \{u\})$. If u is a nondegenerate critical point of J then $i_t(u) = t^{\mu(u)}$, where $\mu(u)$ is the (numerical) Morse index of u and it is given by the dimension of the maximal subspace on which the bilinear form $J''(u)[\cdot, \cdot]$ is negatively definite.

It is useful to recall the following result (see [33]).

Remark 2.3. Let X and Y be topological spaces. If $f : X \to Y$ and $g : Y \to X$ are continuous maps such that $g \circ f$ is homotopic to the identity map on X then $P_t(Y) = P_t(X) + Z(t)$, where $Z(t)$ is a polynomial with non negative coefficients.

Now, let us point out that the transformation $\tau = -I : M \to M$ induces a transformation on $\mathrm{H}^1_g(M)$. We define the linear operator τ^* as follows:

$$\tau^* : \mathrm{H}^1_g(M) \longrightarrow \mathrm{H}^1_g(M), \qquad \tau^*(u(x)) := -u(-x). \tag{2.4}$$

The operator τ^* is selfadjoint with respect to the following scalar product on $\mathrm{H}^1_g(M)$, which is equivalent to the usual one:

$$\langle u, v\rangle_\varepsilon := \frac{1}{\varepsilon^n}\int_M \left(\varepsilon^2 \nabla_g u \nabla_g v + uv\right) d\mu_g, \tag{2.5}$$

which induces the norm

$$\|u\|^2_\varepsilon := \frac{1}{\varepsilon^n}\int_M \left(\varepsilon^2 \left|\nabla_g u\right|^2 + u^2\right) d\mu_g. \tag{2.6}$$

In particular, we have

$$\|\tau^* u\|_{\varepsilon,p} = \|u\|_{\varepsilon,p}, \quad \|\tau^* u\|_\varepsilon = \|u\|_\varepsilon, \quad J_\varepsilon(\tau^* u) = J_\varepsilon(u). \tag{2.7}$$

Here

$$\|u\|^p_{\varepsilon,p} := \frac{1}{\varepsilon^n}\int_M |u|^p d\mu_g \tag{2.8}$$

denotes the norm in $\mathrm{L}^p(M)$, which is equivalent to the usual one. Therefore, in virtue of the Palais Principle, the nontrivial solutions of (1.5) are the critical points of the restriction of J_ε to the τ-invariant Nehari manifold

$$\mathcal{N}^\tau_\varepsilon := \{u \in \mathcal{N}_\varepsilon : u(-x) = -u(x)\} = \mathcal{N}_\varepsilon \cap H^\tau_g, \tag{2.9}$$

where $H^\tau_g := \{u \in \mathrm{H}^1_g(M) : u(-x) = -u(x)\}$.

In fact, since $J - \varepsilon(\tau^* u) = J_\varepsilon(u)$ and τ^* is a selfadjoint operator, we have

$$\left\langle \nabla J_\varepsilon(\tau^* u), \tau^* \varphi\right\rangle_\varepsilon = \left\langle \nabla J_\varepsilon(u), \varphi\right\rangle_\varepsilon \quad \forall \varphi \in \mathrm{H}^1_g(M) \tag{2.10}$$

and so $\nabla J_\varepsilon(u) = \tau^* \nabla J_\varepsilon(\tau^* u) = \tau^* \nabla J_\varepsilon(u)$ if $(\tau^* u)(x) = u(x) = -u(-x)$.

Let us set

$$m_\varepsilon := \inf_{\mathcal{N}_\varepsilon} J_\varepsilon, \quad m^\tau_\varepsilon := \inf_{\mathcal{N}^\tau_\varepsilon} J_\varepsilon \tag{2.11}$$

and let m_∞ be as in (1.6).

It is easy to verify that J_ε satisfies the Palais-Smale condition on $\mathcal{N}_\varepsilon^\tau$. Then, there exists v_ε minimizer of m_ε^τ and v_ε is a critical point of J_ε on $\mathrm{H}_g^1(M)$. Thus v_ε^+ and v_ε^- belong to $\mathcal{N}_\varepsilon$, then $m_\varepsilon^\tau = J_\varepsilon(v_\varepsilon) \geq 2m_\varepsilon$. We recall that $\lim_{\varepsilon\to 0} m_\varepsilon = m_\infty$ as it has been shown in [2, Remark 5.9].

It is well known that there exists a unique positive spherically symmetric (with respect to the origin) function $U \in \mathrm{H}^1(\mathbb{R}^n)$ minimizer of m_∞. Obviously this fact implies that $-\Delta U + U = U^{p-1}$ in $\mathbb{R}^n$ and for any $\varepsilon > 0$ we can define a family of functions $U_\varepsilon(x) := U(x/\varepsilon)$ satisfying the following equation $-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{p-1}$ in $\mathbb{R}^n$.

On the tangent bundle of any compact connected Riemannian manifold M, it is defined the exponential map $\exp : TM \to M$ which is a C^∞-map. Then for ρ sufficiently small (smaller than the injectivity radius of M) the manifold M possesses a special set of charts given by $\exp_x : B(0,\rho) \to B_g(x,\rho)$, where T_xM is identified with $\mathbb{R}^n$ for $x \in M$. Here $B(0,\rho)$ denotes the ball in $\mathbb{R}^n$ centered at 0 with radius ρ and $B_g(x,\rho)$ denotes the ball in M centered at x with radius ρ with the distance given by the metric g. The system of coordinates corresponding to those charts are called *normal coordinates*.

3. The Main Ingredient of the Proof

Let us sketch the proof of our main result.

Since $\lim_{\varepsilon\to 0} m_\varepsilon^\tau = 2m_\infty$ (see Lemma 4.3), given $\delta \in (0, m_\infty/4)$ for ε small enough, we have $0 < 2(m_\infty - \delta) < m_\varepsilon^\tau < 2(m_\infty + \delta)$. Thus $2(m_\infty - \delta)$ is not a critical value of J_ε for any ε. Fixed ε, if the number of critical points of J_ε is finite in $J_\varepsilon^{2(m_\infty+\delta)}$, we can choose δ such that $2(m_\infty + \delta)$ is not a critical value of J_ε.

Let $\mathcal{N}_\varepsilon^\tau/\mathbb{Z}_2$ be the set obtained by identifying antipodal points of the Nehari manifold $\mathcal{N}_\varepsilon^\tau$. It is easy to check that the set $\mathcal{N}_\varepsilon^\tau/\mathbb{Z}_2$ is homeomorphic to the projective space $P^\infty := \partial\Sigma_1/\mathbb{Z}_2$, which is obtained by identifying antipodal points in un unit sphere $\partial\Sigma_1$ in the space H_g^τ.

We are looking for pairs of nontrivial critical points $(u, -u)$ if the functional $J_\varepsilon : H_g^\tau \to \mathbb{R}$, that is we are searching critical points for the functional $\widetilde{J}_\varepsilon : H_g^\tau \setminus \{0\}/\mathbb{Z}_2 \to \mathbb{R}$ defined by $\widetilde{J}_\varepsilon([u]) := J_\varepsilon(u) = J_\varepsilon(-u)$. We use the same arguments as in [33]. The following relation can be proved as in [33, 34] (see [33, Lemma 5.2]):

$$P_t\left(\widetilde{J}_\varepsilon^{2(m_\infty+\delta)}, \widetilde{J}_\varepsilon^{2(m_\infty-\delta)}\right) = tP_t\left(\widetilde{J}_\varepsilon^{2(m_\infty+\delta)} \cap N_\varepsilon^\tau/\mathbb{Z}_2\right). \tag{3.1}$$

By Lemma 4.5 we deduce that

$$M/G \xrightarrow{\widetilde{\Phi}_\varepsilon} \widetilde{J}_\varepsilon^{2(m_\infty+\delta)} \cap \frac{\mathcal{N}_\varepsilon^\tau}{\mathbb{Z}_2} \xrightarrow{\widetilde{\beta}} \frac{M_d}{G}, \tag{3.2}$$

where $\widetilde{\beta} \circ \widetilde{\Phi}_\varepsilon$ is homotopic to the identity map and M_d/G is homotopically equivalent to M_g. Therefore by Remark 2.3 we get

$$P_t\left(\widetilde{J}_\varepsilon^{2(m_\infty+\delta)} \cap \frac{\mathcal{N}_\varepsilon^\tau}{\mathbb{Z}_2}\right) = P_t\left(\frac{M}{G}\right) + Z(t), \tag{3.3}$$

where $Z(t)$ is a polynomial with nonnegative integer coefficients.

By our assumption we have that for ε small enough all the critical points u such that $\widetilde{J}_\varepsilon(u) < 2(m_\infty + \delta)$ are nondegenerate. Moreover the functional $\widetilde{J}_\varepsilon$ satisfies the Palais-Smale condition. Then by Morse theory and relations (3.1) and (3.3) we get at least $P_1(M/G)$ pairs $(u, -u)$ of nontrivial solutions for (1.5). By Remark (4.7) these solutions change sign exactly once. That concludes the proof of Theorem 1.1.

Remark 3.1. By [33, Lemma 5.2] we deduce that

$$P_t\left(H_g^\tau \setminus \frac{\{0\}}{\mathbb{Z}_2}, \widetilde{J}_\varepsilon^{2(m_\infty-\delta)}\right) = tP_t\left(\frac{\mathcal{N}_\varepsilon^\tau}{\mathbb{Z}_2}\right). \tag{3.4}$$

Since P^∞ is homeomorphic to $\mathcal{N}_\varepsilon^\tau/\mathbb{Z}_2$ we get $P_t(\mathcal{N}_\varepsilon^\tau/\mathbb{Z}_2) = P_t(P^\infty)$. Provided the homology is evaluated with $\mathfrak{Z}_2$-coefficients (see, e.g., [35, Theorem 7.4]), we have $P_1(P^\infty) = +\infty$. Then, if all the critical points are nondegenerate, we get infinitely many pairs $(u, -u)$ of nontrivial solutions for (1.5).

4. Technical Results

Let χ_r be a smooth cut-off function such that

$$\chi_r(z) = 1 \quad \text{if } z \in B\left(0, \frac{r}{2}\right), \qquad \chi_r(z) = 0 \quad \text{if } z \in \mathbb{R}^N \setminus B(0, r), \qquad \left|\nabla \chi_r(z)\right| \le 2 \quad \forall z \in \mathbb{R}^N. \tag{4.1}$$

Fixing a point $q \in M$ and $\varepsilon > 0$, let us define the function $w_{\varepsilon,q}$ on M as

$$w_{\varepsilon,q}(x) := U_\varepsilon\left(\exp_q^{-1}(x)\right)\chi_r\left(\exp_q^{-1}(x)\right) \quad \text{if } x \in B_g(q, r) \quad w_{\varepsilon,q}(x) := 0 \text{ otherwise.} \tag{4.2}$$

We choose r smaller than the injectivity radius of M and such that $B_g(q, r) \cap B_g(-q, r) = \emptyset$ for any $q \in M$. For any $\varepsilon > 0$ we can define a positive number $t(w_{\varepsilon,q})$ such that

$$\Phi_\varepsilon(q) := t(w_{\varepsilon,q})w_{\varepsilon,q} \in \mathrm{H}_g^1(M) \cap \mathcal{N}_\varepsilon \quad \text{for any } q \in M. \tag{4.3}$$

Namely, $t(w_{\varepsilon,q})$ verifies

$$t(w_{\varepsilon,q}) = \left[\frac{\int_M \left(\varepsilon^2 \left|\nabla_g w_{\varepsilon,q}\right|^2 + w_{\varepsilon,q}^2\right) d\mu_g}{\int_M w_{\varepsilon,q}^2 d\mu_g}\right]^{1/p-2}. \tag{4.4}$$

In [2, Proposition 4.2] the following lemma has been proved.

Lemma 4.1. *Given $\varepsilon > 0$ the map $\Phi_\varepsilon : M \to H_g^1(M) \cap \mathcal{N}_\varepsilon$ is continuous. Moreover, given $\delta > 0$ there exists $\varepsilon_0(\delta)$ such that if $\varepsilon \in (0, \varepsilon_0(\delta))$ then $\Phi_\varepsilon(q) \in \mathcal{N}_\varepsilon \cap J_\varepsilon^{m_\infty+\delta}$.*

Now, fixing a point $q \in M$ let us define the function

$$\Phi_\varepsilon^\tau(q) := t(w_{\varepsilon,q})w_{\varepsilon,q} - t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q}. \tag{4.5}$$

It holds that

$$\int_M |w_{\varepsilon,q}|^2 p = \int_M |w_{\varepsilon,\tau q}|^2 p, \qquad \int_M |\nabla_g w_{\varepsilon,q}|^2 d\mu_g = \int_M |\nabla_g w_{\varepsilon,\tau q}|^2 d\mu_g. \tag{4.6}$$

By (4.4) and (4.6), we deduce

$$t(w_{\varepsilon,q}) = t(w_{\varepsilon,\tau q}). \tag{4.7}$$

The proof of the next results follows the same arguments as in [8].

Lemma 4.2. *Given $\varepsilon > 0$ the map $\Phi_\varepsilon^\tau : M \to H_g^1(M) \cap \mathcal{N}_\varepsilon^\tau$ is continuous. Moreover, given $\delta > 0$ there exists $\varepsilon_0(\delta)$ such that if $\varepsilon \in (0, \varepsilon_0(\delta))$ then $\Phi_\varepsilon^\tau(q) \in \mathcal{N}_\varepsilon^\tau \cap J_\varepsilon^{2(m_\infty+\delta)}$.*

Proof. Since $U_\varepsilon \chi_r$ is a radially symmetric function, we set $\widetilde{U}_\varepsilon(|z|) := U_\varepsilon(z)\chi_r(z)$. Moreover, since we have

$$\begin{aligned} \left|\exp_{\tau q}^{-1}(\tau x)\right| &= d_g(-x,-q) = d_g(x,q) = \left|\exp_q^{-1}(x)\right|, \\ \left|\exp_q^{-1}(\tau x)\right| &= d_g(-x,q) = d_g(x,-q) = \left|\exp_{\tau q}^{-1}(x)\right|, \end{aligned} \tag{4.8}$$

we get

$$\tau^*\Phi_\varepsilon^\tau(q)(x) \tag{4.9}$$

$$= -t(w_{\varepsilon,q})w_{\varepsilon,q}(-x) + t(w_{\varepsilon,\tau q})w_{\varepsilon,\tau q}(-x) \tag{4.10}$$

$$= -t(w_{\varepsilon,q})\widetilde{U}_\varepsilon\left(\left|\exp_q^{-1}(-x)\right|\right) + t(w_{\varepsilon,\tau q})\widetilde{U}_\varepsilon\left(\left|\exp_q^{-1}(-x)\right|\right)$$

$$= t(w_{\varepsilon,\tau q})\widetilde{U}_\varepsilon\left(\left|\exp_q^{-1}(x)\right|\right) - t(w_{\varepsilon,q})\widetilde{U}_\varepsilon\left(\left|\exp_q^{-1}(\tau x)\right|\right) \tag{4.11}$$

$$= t(w_{\varepsilon,q})\widetilde{U}_\varepsilon\left(\left|\exp_q^{-1}(x)\right|\right) - t(w_{\varepsilon,q})\widetilde{U}_\varepsilon\left(\left|\exp_q^{-1}(x)\right|\right)$$

$$= \Phi_\varepsilon^\tau(q)(x), \tag{4.12}$$

because by (4.7) we have $t(w_{\varepsilon,q}) = t(w_{\varepsilon,\tau q})$. Hence $\Phi_\varepsilon^\tau(q) \in \mathcal{N}_\varepsilon^\tau$.

To get that $\Phi_\varepsilon^\tau(q) \in J_\varepsilon^{2(m_\infty+\delta)}$, it is enough to prove that $J_\varepsilon(\Phi_\varepsilon^\tau(q)) = 2J_\varepsilon(\Phi_\varepsilon(q))$, because by Lemma 4.1 the statement will follow. Since the support of the function $\Phi_\varepsilon^\tau(q)$ is $B_g(q,r) \cup B_g(-q,r)$ and $B_g(q,r) \cap B_g(-q,r) = \emptyset$, by (4.6) and the definition of the function Φ_ε^τ, we get

$$\begin{aligned} J_\varepsilon(\Phi_\varepsilon^\tau(q)) &= \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_M \left|\Phi_\varepsilon^\tau(q)\right|^p d\mu_g \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\left(\int_{B_g(q,r)} \left|\Phi_\varepsilon(q)\right|^p d\mu_g + \int_{B_g(-q,r)} \left|\Phi_\varepsilon(\tau q)\right|^p d\mu_g\right) \\ &= 2J_\varepsilon(\Phi_\varepsilon(q)). \end{aligned} \tag{4.13}$$

That concludes the proof. □

Lemma 4.3. *One has that* $\lim_{\varepsilon\to 0} m_\varepsilon^\tau = 2m_\infty$.

Proof. By Lemma 4.2 and (4.12) we have that for any $\delta > 0$ there exists $\varepsilon_0(\delta)$ such that for any $\varepsilon \in (0, \varepsilon_0(\delta))$ it holds that

$$2m_\varepsilon \le m_\varepsilon^\tau \le J_\varepsilon(\Phi_\varepsilon^\tau(q)) = 2J_\varepsilon(\Phi_\varepsilon(q)) \le 2(m_\infty + \delta). \tag{4.14}$$

Since $\lim_{\varepsilon\to 0} m_\varepsilon = 2m_\infty$ (see [2, Remark 5.9]) we get the claim. □

For any function $u \in \mathcal{N}_\varepsilon^\tau$ we can define a point $\beta(u) \in \mathbb{R}^N$ by

$$\beta(u) := \frac{\int_M x|u^+(x)|^p d\mu_g}{\int_M |u^+(x)|^p d\mu_g}. \tag{4.15}$$

Lemma 4.4. *There exists $\delta_0 > 0$ such that for any $\delta \in (0, \delta_0)$, for any $\varepsilon \in (0, \varepsilon_0(\delta))$ (as in Lemma 4.2), and for any function $u \in \mathcal{N}_\varepsilon^\tau \cap J_\varepsilon^{2(m_\infty+\delta)}$, it holds that $\beta(u) \in M_d$, where $M_d := \{x \in \mathbb{R}^N : d(x, M) < d\}$.*

Proof. Let $u \in \mathcal{N}_\varepsilon^\tau \cap J_\varepsilon^{2(m_\infty+\delta)}$. Since $u(x) = -u(-x)$ we set $M^+ := \{x \in M : u(x) > 0\}$ and $M^- := \{x \in M : u(x) < 0\}$. It is easy to see that $M^+ = \{-x : x \in M^-\}$. Then we have

$$\begin{aligned} J_\varepsilon(u) &= \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\int_M |u|^p d\mu_g \\ &= \left(\frac{1}{2} - \frac{1}{p}\right)\frac{1}{\varepsilon^n}\left(\int_{M^+} |u^+|^p d\mu_g + \int_{M^-} \left|u^-\right|^p d\mu_g\right) = 2J_\varepsilon(u^+). \end{aligned} \tag{4.16}$$

Since $J_\varepsilon(u) \le 2(m_\infty + \delta)$, we have $J_\varepsilon(u^+) \le m_\infty + \delta$ and by [2, Proposition 5.10] we get the claim. □

It is easy to check that $\Phi_\varepsilon^\tau(-q) = -\phi_\varepsilon^\tau(q)$ and $\beta(-u) = -\beta(u)$. Moreover, by Lemmas 4.1 and 4.2, we can define a map $\widetilde{\Phi}_\varepsilon : M/G \to \widetilde{J}_\varepsilon^{2(m_\infty+\delta)} \cap \mathcal{N}_\varepsilon^\tau/\mathbb{Z}_2$ by

$$\widetilde{\Phi}_\varepsilon([q]) := [\Phi_\varepsilon^\tau(q)] = \{\Phi_\varepsilon^\tau(q), \Phi_\varepsilon^\tau(-q)\}. \tag{4.17}$$

By Lemma 4.4 we can define a map $\widetilde{\beta} : \widetilde{J}_\varepsilon^{2(m_\infty+\delta)} \cap \mathcal{N}_\varepsilon^\tau/\mathbb{Z}_2 \to M_d/G$ by

$$\widetilde{\beta}([u]) := [\beta(u)] = \{\beta(u), \beta(-u)\}. \tag{4.18}$$

Lemma 4.5. *There exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ the map*

$$I_\varepsilon := \widetilde{\beta} \circ \widetilde{\Phi}_\varepsilon^\tau : \frac{M}{G} \longrightarrow \frac{M_d}{G} \tag{4.19}$$

is well defined, continuous, and homotopic to the identity map.

Proof. By Lemmas 4.2 and 4.4, I_ε is well defined. In order to show that I_ε is homotopic to the identity, we estimate the following difference:

$$\begin{aligned}
\left|\beta\Phi_\varepsilon^\tau(q) - q\right| &= \frac{\int_M (x-q)\left|(\Phi_\varepsilon^\tau(q))^+\right|^p d\mu_g}{\int_M \left|(\Phi_\varepsilon^\tau(q))^+\right|^p d\mu_g} \\
&= \frac{\int_{B(0,r)} y\left|U(y/\varepsilon)\chi_r(|y|)\right|^p \left|g_q(y)\right|^{1/2} dy}{\int_{B(0,r)} \left|U(y/\varepsilon)\chi_r(|y|)\right|^p \left|g_q(y)\right|^{1/2} dy} \\
&= \frac{\varepsilon\int_{B(0,r/\varepsilon)} z\left|U(z)\chi_r(|\varepsilon z|)\right|^p \left|g_q(\varepsilon z)\right|^{1/2} d\mu_g}{\int_{B(0,r/\varepsilon)} \left|U(z)\chi_r(|\varepsilon z|)\right|^p \left|g_q(\varepsilon z)\right|^{1/2} d\mu_g}.
\end{aligned} \tag{4.20}$$

Hence $|\beta\Phi_\varepsilon^\tau(q) - q|$, $|\beta\Phi_\varepsilon^\tau(-q) + q| \le c\varepsilon$, because $\beta\Phi_\varepsilon^\tau(-q) = -\beta\Phi_\varepsilon^\tau(q)$, for a constant c which does not depend on the point q. Therefore $|I_\varepsilon(q) - q| < c\varepsilon$; that concludes the proof. □

Remark 4.6. We have only to prove that any solution u of (1.5) such that $J_\varepsilon(u) < 2(m_\infty + \delta)$ changes sign exactly once. In fact, assume that the set $\{u \in M : u(x) > 0\}$ has h connected components $M_1, \ldots, M_h$. Set $u_i(x) := u(x)$ if $x \in M_i \cup (-M_i)$ and $u_i(x) := 0$ otherwise. We have $u_i \in \mathcal{N}_\varepsilon^\tau$ and

$$\frac{3}{2} h m_\infty \le m_\varepsilon^\tau \le J_\varepsilon(u) = \sum_{i=1}^h J_\varepsilon(u_i) \le 2(m_\infty + \delta) < 3m_\infty. \tag{4.21}$$

Then $h = 1$. This concludes the proof.

References

[1] J. Byeon and J. Park, "Singularly perturbed nonlinear elliptic problems on manifolds," *Calculus of Variations and Partial Differential Equations*, vol. 24, no. 4, pp. 459–477, 2005.

[2] V. Benci, C. Bonanno, and A. M. Micheletti, "On the multiplicity of solutions of a nonlinear elliptic problem on Riemannian manifolds," *Journal of Functional Analysis*, vol. 252, no. 2, pp. 464–489, 2007.

[3] N. Hirano, "Multiple existence of solutions for a nonlinear elliptic problem on a Riemannian manifold," *Nonlinear Analysis. Theory, Methods & Applications*, vol. 70, no. 2, pp. 671–692, 2009.

[4] D. Visetti, "Multiplicity of solutions of a zero mass nonlinear equation on a Riemannian manifold," *Journal of Differential Equations*, vol. 245, no. 9, pp. 2397–2439, 2008.

[5] E. N. Dancer, A. M. Micheletti, and A. Pistoia, "Multipeak solutions for some singularly perturbed nonlinear elliptic problems on Riemannian manifolds," *Manuscripta Mathematica*, vol. 128, no. 2, pp. 163–193, 2009.

[6] A. M. Micheletti and A. Pistoia, "The role of the scalar curvature in a nonlinear elliptic problem on Riemannian manifolds," *Calculus of Variations and Partial Differential Equations*, vol. 34, no. 2, pp. 233–265, 2009.

[7] A. M. Micheletti and A. Pistoia, "Nodal solutions for a singularly perturbed nonlinear elliptic problem on Riemannian manifolds," *Advanced Nonlinear Studies*, vol. 9, no. 3, pp. 565–577, 2009.

[8] M. Ghimenti and A. M. Micheletti, "On the number of nodal solutions for a nonlinear elliptic problem on symmetric Riemannian manifolds," to appear in *Electronic Journal of Differential Equations*.

[9] V. Benci and G. Cerami, "Positive solutions of some nonlinear elliptic problems in exterior domains," *Archive for Rational Mechanics and Analysis*, vol. 99, no. 4, pp. 283–300, 1987.

[10] D. Cao, N. E. Dancer, E. S. Noussair, and S. Yan, "On the existence and profile of multi-peaked solutions to singularly perturbed semilinear Dirichlet problems," *Discrete and Continuous Dynamical Systems*, vol. 2, no. 2, pp. 221–236, 1996.

[11] E. N. Dancer and S. Yan, "Effect of the domain geometry on the existence of multipeak solutions for an elliptic problem," *Topological Methods in Nonlinear Analysis*, vol. 14, no. 1, pp. 1–38, 1999.

[12] E. N. Dancer and S. Yan, "A singularly perturbed elliptic problem in bounded domains with nontrivial topology," *Advances in Differential Equations*, vol. 4, no. 3, pp. 347–368, 1999.

[13] E. N. Dancer and J. Wei, "On the effect of domain topology in a singular perturbation problem," *Topological Methods in Nonlinear Analysis*, vol. 11, no. 2, pp. 227–248, 1998.

[14] M. del Pino, P. L. Felmer, and J. Wei, "Multi-peak solutions for some singular perturbation problems," *Calculus of Variations and Partial Differential Equations*, vol. 10, no. 2, pp. 119–134, 2000.

[15] M. del Pino, P. L. Felmer, and J. Wei, "On the role of distance function in some singular perturbation problems," *Communications in Partial Differential Equations*, vol. 25, no. 1-2, pp. 155–177, 2000.

[16] M. Grossi and A. Pistoia, "On the effect of critical points of distance function in superlinear elliptic problems," *Advances in Differential Equations*, vol. 5, no. 10–12, pp. 1397–1420, 2000.

[17] Y. Y. Li and L. Nirenberg, "The Dirichlet problem for singularly perturbed elliptic equations," *Communications on Pure and Applied Mathematics*, vol. 51, no. 11-12, pp. 1445–1490, 1998.

[18] W.-M. Ni and J. Wei, "On the location and profile of spike-layer solutions to singularly perturbed semilinear Dirichlet problems," *Communications on Pure and Applied Mathematics*, vol. 48, no. 7, pp. 731–768, 1995.

[19] J. Wei, "On the interior spike solutions for some singular perturbation problems," *Proceedings of the Royal Society of Edinburgh. Section A*, vol. 128, no. 4, pp. 849–874, 1998.

[20] G. Cerami and J. Wei, "Multiplicity of multiple interior peak solutions for some singularly perturbed Neumann problems," *International Mathematics Research Notices*, no. 12, pp. 601–626, 1998.

[21] E. N. Dancer and S. Yan, "Multipeak solutions for a singularly perturbed Neumann problem," *Pacific Journal of Mathematics*, vol. 189, no. 2, pp. 241–262, 1999.

[22] M. del Pino, P. L. Felmer, and J. Wei, "On the role of mean curvature in some singularly perturbed Neumann problems," *SIAM Journal on Mathematical Analysis*, vol. 31, no. 1, pp. 63–79, 1999.

[23] M. Grossi, A. Pistoia, and J. Wei, "Existence of multipeak solutions for a semilinear Neumann problem via nonsmooth critical point theory," *Calculus of Variations and Partial Differential Equations*, vol. 11, no. 2, pp. 143–175, 2000.

[24] C. Gui, "Multipeak solutions for a semilinear Neumann problem," *Duke Mathematical Journal*, vol. 84, no. 3, pp. 739–769, 1996.

[25] C. Gui and J. Wei, "Multiple interior peak solutions for some singularly perturbed Neumann problems," *Journal of Differential Equations*, vol. 158, no. 1, pp. 1–27, 1999.

[26] C. Gui and J. Wei, "On multiple mixed interior and boundary peak solutions for some singularly perturbed Neumann problems," *Canadian Journal of Mathematics*, vol. 52, no. 3, pp. 522–538, 2000.

[27] C. Gui, J. Wei, and M. Winter, "Multiple boundary peak solutions for some singularly perturbed Neumann problems," *Annales de l'Institut Henri Poincaré*, vol. 17, no. 1, pp. 47–82, 2000.

[28] Y. Y. Li, "On a singularly perturbed equation with Neumann boundary condition," *Communications in Partial Differential Equations*, vol. 23, no. 3-4, pp. 487–545, 1998.

[29] W.-M. Ni and I. Takagi, "Locating the peaks of least-energy solutions to a semilinear Neumann problem," *Duke Mathematical Journal*, vol. 70, no. 2, pp. 247–281, 1993.

[30] W.-M. Ni and I. Takagi, "On the shape of least-energy solutions to a semilinear Neumann problem," *Communications on Pure and Applied Mathematics*, vol. 44, no. 7, pp. 819–851, 1991.

[31] J. Wei, "On the boundary spike layer solutions to a singularly perturbed Neumann problem," *Journal of Differential Equations*, vol. 134, no. 1, pp. 104–133, 1997.

[32] J. Wei, "On the interior spike layer solutions to a singularly perturbed Neumann problem," *The Tôhoku Mathematical Journal*, vol. 50, no. 2, pp. 159–178, 1998.

[33] V. Benci and G. Cerami, "Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology," *Calculus of Variations and Partial Differential Equations*, vol. 2, no. 1, pp. 29–48, 1994.

[34] V. Benci, "Introduction to Morse theory: a new approach," in *Topological Nonlinear Analysis*, vol. 15 of *Progress in Nonlinear Differential Equations and Their Applications*, pp. 37–177, Birkhäuser, Boston, Mass, USA, 1995.

[35] G. W. Whitehead, *Elements of Homotopy Theory*, vol. 61 of *Graduate Texts in Mathematics*, Springer, New York, NY, USA, 1978.