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Research Article

Stochastic Finite Element Technique for Stochastic One-Dimension Time-Dependent Differential Equations with Random Coefficients

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The stochastic finite element method (SFEM) is employed for solving stochastic one-dimension time-dependent differential equations with random coefficients. SFEM is used to have a fixed form of linear algebraic equations for polynomial chaos coefficients of the solution process. Four fixed forms are obtained in the cases of stochastic heat equation with stochastic heat capacity or heat conductivity coefficients and stochastic wave equation with stochastic mass density or elastic modulus coefficients. The relation between the exact deterministic solution and the mean of solution process is numerically studied.

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1. Introduction

The objective of solving a stochastic differential equation is to obtain the p.d.f. and the different moments of the solution process. This can be achieved through many methods and techniques, for example the stochastic averaging [1–3], stochastic linearization [4–6], Adomian's decomposition method [7, 8], and stochastic finite element method [9–12].

In this paper, SFEM is applied on stochastic heat and wave equations. The stochastic coefficients are decomposed by Karhunen-Loeve (K-L) expansion. The obtained set of ordinary differential equations is solved using the θ-dependent family. Then the solution process at every time step is projected on two-dimension first-order polynomial chaos. The mean of the solution process is obtained under different values of the point variance of stochastic coefficient.

2. The Karhunen-Loeve decomposition

The use of K-L expansion with orthogonal deterministic basis functions and uncorrelated random coefficients gained interest because of its biorthogonal property, that is, both the deterministic basis functions and the corresponding random coefficients are orthogonal. Let $\overline{\omega}(x)$ denote the mean value of $\omega(x,\theta)$ and $C(x_1,x_2)$ its covariance function which is bounded and positive definite. It has spectral decomposition as [13]

$$C(x_1,x_2) = \sum_{n=1}^{\infty} \lambda_n f_n(x_1) f_n(x_2), \tag{2.1}$$

where λ_n and $f_n(x)$ are the eigenvalues and the eigenvectors of the covariance kernel, respectively. They are the solutions of the homogeneous Fredholm integral equation of second kind given by

$$\int_D C(x_1,x_2) f_n(x_1)\,dx_1 = \lambda_n f_n(x_2). \tag{2.2}$$

Clearly, $\omega(x,\phi)$ can be written as

$$\omega(x,\phi) = \overline{\omega}(x) + \alpha(x,\phi), \tag{2.3}$$

where $\alpha(x,\phi)$ is a process with zero mean and covariance function $C(x_1,x_2)$. Finally, the K-L decomposition of the field $\alpha(x,\phi)$ is given by

$$\alpha(x,\phi) = \sum_{n=1}^{\infty} \zeta_n(\phi)\sqrt{\lambda_n} f_n(x), \tag{2.4}$$

where $\zeta_n(\phi)$ is a set of uncorrelated random variables. For example, consider a homogeneous Gaussian process with exponential covariance

$$C(x_1,x_2) = \sigma^2 e^{-|x_1-x_2|}, \quad x \in [0,1]. \tag{2.5}$$

The eigenfunctions of this covariance kernel are

$$f_i(x) = \frac{\omega_i \operatorname{Cos}(\omega_i x) + \operatorname{Sin}(\omega_i x)}{\sqrt{(\omega^2/2)(1+\operatorname{Sin}(2\omega)/2\omega) + (1/2)(1-\operatorname{Sin}(2\omega)/2\omega) + \operatorname{Sin}^2(\omega)}}, \tag{2.6}$$

where ω_i is the solution of nonlinear equation

$$2\omega \operatorname{Cos}(\omega) + (1-\omega^2)\operatorname{Sin}(\omega) = 0. \tag{2.7}$$

Then the eigenvalues can be evaluated from the relation

$$\omega_i^2 = \frac{2-\lambda_i}{\lambda_i}\sigma^2. \tag{2.8}$$

3. The polynomial chaos

The polynomial chaos is a particular basis of the random variables space based on Hermite polynomial of independent standard random variables $\zeta_1, \zeta_2, \ldots, \zeta_\infty$. Classically, the one-dimension Hermite polynomial is defined by

$$h_n(x) = (-1)^n \frac{d^n\left(e^{-(1/2)x^2}\right)}{dx^n} e^{(1/2)x^2}. \tag{3.1}$$

The multivariable Hermite polynomial can be defined as tensor product of Hermite polynomial. Consider the multi-index

$$\alpha = \{\alpha_1, \ldots, \alpha_m\} \quad \alpha_i \geq 0,\ i = 1, \ldots, m. \tag{3.2}$$

The multivariable Hermite polynomial associated with this sequence is

$$H_\alpha = \prod_{i=1}^{M} h_{\alpha_i}(\zeta_i). \tag{3.3}$$

Finally, any random variable $k(\phi)$ with finite variance can be expressed as [13]

$$k(\phi) = \sum_{i=0}^{\infty} a_i H_i(\zeta), \tag{3.4}$$

where a_i are deterministic constants and H_i are enumeration of the H_α. The expansion is convergent in the mean square sense. In the application of polynomial chaos, the dimension is selected according to the number of random variables in K-L expansion [14].

4. Stochastic heat equation

The unsteady stochastic heat equation for a spatially varying medium, in the absence of convection, is [15]

$$c\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right) + DU = f(x,t), \quad 0 \leq x \leq L \tag{4.1}$$

subjected to the following boundary and initial conditions

$$U(0,t) = u_0, \qquad A\frac{\partial U}{\partial x}\bigg|_{x=L} = p(t), \tag{4.2}$$

$$U(x,0) = g(x). \tag{4.3}$$

The variational form of (4.1) over a typical element is

$$\int_{x_A}^{x_B} \left(c\frac{\partial U}{\partial t}V + A\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + DUV\right)dx - Q_1^{(e)}V(x_A) - Q_2^{(e)}V(x_B) = \int_{x_A}^{x_B} f(x,t)V\,dx, \tag{4.4}$$

where

$$-A\frac{\partial U}{\partial x}\bigg|_{x=x_A} = Q_1^{(e)}, \qquad A\frac{\partial U}{\partial x}\bigg|_{x=x_B} = Q_2^{(e)}. \tag{4.5}$$

Let the approximation of solution over the element be given by

$$U_e(x,t) = \sum_{i=1}^{2} U_i(t)\psi_i(x), \tag{4.6}$$

where $\{\psi_i\}$ are linear interpolating functions. Substituting by (4.6) into (4.4) and by $V = \psi_j(x)$, we get

$$\begin{aligned}\sum_{i=1}^{2} \dot{U}_i(t) \int_{x_A}^{x_B} c\psi_i(x)\psi_j(x)dx + \sum_{i=1}^{2} U_i(t) \int_{x_A}^{x_B} \left(A\psi_i'(x)\psi_j'(x) + D\psi_i(x)\psi_j(x)\right)dx \\ = \int_{x_A}^{x_B} f(x,t)\psi_j(x)dx + Q_j^{(e)}.\end{aligned} \tag{4.7}$$

Let

$$C_{ij}^{(e)} = \int_{x_A}^{x_B} c\psi_i(x)\psi_j(x)dx, \tag{4.8}$$

$$K_{ij}^{(e)} = \int_{x_A}^{x_B} \left(A\psi_i'(x)\psi_j'(x) + D\psi_i(x)\psi_j(x)\right)dx, \tag{4.9}$$

$$F_i^{(e)} = \int_{x_A}^{x_B} f(x,t)\psi_i(x)dx + Q_i^{(e)}. \tag{4.10}$$

Assembling of the elements matrices, we obtain

$$C\dot{U} + KU = F. \tag{4.11}$$

4.1. Case 1. Stochastic heat capacity coefficient. Let the heat capacity coefficient be stochastic process; the two-dimension K-L expansion for that process is

$$c = \bar{c}(x) + \zeta_1\sqrt{\lambda_1}f_1(x) + \zeta_2\sqrt{\lambda_2}f_2(x). \tag{4.12}$$

Equation (4.8) will be divided into the following parts:

$$\begin{aligned} C_{ij}{}_{(0)}^{(e)} &= \int_{x_A}^{x_B} \bar{c}\psi_i(x)\psi_j(x)dx, \\ C_{ij}{}_{(1)}^{(e)} &= \int_{x_A}^{x_B} \sqrt{\lambda_1}f_1(x)\psi_i(x)\psi_j(x)dx, \\ C_{ij}{}_{(2)}^{(e)} &= \int_{x_A}^{x_B} \sqrt{\lambda_2}f_2(x)\psi_i(x)\psi_j(x)dx. \end{aligned} \tag{4.13}$$

Hence, (4.11) will be in the form

$$\left(C_0 + \zeta_1 C_1 + \zeta_2 C_2\right)\dot{U} + KU = F. \tag{4.14}$$

The previous equation is time-dependent system of ordinary differential equations which can be approximated to obtain a system of algebraic equations. Using the θ family of approximation which approximates a weighted average of time derivative of a dependent variable at two consecutive time steps [16], that is,

$$\theta\{\dot{U}\}_{n+1}+(1-\theta)\{\dot{U}\}_{n}=\frac{\{U\}_{n+1}-\{U\}_{n}}{\Delta t},\qquad 0\le\theta\le 1,\tag{4.15}$$

where $\{\cdot\}_n$ refers to the value of the enclosed vector quantity at time $t=t_n$ and $\Delta t = t_{n+1}-t_n$ is the time step. From (4.15), we can obtain a number of well-known difference schemes by choosing different values of θ like

$$\theta=\begin{cases}0 & \text{forward Euler scheme,}\\ \dfrac{1}{2} & \text{Crank-Niclson scheme,}\\ \dfrac{2}{3} & \text{Galerkin method,}\\ 1 & \text{backward Euler scheme.}\end{cases}\tag{4.16}$$

Applying the time approximation (4.15) on (4.14) and rearranging the terms to write $\{U\}_{n+1}$ in terms of $\{U\}_n$, we get

$$\begin{aligned}&\left(C_0+\zeta_1C_1+\zeta_2C_2+\theta K\Delta t\right)\{U\}_{n+1}\\&\quad=\left(C_0+\zeta_1C_1+\zeta_2C_2-(1-\theta)K\Delta t\right)\{U\}_{n}+\Delta t\left(\theta\{F\}_{n+1}+(1-\theta)\{F\}_{n}\right).\end{aligned}\tag{4.17}$$

Projecting the solution at every time step on two-dimension first-order chaos polynomial, we get

$$\begin{aligned}\{U\}_{n+1}&=\{a_0\}_{n+1}H_0+\{a_1\}_{n+1}H_1+\{a_2\}_{n+1}H_2,\\ \{U\}_{n}&=\{a_0\}_{n}H_0+\{a_1\}_{n}H_1+\{a_2\}_{n}H_2.\end{aligned}\tag{4.18}$$

Substituting by (4.18) into (4.17) and making inner product with (H_i), $i=0,1,2$

$$\begin{aligned}&\begin{bmatrix}C_0+\theta K\Delta t & C_1 & C_2\\ C_1 & C_0+\theta K\Delta t & 0\\ C_2 & 0 & C_0+\theta K\Delta t\end{bmatrix}\begin{bmatrix}\{a_0\}_{n+1}\\ \{a_1\}_{n+1}\\ \{a_2\}_{n+1}\end{bmatrix}\\&\quad=\Delta t\begin{bmatrix}\theta\{F\}_{n+1}+(1-\theta)\{F\}_{n}\\ \{0\}\\ \{0\}\end{bmatrix}\\&\qquad+\begin{bmatrix}C_0-(1-\theta)K\Delta t & C_1 & C_2\\ C_1 & C_0-(1-\theta)K\Delta t & 0\\ C_2 & 0 & C_0-(1-\theta)K\Delta t\end{bmatrix}\begin{bmatrix}\{a_0\}_{n}\\ \{a_1\}_{n}\\ \{a_2\}_{n}\end{bmatrix}.\end{aligned}\tag{4.19}$$

Equation (4.19) is used to obtain polynomial chaos coefficients of the solution process over all nodes of the domain at every time step. To complete the applicability of this form, we project the initial condition (4.3) on polynomial chaos basis to get the coefficients $\{a_i\}_0$ at $t = 0$. Hence [17],

$$\begin{gathered}\{a_0\}_0 = \{g(x_j)\}, \quad j = 1,2,3,\dots,N+1, \\ \{a_1\}_0 = \{a_2\}_0 = \{0\}.\end{gathered} \tag{4.20}$$

The coefficients $\{a_1\}_0$ and $\{a_2\}_0$ are assumed to be zero because of deterministic initial condition. Therefore, (4.19) can be used to obtain the polynomial chaos coefficients of the solution $\{a_i\}_1$ at time $t = t_1$. Finally, the polynomial chaos coefficients at time $t = t_{n+1}$ can be obtained in terms of the known coefficients at time $t = t_n$. At this stage, the polynomial chaos coefficients of the solution are known for each node at every time step. Finally, the mean and variance of the solution process can be obtained from the relations

$$\begin{aligned}E\{U(x,\theta)\} &= \{a_0\} \text{ (mean)}, \\ \operatorname{Var}\{U(x,\theta)\} &= \{a_1^2\} + \{a_2^2\} \text{ (variance)}.\end{aligned} \tag{4.21}$$

4.2. Case 2. Stochastic heat conductivity coefficient. Let the heat conductivity coefficient be stochastic process with two-dimension K-L expansion in the form

$$A = \overline{A}(x) + \zeta_1\sqrt{\lambda_1} f_1(x) + \zeta_2\sqrt{\lambda_2} f_2(x). \tag{4.22}$$

Equation (4.9) will be divided into the following parts:

$$\begin{aligned}K_{ij(0)}^{(e)} &= \int_{x_A}^{x_B} \left(\overline{A}(x)\psi_i'(x)\psi_j'(x) + D\psi_i(x)\psi_j(x)\right) dx, \\ K_{ij(1)}^{(e)} &= \int_{x_A}^{x_B} \left(\sqrt{\lambda_1} f_1(x)\psi_i'(x)\psi_j'(x)\right) dx, \\ K_{ij(2)}^{(e)} &= \int_{x_A}^{x_B} \left(\sqrt{\lambda_2} f_2(x)\psi_i'(x)\psi_j'(x)\right) dx.\end{aligned} \tag{4.23}$$

Then (4.11) becomes

$$C\dot{U} + (K_0 + \zeta_1 K_1 + \zeta_2 K_2)U = F. \tag{4.24}$$

Applying the time approximation (4.15) on (4.24), we obtain

$$(C+\theta\Delta t(K_0+\zeta_1K_1+\zeta_2K_2))\{U\}_{n+1} = (C-(1-\theta)\Delta t(K_0+\zeta_1K_1+\zeta_2K_2))\{U\}_n + \Delta t(\theta\{F\}_{n+1}+(1-\theta)\{F\}_n). \tag{4.25}$$

Substituting by (4.18) into (4.25), we get

$$\begin{aligned}
&\begin{bmatrix} C+\theta K_0\Delta t & \theta K_1\Delta t & \theta K_2\Delta t \\ \theta K_1\Delta t & C+\theta K_0\Delta t & 0 \\ \theta K_2\Delta t & 0 & C+\theta K_0\Delta t \end{bmatrix} \begin{bmatrix} \{a_0\}_{n+1} \\ \{a_1\}_{n+1} \\ \{a_2\}_{n+1} \end{bmatrix} \\
&\quad = \Delta t \begin{bmatrix} \theta\{F\}_{n+1}+(1-\theta)\{F\}_n \\ \{0\} \\ \{0\} \end{bmatrix} \\
&\qquad + \begin{bmatrix} C-(1-\theta)K_0\Delta t & -(1-\theta)K_1\Delta t & -(1-\theta)K_2\Delta t \\ -(1-\theta)K_1\Delta t & C-(1-\theta)K_0\Delta t & 0 \\ -(1-\theta)K_2\Delta t & 0 & C-(1-\theta)K_0\Delta t \end{bmatrix} \begin{bmatrix} \{a_0\}_n \\ \{a_1\}_n \\ \{a_2\}_n \end{bmatrix}.
\end{aligned} \tag{4.26}$$

We can get the polynomial chaos coefficient of the solution process at every time step as in the previous case.

5. Stochastic wave equation

Consider the following stochastic wave equation that governs the axial motions in a rod with the mass density ρ, elastic modulus E, and unit cross-sectional area $a = 1$, namely [18],

$$c\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right) + DU = f(x,t), \quad 0 \le x \le L,\ 0 \le t, \tag{5.1}$$

where $c = \rho a$ and $A = Ea$, subjected to boundary conditions

$$U(0,t) = U(L,t) = 0, \tag{5.2}$$

and initial conditions

$$U(x,0) = g(x) \text{ (initial displacement)} \tag{5.3}$$

$$\frac{\partial U}{\partial t}(x,0) = v(x) \text{ (initial velocity).} \tag{5.4}$$

The variational form of (5.1) over a typical element is

$$\int_{x_A}^{x_B}\left(c\frac{\partial^2 U}{\partial t^2}V + A\frac{\partial U}{\partial x}\frac{\partial V}{\partial x} + DUV\right)dx = \int_{x_A}^{x_B} f(x,t)V\,dx - Q_2V(x_B) - Q_1V(x_A). \tag{5.5}$$

Substituting by (4.6) into (5.5), we obtain

$$\sum_{i=1}^{2} \ddot{U}_i \int_{x_A}^{x_B} c\psi_i\psi_j dx + \sum_{i=1}^{2} U_i \int_{x_A}^{x_B} \left(A\psi_i'\psi_j' + D\psi_i\psi_j\right) dx = \int_{x_A}^{x_B} f(x,t)\psi_j dx + Q_j^{(e)}. \tag{5.6}$$

Let

$$C_{ij}^{(e)} = \int_{x_A}^{x_B} c\psi_i(x)\psi_j(x)dx, \tag{5.7}$$

$$K_{ij}^{(e)} = \int_{x_A}^{x_B} \left(A\psi_i'(x)\psi_j'(x) + D\psi_i(x)\psi_j(x)\right) dx, \tag{5.8}$$

$$F_i^{(e)} = \int_{x_A}^{x_B} f(x,t)\psi_i(x)dx + Q_i^{(e)}. \tag{5.9}$$

By assembling the elements matrices, we get

$$C\ddot{U} + KU = F. \tag{5.10}$$

5.1. Case 1. Stochastic mass density. Let the mass density coefficient be a stochastic process with two-dimension K-L expansion in the form

$$c = \bar{c}(x) + \zeta_1\sqrt{\lambda_1} f_1(x) + \zeta_2\sqrt{\lambda_2} f_2(x). \tag{5.11}$$

Equation (5.7) will be divided into the following parts:

$$\begin{aligned} C_{ij}{}_{(0)}^{(e)} &= \int_{x_A}^{x_B} \bar{c}\psi_i(x)\psi_j(x)dx, \\ C_{ij}{}_{(1)}^{(e)} &= \int_{x_A}^{x_B} \sqrt{\lambda_1} f_1(x)\psi_i(x)\psi_j(x)dx, \\ C_{ij}{}_{(2)}^{(e)} &= \int_{x_A}^{x_B} \sqrt{\lambda_2} f_2(x)\psi_i(x)\psi_j(x)dx. \end{aligned} \tag{5.12}$$

Then (5.10) becomes

$$\left(C_0 + \zeta_1 C_1 + \zeta_2 C_2\right)\ddot{U} + KU = F. \tag{5.13}$$

Applying Euler method for time approximation of the second derivative on (5.13) and rearranging the terms, we obtain

$$\begin{aligned}(C_0+\zeta_1C_1+\zeta_2C_2)\{U\}_{n+2} = {} & \left(2(C_0+\zeta_1C_1+\zeta_2C_2)-(\Delta t)^2K\right)\{U\}_{n+1} \\ & -(C_0+\zeta_1C_1+\zeta_2C_2)\{U\}_n+(\Delta t)^2\{F\}_{n+1}.\end{aligned} \tag{5.14}$$

Projecting the solution on two-dimension first-order chaos polynomial as in (4.18) and making inner product with H_i, we get

$$\begin{aligned}\begin{bmatrix} C_0 & C_1 & C_2 \\ C_1 & C_0 & 0 \\ C_2 & 0 & C_0 \end{bmatrix}\begin{bmatrix} \{a_0\}_{n+2} \\ \{a_1\}_{n+2} \\ \{a_2\}_{n+2} \end{bmatrix} = {} & \begin{bmatrix} 2C_0-(\Delta t)^2K & 2C_1 & 2C_2 \\ 2C_1 & 2C_0-(\Delta t)^2K & 0 \\ 2C_2 & 0 & 2C_0-(\Delta t)^2K \end{bmatrix}\begin{bmatrix} \{a_0\}_{n+1} \\ \{a_1\}_{n+1} \\ \{a_2\}_{n+1} \end{bmatrix} \\ & -\begin{bmatrix} C_0 & C_1 & C_2 \\ C_1 & C_0 & 0 \\ C_2 & 0 & C_0 \end{bmatrix}\begin{bmatrix} \{a_0\}_{n} \\ \{a_1\}_{n} \\ \{a_2\}_{n} \end{bmatrix} - (\Delta t)^2\begin{bmatrix} \{F\}_{n+1} \\ \{0\} \\ \{0\} \end{bmatrix}.\end{aligned} \tag{5.15}$$

The coefficients of polynomial chaos $\{a_i\}_0$ are defined by (4.20). Applying Euler method at $t=0$ on initial velocity (5.4), we get

$$\frac{\{U\}_{n+1}-\{U\}_n}{\Delta t} = v(x), \tag{5.16}$$

then

$$\begin{aligned}\{a_0\}_1 &= \Delta t\{v(x_j)\}+\{a_0\}_0, \quad j=1,2,\dots,N+1, \\ \{a_1\}_1 &= \{a_2\}_1 = \{0\}.\end{aligned} \tag{5.17}$$

Therefore, (5.15) can be used to obtain the polynomial chaos coefficients of the solution at $t=t_2$. Finally, the polynomial chaos coefficients of the solution at time $t=t_{n+2}$ are obtained in terms of the known coefficients at times $t=t_{n+1}$ and $t=t_n$.

5.2. Case 2. Stochastic elastic modulus. Let the elastic modulus coefficient be a stochastic process with two-dimension K-L expansion in the form

$$A = \overline{A}(x)+\zeta_1\sqrt{\lambda_1}f_1(x)+\zeta_2\sqrt{\lambda_2}f_2(x). \tag{5.18}$$

Equation (5.8) will be divided into the following parts:

$$\begin{aligned}K_{ij(0)}^{(e)} &= \int_{x_A}^{x_B}\bar{c}\psi_i(x)\psi_j(x)dx, \\ K_{ij(1)}^{(e)} &= \int_{x_A}^{x_B}\sqrt{\lambda_1}f_1(x)\psi_i(x)\psi_j(x)dx, \\ K_{ij(2)}^{(e)} &= \int_{x_A}^{x_B}\sqrt{\lambda_2}f_2(x)\psi_i(x)\psi_j(x)dx.\end{aligned} \tag{5.19}$$

Then (5.10) becomes

$$C\ddot{U} + (K_0 + \zeta_1 K_1 + \zeta_2 K_2)U = F. \tag{5.20}$$

Applying Euler method and rearranging the terms, we get

$$C\{U\}_{n+2} = (2C - (\Delta t)^2(K_0 + \zeta_1 K_1 + \zeta_2 K_2))\{U\}_{n+1} - C\{U\}_n + (\Delta t)^2\{F\}_{n+1}. \tag{5.21}$$

Projecting the solution on two-dimension first-order chaos polynomial and making inner product with H_i, we get

$$\begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \{a_0\}_{n+2} \\ \{a_1\}_{n+2} \\ \{a_2\}_{n+2} \end{bmatrix} = \begin{bmatrix} 2C - (\Delta t)^2 K_0 & -(\Delta t)^2 K_1 & (\Delta t)^2 K_2 \\ (\Delta t)^2 K_1 & 2C - (\Delta t)^2 K_0 & 0 \\ (\Delta t)^2 K_2 & 0 & 2C - (\Delta t)^2 K_0 \end{bmatrix} \begin{bmatrix} \{a_0\}_{n+1} \\ \{a_1\}_{n+1} \\ \{a_2\}_{n+1} \end{bmatrix}$$
$$- \begin{bmatrix} C & 0 & 0 \\ 0 & C & 0 \\ 0 & 0 & C \end{bmatrix} \begin{bmatrix} \{a_0\}_{n} \\ \{a_1\}_{n} \\ \{a_2\}_{n} \end{bmatrix} - (\Delta t)^2 \begin{bmatrix} \{F\}_{n+1} \\ \{0\} \\ \{0\} \end{bmatrix}. \tag{5.22}$$

We can get the chaos polynomial coefficients of the solution at every time step as in the previous case.

6. Numerical examples

In this section, we will apply the fixed forms on the following examples with studying the effect of stochastic parameters on the solution moments. The approach of the mean of stochastic solution to the exact deterministic one is studied numerically.

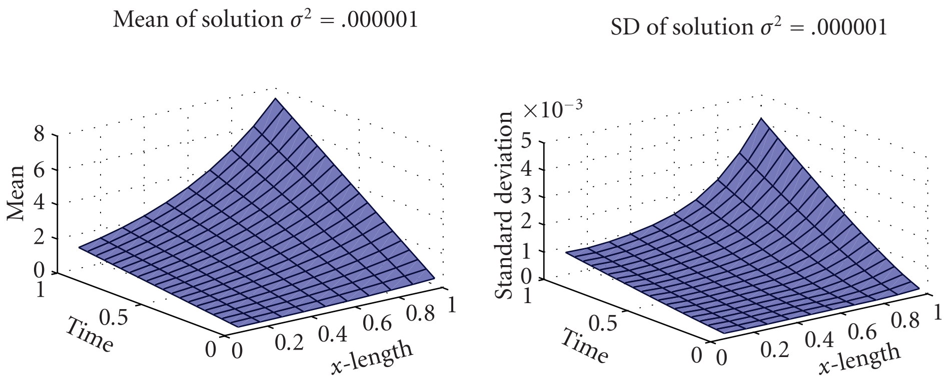
Example 6.1.

$$c\frac{\partial U}{\partial t} - \frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right) + U = (1 - 3t)\exp(2x), \quad 0 \le x \le 1, \tag{6.1}$$

subjected to boundary and initial conditions

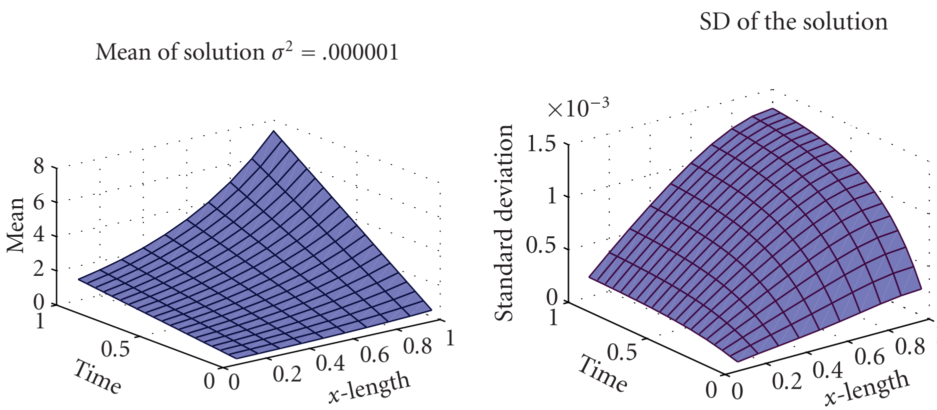
$$U(0,t) = t, \qquad \frac{\partial U}{\partial x}(1,t) = 2t\exp(2)$$
$$U(x,0) = 0. \tag{6.2}$$

Let the stochastic process be with mean one and exponential covariance in which λ_i and f_i are described by (2.6)–(2.8). Figures 6.1 and 6.2 show the mean and standard deviation of the solution process at very small point variance $\sigma^2 = .000001$ of the stochastic process c and A.



(a) (b)

Figure 6.1. The solution moments at stochastic heat conductivity A.



(a) (b)

Figure 6.2. The solution moments at stochastic heat capacity c.

From Figures 6.1 and 6.2, the type of stochastic coefficient affects only the standard deviation of the solution process. Figures 6.3 and 6.4 illustrate the effect of variation of σ^2 on the solution parameters.

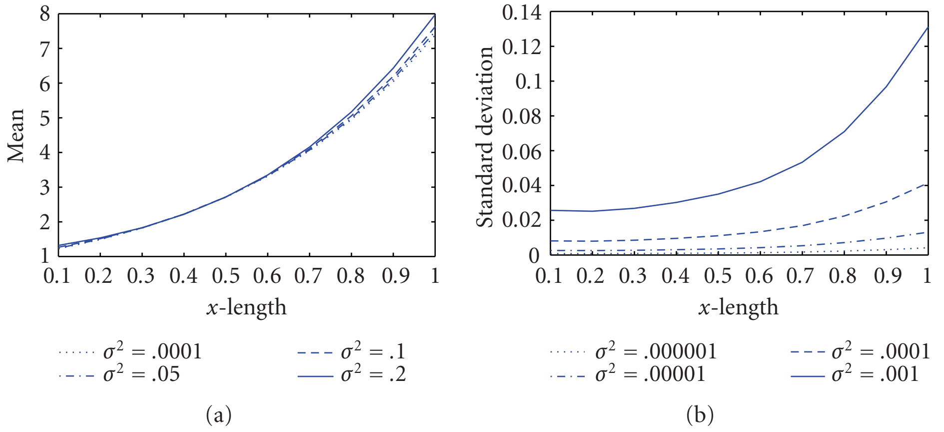


Figure 6.3. The effect of σ^2 on the solution moments at stochastic heat conductivity A. (a) Effect of σ^2 on the mean of solution at $t = 1$ second. (b) Effect of σ^2 on SD of the solution at $t = 1$ second.

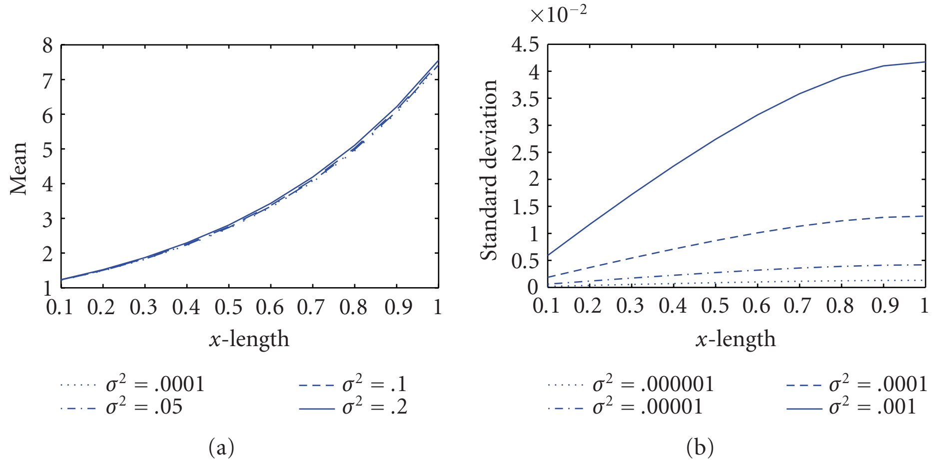


Figure 6.4. The effect of σ^2 on the solution moments at stochastic heat capacity c. (a) Effect of σ^2 on the mean of solution at $t = 1$ second. (b) Effect of σ^2 on SD of the solution at $t = 1$ second.

From Figures 6.3 and 6.4, the variation of σ^2 is more effective on standard deviation than the mean of the solution process.

The exact deterministic solution of this example at $c = A = 1$ is

$$U = t \exp(2x). \tag{6.3}$$

Table 6.1 illustrates the approach of the mean of stochastic solution to this exact deterministic one.

Table 6.1

x-domain	Mean of the solution process at different values of σ^2			Exact deterministic solution
	$\sigma^2 = .005$	$\sigma^2 = .001$	$\sigma^2 = .0005$	
0.1	1.2234	1.2217	1.2215	1.221403
0.2	1.4927	1.4919	1.4918	1.491825
0.3	1.8227	1.8220	1.8219	1.822188
0.4	2.2259	2.2253	2.2253	2.225541
0.5	2.7188	2.7180	2.7180	2.718282
0.6	3.3209	3.3200	3.3198	3.320117
0.7	4.0576	4.0553	4.0550	4.055199
0.8	4.9579	4.9536	4.9531	4.953032
0.9	6.0581	6.0509	6.0500	6.049647
1.0	7.4019	7.3912	7.3899	7.389056

The mean of the stochastic solution approaches the exact deterministic solution as the point variance of stochastic coefficient decreases.

Example 6.2.

$$\frac{\partial^2 U}{\partial t^2} - \frac{\partial}{\partial x}\left(A\frac{\partial U}{\partial x}\right) - 4U = 0, \quad 0 \le x \le \frac{\pi}{2},\ 0 \le t, \tag{6.4}$$

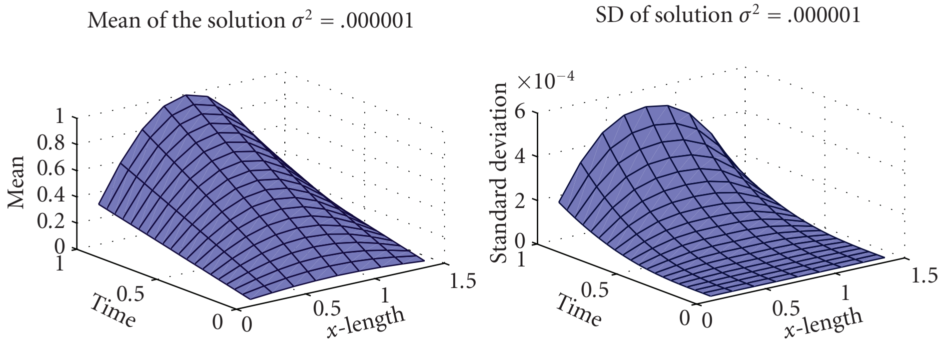
subjected to boundary conditions

$$U(0,t) = U\left(\frac{\pi}{2},t\right) = 0, \tag{6.5}$$

and initial conditions

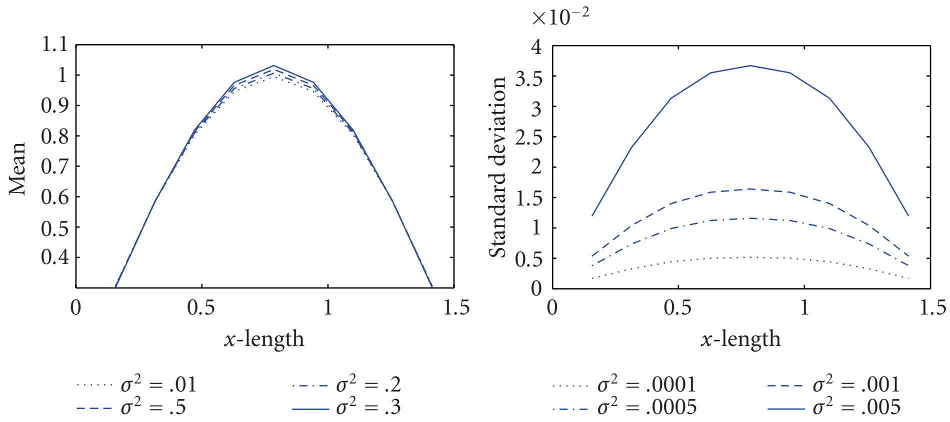
$$\begin{aligned} U(x,0) &= 0, \\ \frac{\partial U}{\partial t}(x,0) &= \sin(2x). \end{aligned} \tag{6.6}$$

Let A be stochastic process with mean one and exponential covariance, in which λ_i and f_i are described by (2.6)–(2.8). Figure 6.5 shows the mean and standard deviation of the solution process at very small point variance $\sigma^2 = .000001$ of the stochastic process A. Figure 6.6 illustrates the effect of variation of σ^2 on the solution parameters.



(a) (b)

Figure 6.5



(a) Effect of σ^2 on mean at $t = 1$ second

(b) Effect of σ^2 on SD at $t = 1$ second

Figure 6.6

From Figure 6.6, the variation of σ^2 is more effective on standard deviation than the mean of the solution process.

The exact deterministic solution of this example at $A = 1$ is

$$U = t\,\mathrm{Sin}(2x). \tag{6.7}$$

Table 6.2 illustrates the approach of the mean of stochastic solution to this exact deterministic one.

Table 6.2

x-domain	Mean of the solution process at different values of σ^2			Exact deterministic solution
	$\sigma^2 = .01$	$\sigma^2 = .005$	$\sigma^2 = .0005$	
$\pi/20$	0.3088	0.3088	0.3089	0.309017
$2\pi/20$	0.5876	0.5876	0.5876	0.587785
$3\pi/20$	0.8092	0.8091	0.8089	0.809017
$4\pi/20$	0.9517	0.9512	0.9509	0.951056
$5\pi/20$	1.0008	1.0003	0.9998	1.000000

The mean of the stochastic solution approaches the exact deterministic solution as the point variance of stochastic coefficient decreases.

7. Conclusion

The stochastic finite element method based on K-L decomposition and projection of the solution on chaos polynomials is an effective and easy method for solving the stochastic one-dimension time-dependent partial differential equation. Two fixed forms are obtained for chaos polynomial coefficients of the solution in the case of stochastic heat equation with stochastic heat capacity (4.19) or stochastic heat conductivity (4.26) coefficients. Another two fixed forms are obtained for chaos polynomials coefficients of the solution in the case of stochastic wave equation with stochastic mass density (5.15) or stochastic elastic modulus (5.22) coefficients. The stochastic parameter σ^2 has a great effect on the standard deviation of the solution process but has a very small effect on the mean of solution process. The mean of the stochastic solution approaches the exact deterministic solution as the point variance of stochastic coefficient decreases.

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