

Economic Dynamic Model for Small Islands*

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INTRODUCTION

The study constructs an economic dynamic model for small islands based on the Easter Island study by Brander and Taylor (1998) (BT henceforth). It deals with analytical as well as numerical representations of evolutions of a small island economy, evolutions of which are characterized by a two-fold process of transformation: environmental degradation and population growth. Its main purpose is to investigate under what circumstances, the small island economy can survive and under what other circumstances, it must collapse.

Archeological studies have suggested that many Pacific islands followed similar evolutionary patterns of natural resource and population dynamics; rapid population growth, resource degradation, economic decline and then population collapse. Brander and Taylor's study reconsiders archeological and anthropological evidences from economic point of view. In particular, BT present a general equilibrium model of renewable resource and population dynamics to explain the rise and fall of Easter Island for 1400 years between about the 4th century and the middle of the 18th century. It indicates that an economic model linking resource-dynamics and population-dynamics may explain, not only the sources of past historical evolutions discovered in those small islands but also a possibility of sustainable growth of our world economy in which rapid-increasing population and rapid-degrading environment become a serious problem. And, in fact, the research from the economic perspective for such explanations has already begun; see Erickson and Gowdy (2000), Dalton and Coats (2000) and Pezzy and Anderies (2000).

In the existing literature of Easter Island, however, not much has been revealed with respect to a "history" of Easter Island. It has been believed that a small group of Polynesians arrived at the island around 400 A.D.,

deforestation occurred around 1000 A.D., most of statues were carved during 1000–1400 A.D., and so forth. Based on this "conventional wisdom", BT as well as other researchers investigated a dynamic pattern of natural resource and populations. However, the "wisdom" is still one of possible hypotheses and not fully confirmed yet. In particular, according to Intoh [2000], recent re-considerations of archeological evidences on the island suggest that the small group settled in the island between 410 and 1270 A.D.‡ This new finding is inconsistent with the traditional wisdom. In other word, we may have a different history of Easter Island, even though the available historical evidences are the same. It is thus imperative to construct a model for small islands that can generate various patterns of dynamics in order to deal with such ambiguous characteristics of archeological evidences.

In this study, we reconstruct BT's continuous-time model of Easter Island in discrete steps for three reasons. First concerns the theoretical reason. It is of sure interest to see whether theoretical results obtained in a continuous-time model are carried over to a discrete-time model as well. Second concerns the numerical reason. Since there are no general results on global dynamics for non-linear structure, numerical simulations are a useful procedure for dynamic characterization. However the simulation analysis requires a discretization of the continuous-time model. Thus constructing a non-linear model in discrete steps is a natural direction to go. Lastly and more importantly, a discretization of a continuous-time model allows us to choose a length of "unit time" that is a matter of fundamental importance for a construction of dynamic model. It further allows for the possibility of rich dynamics ranging from stable dynamics, periodic oscillations and aperiodic behavior involving more extreme or moderate "feast and famine" pattern of cyclical evolutions. Our departure from BT's continuous-

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‡700 A.D., an average year, is often used for the arrival year in the archeological analysis of Easter Island, but only for the sake of convenience.

time model seems a minor modification, but our arrival on the main results obtained in this study shows a major difference; the choice of time domain can be critical in achieving the non-sustainable growth (i.e. a pattern of overshoot-and-collapse) or the sustainable growth (i.e. population stability and natural resource conservation).

The paper proceeds as follows. The second section recapitulates BT's model as a basic economic model of small islands. Third section develops a discrete-time version and considers its dynamic characterization. Fourth section makes numerical simulations to complement the analytical considerations in the third section. The fifth section concludes the study.

ECONOMIC MODEL FOR SMALL ISLANDS

Since our analysis is based on BT's dynamic model of renewable resource and population, we recapitulate their model in this section. See BT's paper for more details.

The model describes the dynamics of an economy with two goods; the harvest of the renewable resource and some other goods called manufactured goods. The model functions as follows. At time t , the stock of natural resource $S(t)$ and the size of population $L(t)$ are given. Producers determine their demands for labor and supplies of goods so as to maximize their profits. The manufactured goods are produced with constant returns to scale using only labor while harvesting of the resource is carried out according to the Schaefer harvesting production function,

$$H^s(t) = \alpha S(t)L_H(t),$$

where $H^s(t)$ is the harvest supplied by the agricultural sector, $L_H(t)$ is the labor used in resource harvesting and α is a positive constant. A representative consumer is endowed with one unit of labor and is assumed to have a Cobb–Douglas utility function,

$$u(h, m) = h^\beta m^{1-\beta},$$

where h and m are individual consumption of the resource good and of manufactures, and $\beta \in (0, 1)$. Each consumer supplies one unit of labor and demands both goods so as to maximize his utility. Prices are adjusted to establish temporary equilibrium in each of three markets; harvest market, manufactured good market and labor market. As a result of the specified utility function and production functions, the fixed proportion of the total population is employed in the agricultural section, $L_H(t) = \beta L(t)$ and thus the resource harvest is $H(t) = \alpha\beta S(t)L(t)$ at the temporary equilibrium state. After finishing transactions in each market, new values of natural resource and the size of population are determined at the next instant of time. With those new values, the process repeats itself until the temporary equilibrium is attained.

Dynamics of temporary equilibrium is described as follows. The change in the stock at time t is determined by the natural growth rate $G(S(t))$ minus the harvest rate, $H(t)$;

$$\frac{dS(t)}{dt} = G(S(t)) - H(t).$$

The logistic functional form for G is assumed, $G(S(t)) = r(1 - S(t)/K)S(t)$ where K is the maximum possible size for the resource stock, and r is an intrinsic growth rate of natural resource, and both are positive constants. A change of population depends on a difference between an underlying mortality rate and fertility rate. The base or exogenous mortality rate, denoted as n , is assumed to be a positive constant.[¶] Following the formulation of Malthusian population dynamics, BT assume that per capita consumption of the resource good increase fertility and/or decreases mortality. BT let $\phi(H(t)/L(t))$ be a fertility function where ϕ is positive constant. In consequence, the population growth rate is

$$\frac{1}{L(t)} \frac{dL(t)}{dt} = \left(-n + \phi \frac{H(t)}{L(t)} \right),$$

where the first factor on the right-hand side is the exogenous mortality rate and the second is the endogenous fertility rate. Thus substituting the logistic function into the natural growth rate and the optimal harvest, $H(t) = \alpha\beta S(t)L(t)$, into the fertility function yield the dynamic system,

$$\begin{cases} \frac{dS(t)}{dt} = \left\{ r \left(1 - \frac{S(t)}{K} \right) - \alpha\beta L(t) \right\} S(t), \\ \frac{dL}{dt} = (-n + \alpha\beta\phi S(t))L(t). \end{cases} \quad (1)$$

This is a two-dimensional dynamic system of differential equations and is a variation of the Lotka–Volterra predator–prey model, in which the human is the “predator” and the resource stock is the “prey”. The following is the main result of BT.

THEOREM 1 (*Proposition 4 of Brander and Taylor, 1998*)
When an interior steady state exists, the local behavior of the system is as follows. (i) Steady state 1 ($L = 0, S = 0$) is an unstable saddle point steady state allowing an approach along the $S = 0$ axis. (ii) Steady state 2 ($L = 0, S = K$) is an unstable saddlepoint steady state allowing an approach along the $L = 0$ axis. (iii) Steady State 3 ($L > 0, S > 0$) is a stable steady state and a “spiral node” with cyclical convergence if $nr/K\phi\alpha\beta + 4(n - K\phi\alpha\beta) < 0$ and an “improper node” allowing monotonic convergence if not.

[¶]Only for the sake of simplicity, we denote the base rate as n while BT use $d - b$ where d is the mortality rate, b is the fertility rate.

DISCRETE-TIME DYNAMIC MODEL

Discretisation

Euler's difference scheme for the continuous system Eq. (1) takes the form,

$$\begin{cases} \frac{S(t+\Delta t)-S(t)}{\Delta t} = \left\{ r\left(1 - \frac{S(t)}{K}\right) - \alpha\beta L(t) \right\} S(t), \\ \frac{L(t+\Delta t)-L(t)}{\Delta t} = \{-n + \phi\alpha\beta S(t)\} L(t), \end{cases} \quad (2)$$

where Δt denotes time step. As $\Delta t \rightarrow 0$, the discrete system converges to the continuous system. Roughly speaking, a discrete system can give rise the same dynamics as a continuous systems if Δt is small. It also has a possibility for generating qualitatively different dynamics if Δt is large. In this sense, the discrete system with $\Delta t > 0$ generalizes the corresponding continuous system. In the following, we first simplify the discretised system Eq. (2) by changing variables,[§] explore what dynamics the simplified model can produce and examine what implications would the values of model's parameters have on the dynamic behavior of the small island economy.

Eq. (2) can be, after arranging terms, written as

$$(S_{t+1}, L_{t+1}) = F(S_t, L_t) : \mathbb{R}^2 \rightarrow \mathbb{R}^2$$

where

$$\begin{cases} S_{t+1} = \left\{ (1 + r\Delta t) - \frac{r\Delta t}{K} S_t - \alpha\beta\Delta t L_t \right\} S_t, \\ L_{t+1} = \{(1 - n\Delta t) + \alpha\beta\phi\Delta t S_t\} L_t. \end{cases} \quad (3)$$

The state of the system at time t , (S_t, L_t) , is mapped to $(S_{t+1}, L_{t+1}) = F(S_t, L_t)$, the state of the system one unit of time afterwards. In order that the dynamics system generates economically feasible solutions, it is required that not only $(S_t, L_t) \geq 0$ but also $(S_{t+1}, L_{t+1}) \geq 0$ which are alternatively put as

$$(1 + r\Delta t) - \frac{r\Delta t}{K} S_t - \alpha\beta\Delta t L_t \geq 0 \quad \text{and}$$

$$(1 - n\Delta t) + \alpha\beta\phi\Delta t S_t \geq 0.$$

By the second equation of Eq. (3), $L_{t+1} = (1 - n\Delta t)L_t$ describes the population change when no economic activities take place (i.e. $H_t = 0$). Put it differently,

$$\frac{L_{t+1} - L_t}{L_t} = -n\Delta t,$$

[§]A variable $x(t)$ in continuous time can be written by $x(t_k)$ in discrete time. Set $t_k = \Delta t \cdot k$ ($k = 1, 2, \dots$). Then given $\Delta t > 0$, the variable can be expressed as follows; $x(t_k) = x(\Delta t \cdot k) = x_k$ and $x(t_k + \Delta t) = x(\Delta t \cdot (k + 1)) = x_{k+1}$. Thus, by the same token, the discretized dynamic system, Eq. (2) can be written as

$$S_{k+1} = \left\{ (1 + r\Delta t) - \frac{r\Delta t}{K} S_k - \alpha\beta\Delta t L_k \right\} S_k,$$

$$L_{k+1} = \{(1 - n\Delta t) + \alpha\beta\phi\Delta t S_k\} L_k.$$

Length of one period is equal to Δt . For notational convenience, replacing k with t yields the following discrete-time system.

which implies $n\Delta t$ is the exogenous mortality rate per the time-step. Here, $n\Delta t \geq 1$ means an extinction of humans in one period as a negative population size is not acceptable. Humans can become extinct if "one period" is so long, in addition to the absence of natural resource. However, in this study, to avoid such an extreme and uninteresting case, we mainly focus on a "moderate case" in which "one period" is less than the inverse of the exogenous instantaneous (or point) growth rate of population (i.e. $\Delta t < 1/n$).

Assumption $(1 - n\Delta t) \geq 0$.

This assumption makes $L_{t+1} \geq 0$ for all t . The remaining conditions, $(S_t, L_t) \geq 0$ and $S_{t+1} \geq 0$, are satisfied if the dynamic system Eq. (3) is an into-mapping from the following region \mathbb{R} to itself,

$$\mathbb{R} = \{(S, L) | S \geq 0, L \geq 0, \text{ and } g(S) \geq L\},$$

where

$$g(S) \equiv \frac{1 + r\Delta t}{\alpha\beta\Delta t} - \frac{r}{K\alpha\beta} S.$$

The following theorem clarifies under what parameter configurations the dynamic system Eq. (3) can generate economically feasible solutions.

THEOREM 2 $F(\mathbb{R}) \subset \mathbb{R}$ if $0 \leq r\Delta t \leq 3$ and $(n, K\alpha\beta\phi) \in \mathbb{N}$ where \mathbb{N} is defined by

$$\mathbb{N} = \left\{ (n, K\alpha\beta\phi) | 0 \leq n \leq \frac{1}{\Delta t}, 0 \leq K\alpha\beta\phi \leq \frac{r(1 + \sqrt{n\Delta t})^2}{(1 + r\Delta t)} \right\}.$$

Proof See Appendix A. \square

Stationary State

Since Eq. (3) is resemble to a Lotka–Volterra predator–prey model, it is expected that it can generate various dynamics according to various values of parameters; dynamics includes stable stationary points, periodic points and aperiodic points. We will find stationary states of the model and then investigate a possibility of such various dynamics.

Consider the $\Delta S = 0$ locus and the $\Delta L = 0$ locus where $\Delta S = S_{t+1} - S_t$ and $\Delta L = L_{t+1} - L_t$. An intersection of those loci determine a stationary state of the discrete-time system. By the first equation of Eq. (3), ΔS is $\Delta S = \left(r - \frac{r}{K} S_t - \alpha\beta L_t\right) \Delta t S_t$.

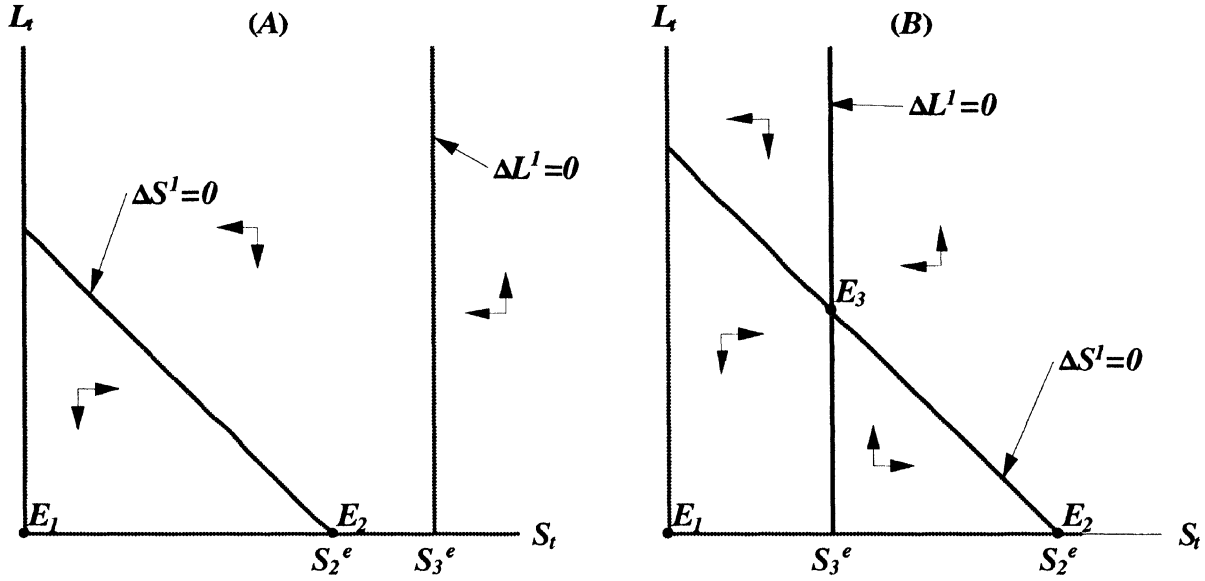


FIGURE 1 Determination of stationary state: (A) $K\alpha\beta\phi \leq n$ and (B) $K\alpha\beta\phi > n$.

This indicates that there are two $\Delta S = 0$ loci on which S_t is constant; one is $L_t = (r/\alpha\beta) - (r/\alpha\beta K)S_t$ which is denoted as $\Delta S^1 = 0$ and the other is $S_t = 0$ which is denoted as $\Delta S^2 = 0$. On the nonnegative (S_t, L_t) quadrant, the $\Delta S^1 = 0$ locus has a negative slope and crosses the horizontal axes at $S = K$ while the $\Delta S^2 = 0$ locus is identical with the vertical axis. By the second equation of Eq. (3), ΔL is

$$\Delta L = (-n + \alpha\beta\phi S_t)\Delta t L_t.$$

This also implies that there are two $\Delta L = 0$ loci on which L_t is constant; one is $S_t = n/\alpha\beta\phi$ which is denoted as $\Delta L^1 = 0$ and the other is $L_t = 0$ which is denoted as $\Delta L^2 = 0$. The former locus is a vertical line crossing the horizontal axis at $S = n/\alpha\beta\phi$ while the latter is identical with the horizontal axis. As illustrated in Fig. 1, the nonnegative quadrant of S_t and L_t is partitioned into three regions by the $\Delta S^1 = 0$ and $\Delta L^1 = 0$ loci if $K\alpha\beta\phi \leq n$ and four regions if $K\alpha\beta\phi < n$. There it can be seen that the $\Delta S^i = 0$ locus crosses the $\Delta L^i = 0$ locus ($i = 1, 2$) at each of the following three points;

$$\begin{cases} E_1 = (S_1^e, L_1^e) \text{ where } S_1^e = L_1^e = 0, \\ E_2 = (S_2^e, L_2^e) \text{ where } S_2^e = K \text{ and } L_2^e = 0, \\ E_3 = (S_3^e, L_3^e) \text{ where } S_3^e = \frac{n}{\alpha\beta\phi} \text{ and } L_3^e = \frac{r}{\alpha\beta} \left\{ 1 - \frac{n}{K\alpha\beta\phi} \right\}, \end{cases} \quad (4)$$

where E_1 is a solution of $\Delta S^2 = \Delta L^2 = 0$, E_2 is a solution of $\Delta S^1 = \Delta L^2 = 0$, and E_3 is a solution of $\Delta S^1 = \Delta L^1 = 0$. Since the loci of $\Delta S^2 = 0$ and $\Delta L^1 = 0$ are vertical lines, those do not intersect each other.

DYNAMICS

Arrows in Fig. 1 indicate directions in which variables S_t and L_t move. Those seem to suggest monotonic convergence to E_2 in Fig. 1A and cyclic fluctuations around E_3 in Fig. 1B. To formally analyze dynamic behavior in a neighborhood of each of these three stationary points, we make Taylor expansion of the nonlinear dynamic system Eq. (3) and then check eigenvalues of the resultant Jacobi matrix. The Jacobi matrix evaluated at each of three stationary points is

$$J_i = \begin{pmatrix} 1 + r\Delta t - 2\frac{r\Delta t}{K}S_i^e - \alpha\beta\Delta t L_i^e & -\alpha\beta\Delta t S_i^e \\ -\alpha\beta\phi\Delta t L_i^e & 1 - n\Delta t + \alpha\beta\phi\Delta t S_i^e \end{pmatrix} \quad (5)$$

where $i = 1, 2, 3$. It is well known that the stationary state is asymptotically stable if and only if, for eigenvalues λ_1 and λ_2 of the Jacobi matrix, the following conditions hold,

$$|\lambda_1| < 1 \quad \text{and} \quad |\lambda_2| < 1.$$

We will detect under what circumstances those stability conditions are satisfied.

Stability of E_1 And E_2

We start with stationary state E_1 . Inserting $S_1^e = L_1^e = 0$ into the Jacobi matrix Eq. (5) yields

$$J_1 = \begin{pmatrix} 1 + r\Delta t & 0 \\ 0 & 1 - n\Delta t \end{pmatrix},$$

in which the eigenvalues are the diagonal elements $\lambda_1 =$

$1 + r\Delta t > 1$ and $0 < \lambda_2 = 1 - n\Delta t < 1$. Thus stationary state E_1 is a saddle point which a trajectory leaves on the horizontal axis to E_2 and approaches on the vertical axis to E_1 in the (S_t, L_t) plane. To summarize, we have,

THEOREM 3 E_1 is an unstable saddle point for which the horizontal axis is an unstable manifold and the vertical axis is a stable manifold.

Turn to stationary point E_2 . Inserting $S_2^e = K$ and $L_2^e = 0$ of Eq. (4) into Eq. (5) yields the Jacobi matrix evaluated at E_2 ,

$$J_2 = \begin{pmatrix} 1 - r\Delta t & -\alpha\beta\Delta t K \\ 0 & 1 - n\Delta t + \alpha\beta\phi\Delta t K \end{pmatrix}.$$

Its eigenvalues are also the diagonal elements $\lambda_1 = 1 - r\Delta t < 1$ and $\lambda_2 = 1 + (K\alpha\beta\phi - n)\Delta t > 0$ in which the direction of inequality is due to Assumption. Since the positive growth rate of natural resource per the time-step has an upper bound (i.e. $0 \leq r\Delta t \leq 3$) by Theorem 2, it can verified that

$$\begin{aligned} -1 \leq \lambda_1 \leq 1 \text{ if } 0 < r\Delta t \leq 2 \quad \text{and} \\ \lambda_1 < -1 \text{ if } 2 < r\Delta t \leq 3. \end{aligned}$$

Further, the following is straightforward,

$$\lambda_2 < 1 \text{ if } K\alpha\beta\phi \leq n \quad \text{and} \quad \lambda_2 > 1 \text{ if } K\alpha\beta\phi > n.$$

E_2 in Fig. 1B, in which $K\alpha\beta\phi > n$ holds, is definitely unstable as the dominant root is greater than unity whereas E_2 in Fig. 1A, in which $K\alpha\beta\phi \leq n$ holds, can be stable or unstable according to the natural growth rate over the time-step (i.e. $r\Delta t$) is less or greater than 2. Thus dynamic behavior in a vicinity of E_2 is summarized as follows.

THEOREM 4 (1) Provided that $K\alpha\beta\phi \leq n$, E_2 is a stable point if $0 < r\Delta t \leq 2$ and a saddle point if $2 < r\Delta t \leq 3$; (2) Provided that $K\alpha\beta\phi > n$, E_2 is a saddle point if $0 < r\Delta t \leq 2$ and an unstable point if $2 < r\Delta t \leq 3$.

Theorem 4(1) indicates that

$$K\alpha\beta\phi \leq n \tag{6}$$

is the condition for extinction of humans (i.e. $L_2^e = 0$ at E_2). We give an intuitive explanation for this condition. Since the endogenous fertility rate, $\phi(H/L) = S\alpha\beta\phi$, positively depends on the actual size of the natural resource, and K is the carrying capacity of the natural stocks, $K\alpha\beta\phi$ is the maximum fertility rate when the size of the natural resource reaches at its maximum (i.e. $S = K$). Since n is the exogenous mortality rate of population, the inequality of Eq. (6) indicates that the maximum endogenous fertility rate is less than or equal to

the exogenous mortality rate. Therefore, under this condition, the humans will extinct sooner or later regardless of their economic activities. This is what Theorem 4(1) implies. The stationary state E_3 is identical with E_2 if the equality of Eq. (6) holds whereas it becomes economically meaningless if the inequality of Eq. (6) strictly holds (i.e. $K\alpha\beta\phi < n \Rightarrow S_3^e > K$ and $L_3^e < 0$).^{||}

Stability of E_3

Since we are interested in the survival of humans, we confine the following analysis to a case in which the maximum endogenous fertility rate is greater than the exogenous mortality rate,

$$K\alpha\beta\phi > n \tag{7}$$

which we call the *survival* condition. The consistent requirement, the natural resource is less than the carrying capacity (i.e. $S_3^e < K$) and the population size is positive (i.e. $L_3^e > 0$), holds under the survival condition. The population size can increase or decrease depending on the relative magnitude between the actual fertility rate, $\alpha\beta\phi S_t$, and the exogenous mortality rate, n . This is a source of dynamics in our model.

Inserting (S_3^e, L_3^e) into Eq. (5) yields the Jacobi matrix evaluated at the stationary point E_3 ,

$$J_3 = \begin{pmatrix} 1 - \frac{nr\Delta t}{K\alpha\beta\phi} & -\frac{n\Delta t}{\phi} \\ r\Delta t\left(\phi - \frac{n}{K\alpha\beta}\right) & 1 \end{pmatrix}.$$

Its characteristic polynomial is $f(\lambda) = \lambda^2 - \text{tr } J_3 \lambda + \det J_3$ and its eigenvalues are roots of $f(\lambda) = 0$,

$$\lambda_{1,2} = \frac{\text{tr } J_3 \pm \sqrt{(\text{tr } J_3)^2 - 4\det J_3}}{2},$$

where

$$\begin{aligned} \text{tr } J_3 &= \lambda_1 + \lambda_2 = 2 - \frac{nr\Delta t}{K\alpha\beta\phi}, \\ \det J_3 &= \lambda_1 \lambda_2 = \frac{K\alpha\beta\phi(1 + nr\Delta t^2) - nr\Delta t(1 + n\Delta t)}{K\alpha\beta\phi}. \end{aligned} \tag{8}$$

The roots are real if the discriminant $D = (\text{tr } J_3)^2 - 4\det J_3$ is nonnegative and complex if D is negative. Thus points on the $D = 0$ locus construct a line of demarcation between real roots and complex roots. Given r , solving

^{||}After the humans become extinct (i.e. $L_t = 0$ for all t), the natural resource will develop according to the following logistic map, $x_{t+1} = a(1 - x_t)x_t$ where $x_t = (r\Delta t)/(1 + r\Delta t)K$ and $a = 1 + r\Delta t$. If $a \leq 3$ or $r\Delta t \leq 2$, then natural resource gradually grows and reaches at its carrying capacity. Although it has been well known that the logistic map can generate chaotic dynamic for $a > 3$ or $r\Delta t > 2$, the economic constraint $S_t \leq K$ prevent the logistic map generating such complex dynamic. Numerical simulations indicate that trajectories of S_t subject to $S_t \leq K$ reaches K within finite time periods even if $r\Delta t > 2$ holds.

$D = 0$ for $K\alpha\beta\phi$ yields the demarcation line,

$$f_{\text{dem}}(n) = \frac{n + \sqrt{n(n+r)}}{2}. \quad (9)$$

The stability condition, both eigenvalues are less than unity in absolute values, is hold if and only if the following inequalities are satisfied,[#]

$$-1 \pm \text{tr} J_3 < \det J_3 < 1, \quad (10)$$

where the first inequality ensures that the real roots lie in the interval $(-1, 1)$ and the second ensures the modulus of the complex roots is less than unity. To find the stable region, we construct the lines $f(1) = 0$, $f(-1) = 0$ and $\det J_3 - 1 = 0$ where $f(1) = 1 - \text{tr} J_3 + \det J_3$ and $f(-1) = 1 + \text{tr} J_3 + \det J_3$. Substituting the most right expressions of Eq. (8) into each of those lines gives

$$\begin{cases} f(1) = \frac{nr\Delta t^2}{K\alpha\beta\phi}(K\alpha\beta\phi - n), \\ f(-1) = \frac{4+(n\Delta t)(r\Delta t)}{K\alpha\beta\phi} \left\{ K\alpha\beta\phi - n \frac{2r\Delta t + (n\Delta t)(r\Delta t)}{4+(n\Delta t)(r\Delta t)} \right\}, \\ \det J_3 - 1 = \frac{nr\Delta t^2}{K\alpha\beta\phi} \left\{ K\alpha\beta\phi - \left(n + \frac{1}{\Delta t} \right) \right\}. \end{cases}$$

Therefore, given r and Δt , the stable region of the stationary state E_3 in the $(n, K\alpha\beta\phi)$ plane is surrounded by the following boundary lines,

$$f(1) = 0 \Leftrightarrow K\alpha\beta\phi = f_{\text{div}}(n) \equiv n,$$

$$f(-1) = 0 \Leftrightarrow K\alpha\beta\phi = f_{\text{flip}}(n) \equiv n \frac{2r\Delta t + (n\Delta t)(r\Delta t)}{4 + (n\Delta t)(r\Delta t)},$$

$$\det J_3 = 1 \Leftrightarrow K\alpha\beta\phi = f_{\text{flut}}(n) \equiv n + \frac{1}{\Delta t},$$

which we call the *divergence* boundary, the *flip* boundary and the *flutter* boundary, respectively, following Sonis (1996). Since it can be shown that

$$f_{\text{flip}}(n) \underset{>}{<} f_{\text{div}}(n) \text{ according to } r\Delta t \underset{>}{<} 2, \quad (11)$$

two different stable regions are, under the survival condition defined by

$$\mathbb{M}_1 = \{(n, K\alpha\beta\phi) | 0 \leq n, f_{\text{div}}(n) < K\alpha\beta\phi < f_{\text{flut}}(n)\} \text{ if } r\Delta t \leq 2,$$

$$\mathbb{M}_2 = \{(n, K\alpha\beta\phi) | 0 \leq n, f_{\text{flip}}(n) < K\alpha\beta\phi < f_{\text{flut}}(n)\} \text{ if } r\Delta t > 2.$$

To summarize the dynamic behavior in a vicinity of the stationary state E_3 , we have:

[#]See Gandolfo (1997) for a heuristic proof.

^{**}In particular, two boundaries cross twice for $n = n_1$ and $n = n_2$ if $r\Delta t > 1$. We can assume $n_1 < n_2$ without a loss of generality which implies $n_1 < 1/\Delta t < n_2$. However, $f_{\text{up}}(n)$ is defined only on the interval $[0, 1/\Delta t]$ so that we do not need to consider the second intersection for $n = n_2$.

THEOREM 5 (1) *Provided that $r\Delta t \leq 2$, E_3 is stable for $(n, K\alpha\beta\phi) \in \mathbb{M}_1$; (2) provided that $r\Delta t > 2$, E_3 is stable for $(n, K\alpha\beta\phi) \in \mathbb{M}_2$.*

\mathbb{N} in Theorem 2 concerns the feasibility of solutions as it is the set of parameters $(n, K\alpha\beta\phi)$ for which $F(\mathbb{R}) \subset \mathbb{R}\mathbb{M}_i$ in Theorem 5 concerns the stability of solutions as it is the set of parameters $(n, K\alpha\beta\phi)$ for which $F(\mathbb{R})$ generates stable dynamics. Thus an intersection of \mathbb{M}_i and \mathbb{N} is the set of parameters for which solutions of $F(\mathbb{R})$ are feasible and stable. Returning to the definition of the set \mathbb{N} , we can see that the upper boundary of \mathbb{N} in the $(n, K\alpha\beta\phi)$ plane is

$$f_{\text{up}}(n) \equiv \frac{r}{1+r\Delta t} (1 + \sqrt{n\Delta t})^2,$$

which is defined on the interval $[0, 1/\Delta t]$. To check whether this upper boundary and the flutter boundary intersect, we solve $f_{\text{up}}(n) = f_{\text{flut}}(n)$ for n to have

$$n_{1,2} = \frac{2(r\Delta t)^2 - 1 \pm 2r\Delta t \sqrt{(r\Delta t)^2 - 1}}{\Delta t}. \quad (12)$$

This implies that the upper boundary is tangent to the flutter boundary at $(1/\Delta t, 2/\Delta t)$ if $r\Delta t = 1$. Thus the upper boundary of \mathbb{N} and the flutter boundary of \mathbb{M}_i intersect none if $r\Delta t < 1$ and once in the interval $(0, 1/\Delta t)$ if $r\Delta t > 1$.^{**} By Theorem 2, we have $0 < r\Delta t \leq 3$. By Theorem 5, the stable set is either \mathbb{M}_1 or \mathbb{M}_2 according to $r\Delta t < 2$ or $r\Delta t > 2$. Summing up the foregoing discussions indicates that for detecting the dynamics behavior of E_3 , three distinct cases can be identified which depend on the growth rate of natural resource over the time-step: (1) $0 \leq r\Delta t \leq 1$, (2) $1 < r\Delta t \leq 2$ and (3) $2 < r\Delta t \leq 3$. Dynamics in each of three cases is considered, allowing for the constraints imposed on parameters,

$$0 < n\Delta t \leq 1 \quad \text{and} \quad K\alpha\beta\phi > n.$$

Case 1 $0 \leq r\Delta t \leq 1$. In this case, $f_{\text{up}}(n) \leq f_{\text{flut}}(n)$ for all $n \in [0, 1/\Delta t]$. Thus the upper bound of the $\mathbb{N} \cap \mathbb{M}_1$ is the $K\alpha\beta\phi = f_{\text{up}}(n)$ locus. The dynamics of E_3 is summarized as follows.

THEOREM 6 *Provided that $0 \leq r\Delta t \leq 1$, E_3 is stable for $(n, K\alpha\beta\phi) \in \mathbb{N} \cap \mathbb{M}_1 = \{(n, K\alpha\beta\phi) | 0 \leq n \leq 1/\Delta t, n < K\alpha\beta\phi \leq r/(1+r\Delta t)(1 + \sqrt{n\Delta t})^2\}$.*

In Fig. 2 below, the stable and feasible region $\mathbb{N} \cap \mathbb{M}_1$ is depicted as an shaded area surrounded by the upper boundary, the divergence boundary, the vertical axis and the vertical line, $n = 1/\Delta t$. It is divided into two regions by the demarcation line on which we have a real root of multiplicity two. For any point in a region below the demarcation line and above the divergence boundary (i.e. the dark-shaded region), the eigenvalues are real so that

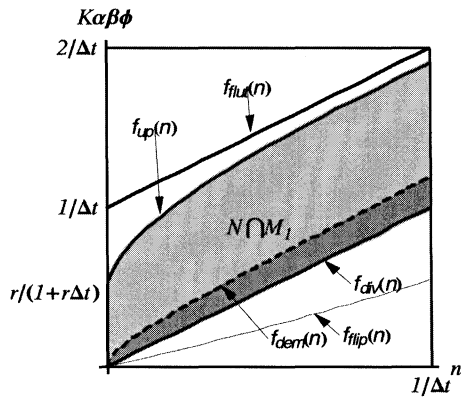


FIGURE 2 Stable and feasible region; $0 \leq r\Delta t \leq 1$.

the dynamic system $F(\mathbb{R})$ shows a monotonic convergence toward the stationary state E_3 . On the other hand, for any point in a region above the demarcation line and below the upper boundary (i.e. the light-shaded region), the eigenvalues are complex so that $F(\mathbb{R})$ shows a damped oscillations around E_3 .

Using those results together with Theorem 6, the following is implied.

COROLLARY 7 For $(n, K\alpha\beta\phi) \in \mathbb{N} \cap \mathbb{M}_1$, E_3 is a stable node with monotonic convergence if $K\alpha\beta\phi \leq (n + \sqrt{n(n+r)})/2$ and a stable focus with damped oscillations if $(n + \sqrt{n(n+r)})/2 < K\alpha\beta\phi$.

Theorems 3–5 and Corollary 7 imply that the discrete-time system can generate the similar dynamics as the continuous-time system if the time step, Δt , is small enough in a sense that Δt is smaller than an inverse of r . Dynamics of the stationary states in the continuous-time model is summarized in Theorem 1 above. Theorem 3, which shows that the stationary state E_1 is a saddle point, is the same as Theorem 1(i). The first part of Theorem 4(2), which shows that the stationary state E_2 is an unstable saddle point, is the same as Theorem 1(ii). Lastly, Corollary 7 just mentioned above is discrete-time counterparts of Theorem 1(iii) and (iv).^{††} Thus BT’s results follow in the discrete-time model.

It is possible to show that the discrete-time model can generate qualitatively different dynamics. To this end, we simulate the model, using parameter values used by BT. The parameter values used in BT can be divided into two; one is a set of time-independent parameters that can be used either of continuous- or discrete-time model and the other is a set of time-dependent parameters that are sensitive to a choice of continuous- or discrete-time model. The carrying capacity of the island, K , the labor harvesting productivity, α , measure of preference for the output of the harvest good, β , and a constant factor of the fertility function, ϕ , are thought to be in the former set so

that we coax parameter values from BT and set $K = 12,000$, $\alpha = 0.00001$, $\beta = 0.4$ and $\phi = 4$ in the following simulations. The exogenous motility rate, n , and the intrinsic growth rate of natural resource, r , are in the latter set and, as will be seen shortly, play critical roles for the emergence of complex dynamics. We take those to be $n = 0.05$ and $r = 0.04$ in the first simulation. Further, we set $\Delta t = 10$ and take the initial values of S_t and L_t as $S_0 = K$ and $L_0 = 100$.^{‡‡} As a result of those specifications, a pair of $K\alpha\beta\phi = 0.192$ and $n = 0.05$ are not in $\mathbb{N} \cap \mathbb{M}_1$.^{†††} Although the stationary state is unstable, the simulation illustrated in Fig. 3A shows an aperiodic oscillation that the continuous-time model cannot generate. As plotted in Fig. 3B, evolutions of the natural resource (the dotted line) and the population (the real line) would resemble to the “history” of Easter Island.

Following BT, we assume that the history of Easter Island starts shortly after 400 A.D. and ended in the 1800s by the outside intervention. A growth-collapse pattern during those 1400 years are fairly described by the simulation if we presume that a time-step equals 100 actual years (i.e. $\Delta t = 100$ years where $\Delta t = 10$ time-units indicate that one-time unit is one decade). In particular, in the simulation, after 4 periods of the beginning of the history (i.e. about 800 A.D. in the actual calendar), the natural resource degradation starts and population begins to increase rapidly. At 9th period (i.e. at 1100 A.D.), population peaks at $L = 30,000$ unity that means that 15,000 people live on the island while the natural resource is degraded severely. At 12th period (i.e. 1600 A.D.) the natural resource was almost gone to nothing and population continues to decline to 2500 from its peak. It is said that the carving stone statues starts around 1100 A.D after the size of population becomes larger than the essential size and stops around 1600 A.D due to the extinction of palm tree forest on the island. Such archeological evidence can go with the simulation results. At the 14th period (i.e. 1800A.D.) the history of endogenous evolution ended. This simulation mimics a feast (i.e. a gradual–rapid population growth and intensive use of natural resource) and famine (i.e. rapid population collapse and environmental degradation) pattern, which is often observed in small island economy. In the simulation $r\Delta t = 0.4$ and $n\Delta t = 0.5$ which means that the natural resource increases by 40% per century (i.e. 4% per decade) without human harvesting and the base motility rate per century is 50% (5% per decade), both of which are supposed to be plausible estimates.

Case 2 $1 < r\Delta t \leq 2$. Case 2 deals with the effect caused by an increase in the intrinsic growth rate per unit time interval on the dynamic behavior. The first inequality condition $1 < \Delta t$ makes the discriminant of Eq. (12) positive, that is, two roots, n_1 and n_2 , are real and distinct.

^{††} An alternative expression of $(n + \sqrt{n(n+r)})/2 < K\alpha\beta\phi$ is $nr/K\alpha\beta\phi + 4(n - K\alpha\beta\phi) < 0$, which is Eq. (14) of BT.

^{‡‡} Two unit of labor force is assumed to be one person. Thus $L_0 = 100$ means that the founding population is 50.

^{†††} Since Theorem 2 provides a sufficient condition for feasibility, it is possible that a trajectory starting at a point outside of \mathbb{N} oscillates around an unstable stationary point if the non-linearity’s of the model are strong enough.

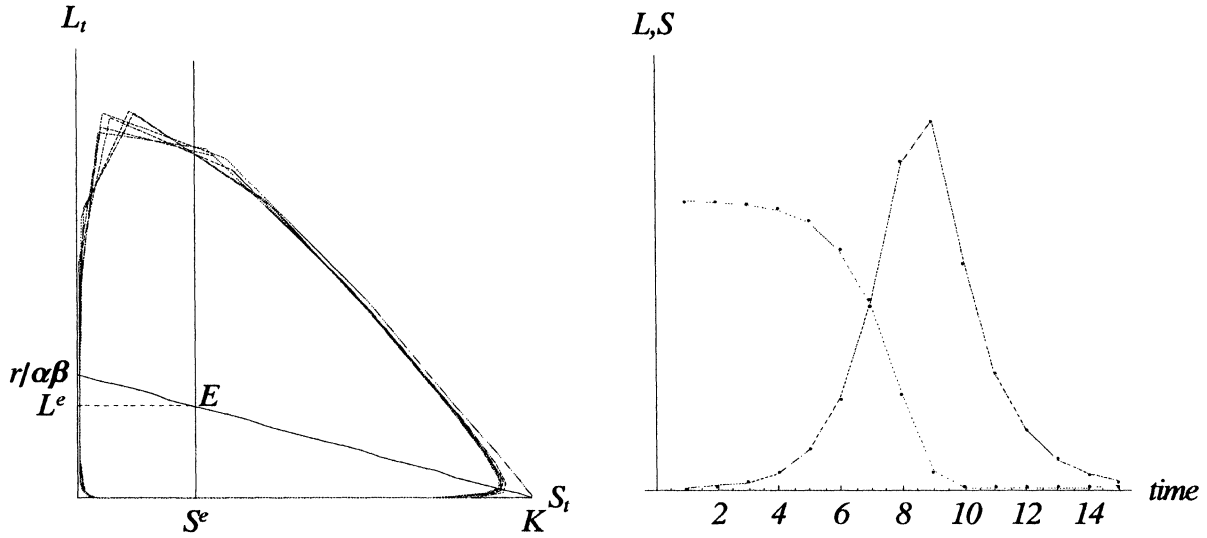


FIGURE 3 Numerical simulation of Easter Island evolution.

The upper boundary and the flutter boundary intersect for $n = n_1 < 1/\Delta t$. Consequently the upper bound of the feasible and stable region is $\min[f_{\text{up}}(n), f_{\text{flut}}(n)]$ in which $f_{\text{up}}(n) < f_{\text{flut}}(n)$ holds for $0 < n < n_1$ and the direction of inequality is reversed for $n_1 < n \leq 1/\Delta t$. Further, since the second inequality condition $r\Delta t \leq 2$ leads to $f_{\text{flip}}(n) \leq f_{\text{div}}(n)$ the divergence boundary is the lower bound of \mathbb{M}_1 . Thus, in Case 2, the feasible and stable region is an intersection of \mathbb{N} and \mathbb{M}_1 .

$$\mathbb{N} \cap \mathbb{M}_1 = \left\{ (n, K\alpha\beta\phi) \mid 0 \leq n \leq \frac{1}{\Delta t}, \right. \\ \left. f_{\text{div}}(n) \leq K\alpha\beta\phi \leq \min[f_{\text{up}}(n), f_{\text{flut}}(n)] \right\}.$$

Since $f_{\text{flut}}(n) < f_{\text{up}}(n)$ for $n > n_1$, we can define a feasible and unstable region denoted as \mathbb{U}_1 ,

$$\mathbb{U}_1 = \left\{ (n, K\alpha\beta\phi) \mid n_1 \leq n \leq \frac{1}{\Delta t}, \right. \\ \left. f_{\text{flut}}(n) \leq K\alpha\beta\phi \leq f_{\text{up}}(n) \right\}.$$

The feasible and stable region (i.e. a dark and light shaded region) and the feasible and unstable region (i.e. lighter shaded region) are depicted in Fig. 4.

Thus the results are summarized as

THEOREM 8 *Provided that $1 < r\Delta t \leq 2$, E_3 is stable for $(n, K\alpha\beta\phi) \in \mathbb{N} \cap \mathbb{M}_1$ and unstable for $(n, K\alpha\beta\phi) \in \mathbb{U}_1$*

^{§§}As in the first simulation, we set $\alpha = 0.00001$, $\beta = 0.4$, $\phi = 4$, $K = 12,000$ which leads to $K\alpha\beta\phi = 0.192$ and take r and Δt to be 0.15 and 10 which leads to $1 < r\Delta t \leq 2$.

^{|||}Line ab is divided into ten segment by points defined as $n(i) = n_b - (i/T)(n_b - n_a)$, $i = 0, 1, \dots, T$ where T is the number of bifurcation steps. $T = 10$ in the simulation. See Sonis (1996) for more precise explanations for this bifurcation cascade.

For $(n, K\alpha\beta\phi) \in \mathbb{N} \cap \mathbb{M}_1$, the dynamic generated in Case 2 is qualitatively the same as the one in Case 1. A trajectory exhibits either a monotonic or oscillatory movement converging towards the stationary state E_3 according to the roots of the characteristic equation are real or complex. For $(n, K\alpha\beta\phi) \in \mathbb{U}_1$, a trajectory generated by Eq. (3) does not converge to the stationary state since the stability condition is violated. In spite of this instability, the trajectory does not explode globally but keeps oscillating in a limited region due to inherent nonlinearity's in the model. To see what happens in the region \mathbb{U}_1 , we have to examine the points on the flutter boundary on which eigenvalues are complex and $|\lambda_1| = |\lambda_2| = 1$ where $\det J_3 = \lambda_1\lambda_2$ is equal to the modulus of roots. The stationary point is *nonhyperbolic* so that the non-linear model and its linear approximation are not topologically equivalent in a vicinity of E_3 . Crossing of the flutter boundary means that the second- and higher-order derivatives of the behavioral functions becomes essential and creates a Hopf bifurcation from the stationary state. To detect the dynamic behavior, we simulate the model choosing the mortality rate as a bifurcation parameter and fixing other parameters at some appropriate values.^{§§} Figure 5 below presents the bifurcations appearing when n is moving along a straight line ab from n_b to n_a in the region \mathbb{U}_1 in Fig. 4.^{|||} For each value of n , the dynamic system starts from the same initial values, $S_0 = K$ and $L_0 = 100$ and is simulated for 20 iterations to see the long-run dynamic behavior. It will be safe to assume one period equal one hundred years as in the first simulation. It can be seen that trajectories goes more sharply ups and downs, as n gets smaller than n_b . This indicates that a change from a heyday to a hell-day in

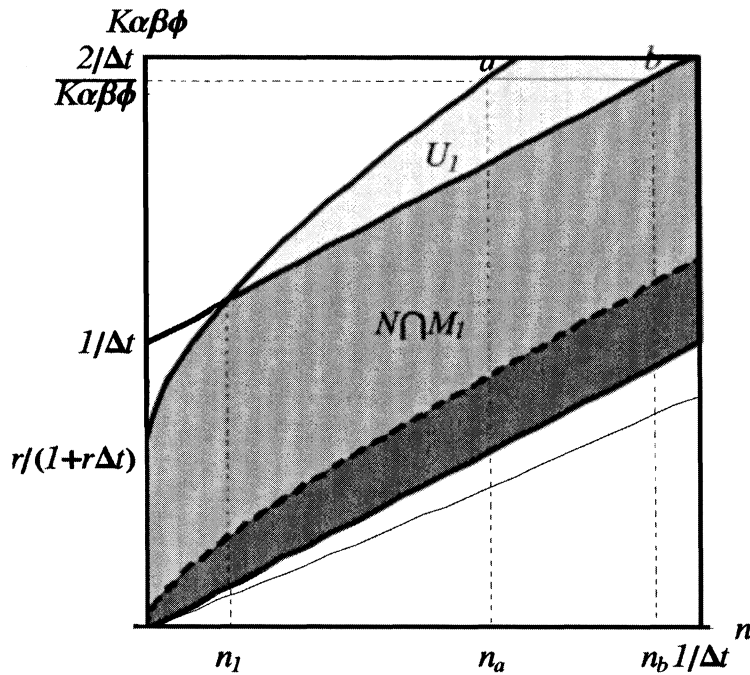


FIGURE 4 Stable and feasible region: $1 < \Delta t \leq 2$.

a feast and famine cycle gets more rapid and drastic as the exogenous mortality rate becomes smaller.

Case 3 $2 < r\Delta t \leq 3$. Suppose $r\Delta t$ becomes larger and lies in an interval $(2,3]$. As will be seen shortly, the higher intrinsic growth rate of natural resource increases a possibility causing complex long-run dynamics. Similar to Case 2, the upper boundary and the flutter boundary intersect at $n = n_1$ and thus $f_{up}(n) < f_{flut}(n)$ holds for $0 < n < n_1$ and $f_{up}(n) > f_{flut}(n)$ holds for $n_1 < n \leq 1/\Delta t$. In addition to this, the flip boundary gets steeper than the divergence boundary for $r\Delta t > 2$. Thus the feasible and

stable region is defined by an intersection of \mathbb{N} and \mathbb{M}_2 ,

$$\mathbb{N} \cap \mathbb{M}_2 = \left\{ (n, K\alpha\beta\phi) \mid 0 \leq n \leq \frac{1}{\Delta t}, \right.$$

$$\left. f_{flip}(n) \leq K\alpha\beta\phi \leq \min[f_{up}(n), f_{flut}(n)] \right\}.$$

The unstable region \mathbb{U}_1 is similarly defined as a region between the upper boundary and the flutter boundary for $n \geq n_1$. Since E_3 is economically feasible only for $K\alpha\beta\phi > f_{div}(n)$ (i.e. the survival condition, $K\alpha\beta\phi > n$),

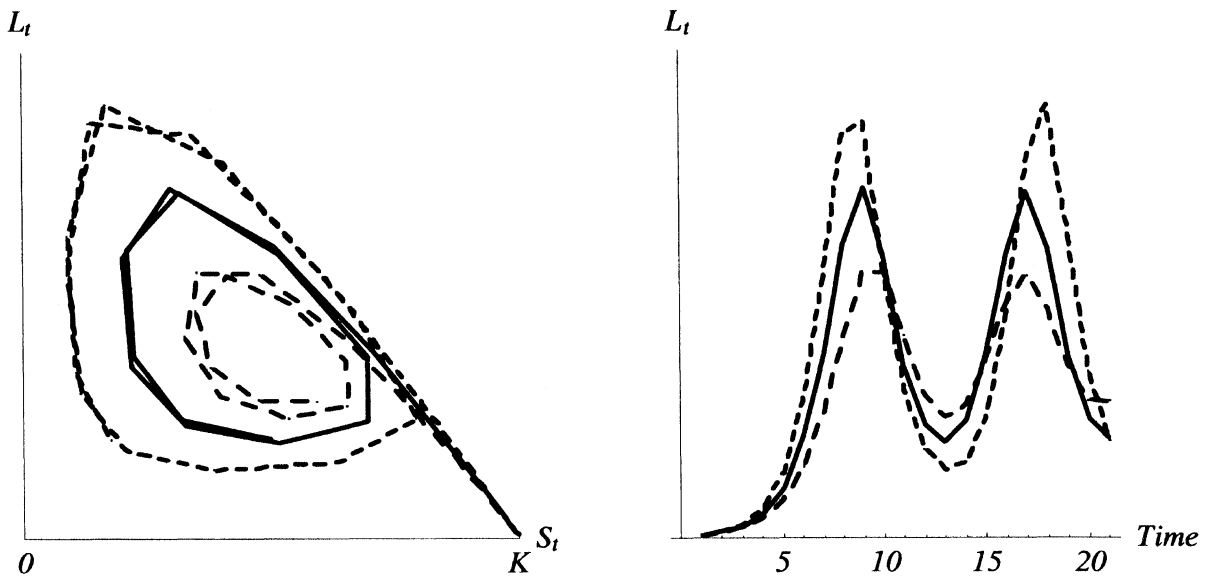


FIGURE 5 Various oscillations.

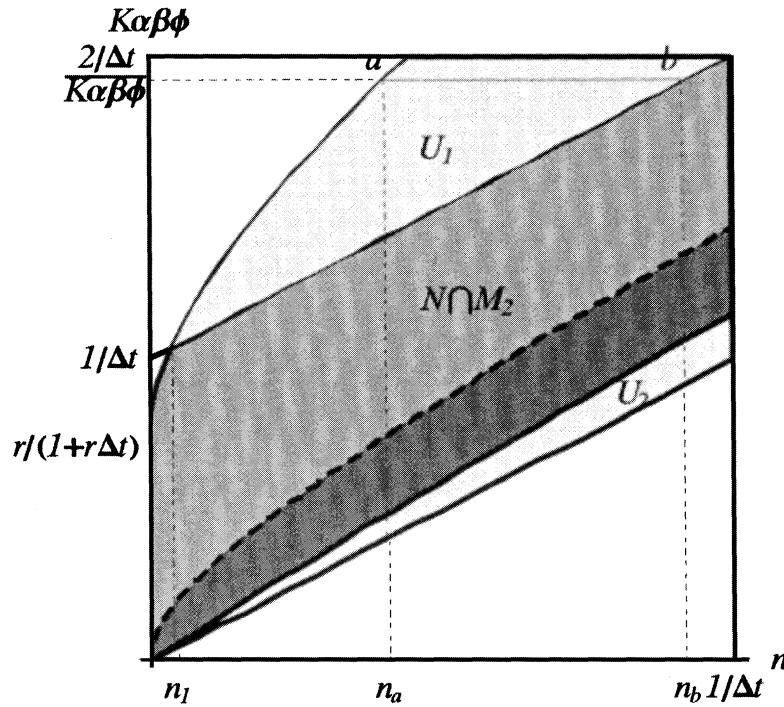


FIGURE 6 Stable and unstable regions; $2 < \Delta t \leq 3$.

another feasible and unstable region, denoted as \mathbb{U}_2 , can be defined as:

$$\mathbb{U}_2 = \left\{ (n, K\alpha\beta\phi) \mid 0 \leq n \leq \frac{1}{\Delta t}, \right. \\ \left. f_{\text{div}}(n) < K\alpha\beta\phi \leq f_{\text{flip}}(n) \right\}.$$

Thus the results are summarized as

THEOREM 9 *Provided that $2 \leq r\Delta t \leq 3$, E_3 is stable for $(n, K\alpha\beta\phi) \in \mathbb{N} \cap \mathbb{M}_2$ and unstable for $(n, K\alpha\beta\phi) \in \mathbb{U}_1 \cup \mathbb{U}_2$.*

When the stationary state is stable, dynamic is similar to the one considered in Case 1 and Case 2, that is, monotonic convergence or damped oscillations. When the stationary state is unstable, the non-linear structure of the model results in far more complex dynamics for which there is no general theoretical results. Numerical simulation is useful to detect such non-linear structure. Crossing of the flutter boundary implies the start of Hopf bifurcation cascade to chaos. In particular, the third simulation is done with $r = 2.5$ and the exogenous motility rate is chosen as a bifurcation parameter. As n decreases from n_b which is on the flutter boundary, the stationary state bifurcates to limit cycles, which represents an increase in the magnitude of population ups and downs. However, the cycles are relatively regular and (quasi-) symmetrical as illustrated in Fig. 7A. As n decreases more,

the cycles are distorted and show irregular oscillations. As n gets to n_a , chaotic or very high-period cycles appears as seen in Fig. 7B in which chaotic attractor is represented.

Crossing of the flip boundary makes E_3 unstable and generates a period-doubling bifurcation, which is a typical route to chaos. A periodic cycle with period two emerges for each point on the flip boundary.^{##} Although S_t exhibits various dynamics from periodic cycles to chaotic movement, it becomes larger than the carrying capacity for some values of n . Thus such dynamic behavior is economically unrealistic. If we perform simulation allowing for the capacity constraint $S_t \leq K$ explicitly, then high order cycles including chaotic behavior disappear but a two-periodic cycle appears.

CONCLUDING REMARKS

In this study, we have demonstrated that a discrete version of BT's continuous-time model for small islands can generate various dynamics ranging from stable as well as periodic trajectories to complex dynamics involving chaos. In particular, on the one hand, Theorem 4 shows a natural result that the humans are extinct soon or later if the maximum fertility rate is less than mortality rate, $K\alpha\beta\phi \leq n$. On the other hand, Theorems 5–9 shows that sources of a diversity of dynamics are degrees of the intrinsic growth rate of natural resource over a unit of time $r\Delta t$ and the base mortality rate, in a unit of time $n\Delta t$ even if the maximum fertility rate is greater than the mortality

^{##}Since substituting the middle expression of (8) into the alternative expression of the flutter boundary, $-1 + \text{tr}J_3 = \det J_3$, gives $(\lambda_1 + 1) \times (\lambda_2 + 1) = 0$, one of the eigenvalues is -1 and the other is $-\det J_3$ on the flip boundary.

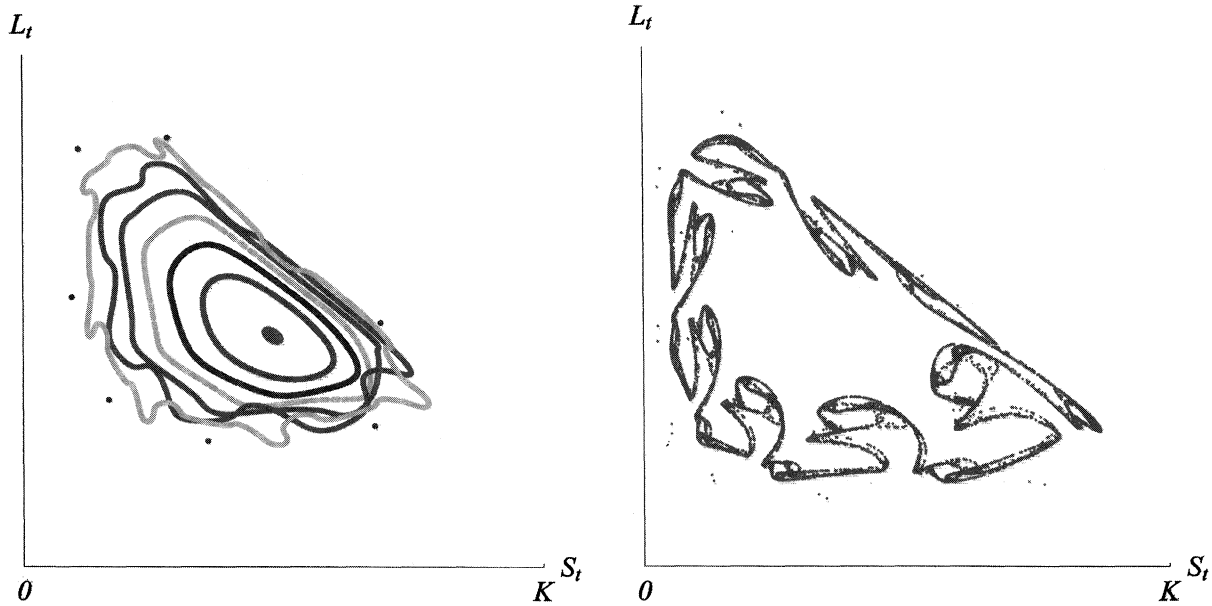


FIGURE 7 (A) invariant cycles, (B) strange attractor.

rate, $K\alpha\beta\phi > n$. To summarize our results, we can say that a discrete-time model can be a good candidate for explaining various pattern of evolutions observed in small islands.

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APPENDIX

In this appendix, we show that $F(\mathbb{R}) \subset \mathbb{R}$ if $0 \leq r\Delta t \leq 3$ and $(n, K\alpha\beta\phi) \in \mathbb{N}$. A boundary line of \mathbb{R} , $L = g(S)$, is downward sloping and crosses the vertical axis at $L_0 = 1 + r\Delta t/\alpha\beta\Delta t$ and the horizontal axis at $S_0 = (1 + r\Delta t)/(r\Delta t)K$. As a first step, we construct a locus of S

and L such that $\{(S, L) | S \geq 0, L \geq 0 \text{ and } \det J = 0\}$ where $\det J$ is the determinant of the Jacobi matrix of the dynamic system Eq. (3),

$$J = \begin{pmatrix} 1 + r\Delta t - 2\frac{r\Delta t}{K}S_t - \alpha\beta\Delta tL_t & -\alpha\beta\Delta tS_t \\ -\alpha\beta\phi\Delta tL_t & (1 - n\Delta t) + \alpha\beta\Delta tS_t \end{pmatrix}$$

Solving $\det J = 0$ for L yields a critical line, denoted as $g(S)$, that divides \mathbb{R} into two regions \mathbb{R}_1 and \mathbb{R}_2 ; $\det J > 0$ in \mathbb{R}_1 and $\det J < 0$ in \mathbb{R}_2 ,

$$h(S) = \frac{[K(1 + r\Delta t) - 2r\Delta tS_t][(1 - n\Delta t) + \alpha\beta\Delta tS_t]}{(1 - n\Delta t)K\alpha\beta\Delta t}$$

As illustrated in Fig. A1, the $L = h(S)$ curve crosses the horizontal axes at point $A = (S_A, L_A)$ where $S_A = S_0/2$, $L_A = 0$ and the vertical axes at point $B = (S_B, L_B)$ where $S_B = 0$ and $L_B = L_0$. The next step is to compute an intersection(s) of those two curves, $L = f(S)$ and $L = g(S)$. It can be verified that, in the non-negative quadrant of (S, L) , two curves intersect once at point B if $K\alpha\beta\phi \leq r(1 - \alpha\beta\phi) \leq r(1 - n\Delta t)/(1 + r\Delta t)$ and twice at point B and point $C = (S_c, L_c)$ where $S_c = [(1 + r\Delta t)K\alpha\beta\phi - r(1 - n\Delta t)]/2r\alpha\beta\phi\Delta t$ and $L_c = [(1 + r\Delta t)K\alpha\beta\phi + r(1 - n\Delta t)]/2r(\alpha\beta)^2\phi\Delta t$ if $K\alpha\beta\phi > r(1 - n\Delta t)/(1 + r\Delta t)$. We are now ready to detect conditions under which those critical points in \mathbb{R} are mapped into \mathbb{R} .

(1) Suppose $K\alpha\beta\phi \leq r(1 - n\Delta t)/(1 + r\Delta t)$. Consider point $A = (S_A, 0)$ in Fig. A1(A) It is mapped to point $A' = (S'_A, 0)$ on the horizontal axis where $S'_A = (1 + r\Delta t/4)S_0$. Hence $0 \leq S'_A \leq S_0$ hold if $0 \leq (1 + r\Delta t/4) \leq 1$, whence $0 \leq S'_A \leq S_0$ if $0 \leq r\Delta t \leq 3$. Consider point $B = (0, L_0)$. It is mapped to point $B' = (0, L'_B)$ on the vertical axis where $L'_B = (1 - n\Delta t)L_0$. Hence $0 \leq L'_B \leq L_0$ since $0 \leq (1 - n\Delta t) \leq 1$ by $n\Delta t \geq 0$ and Assumption.

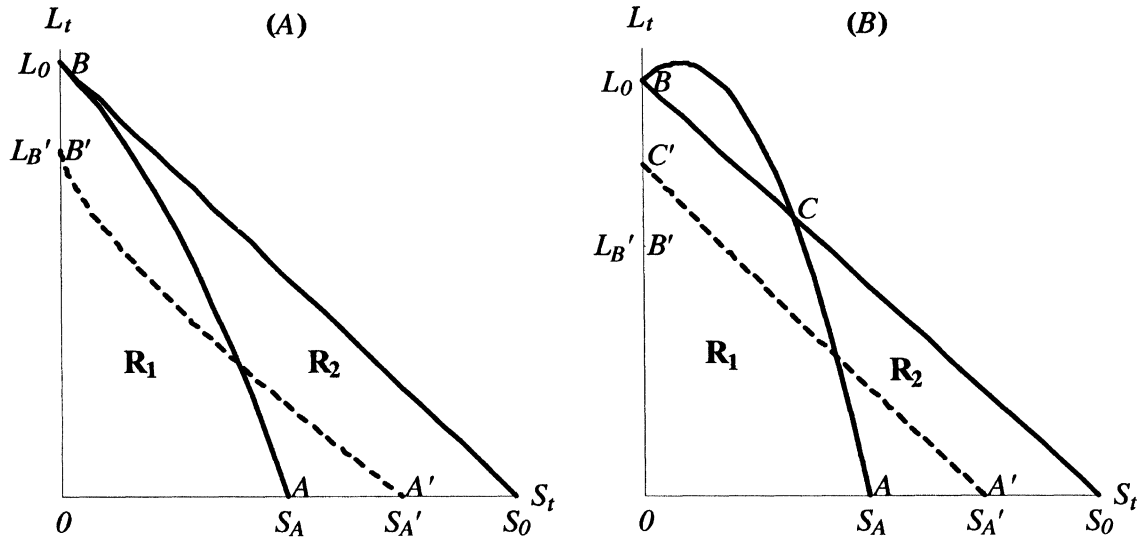


FIGURE A1 Transformation of \mathbb{R} . (A) $K\alpha\beta\phi \leq r(1 - n\Delta t)/(1 + r\Delta t)$; (B) $K\alpha\beta\phi > r(1 - n\Delta t)/(1 + r\Delta t)$.

(2) Suppose $K\alpha\beta\phi > r(1 - n\Delta t)/(1 + r\Delta t)$. By considering points A and B in Fig. A1(B), we can also find that $0 \leq S_{A'} \leq S_0$ if $0 \leq r\Delta t \leq 3$ and $0 \leq L_{B'} \leq L_0$ if $0 \leq n\Delta t \leq 1$. Point C is an intersection of $f(S)$ and $g(S)$. It is mapped to point $C' = (0, L'_C)$ on the vertical axis where $L'_C = \{(1 + r\Delta t)K\alpha\beta\phi + r(1 - n\Delta t)\}^2 / 4rK(\alpha\beta)^2\phi\Delta t$. It can be shown that $L'_C \leq L_0$ if $K\alpha\beta\phi \leq r(1 + \sqrt{n\Delta t}^2)/1 + r\Delta t$.

In either of Fig. A1(A) or (B), the real line AB is mapped to the dotted line $A'B'$, and either of the sub-regions, \mathbb{R}_1 or \mathbb{R}_2 , is mapped into the triangle $OA'B'$. Thus the followings

are the conditions under which $OA'B' \subset \mathbb{R}$, namely, $F(\mathbb{R}) \subset \mathbb{R}$.

- (1) Suppose $K\alpha\beta\phi \leq r(1 - n\Delta t)/(1 + r\Delta t)$, then $F(\mathbb{R}) \subset \mathbb{R}$ if $0 \leq n\Delta t \leq 1$ and $0 \leq r\Delta t \leq 3$.
- (2) Suppose $K\alpha\beta\phi > r(1 - n\Delta t)/(1 + r\Delta t)$, then $F(\mathbb{R}) \subset \mathbb{R}$ if $0 \leq n\Delta t \leq 1$, $0 \leq r\Delta t \leq 3$ and $K\alpha\beta\phi \leq r(1 + \sqrt{n\Delta t}^2)/(1 + r\Delta t)$

Putting (1) and (2) together, we have proved Theorem 2.