Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2012, Article ID 864340, 8 pages
doi:10.1155/2012/864340

Research Article

Nearly Derivations on Banach Algebras

M. Eshaghi Gordji,[1] H. Khodaei,[1] and G. H. Kim[2]

[1] *Department of Mathematics, Semnan University, P.O. Box 35195-363, Semnan, Iran*
[2] *Department of Mathematics, Kangnam University, Yongin, Gyeonggi 446-702, Republic of Korea*

Correspondence should be addressed to M. Eshaghi Gordji, madjid.eshaghi@gmail.com
and G. H. Kim, ghkim@kangnam.ac.kr

Received 11 December 2011; Accepted 23 January 2012

Academic Editor: John Rassias

Let n be a fixed integer greater than 3 and let λ be a real number with $\lambda \neq (n^2 - n + 4)/2$. We investigate the Hyers-Ulam stability of derivations on Banach algebras related to the following generalized Cauchy functional inequality $\| \sum_{\substack{1\le i<j\le n \\ 1\le k_l \neq i,j \le n}} f((x_i + x_j)/2 + \sum_{l=1}^{n-2} x_{k_l}) + f(\sum_{i=2}^{n} x_i) + f(x_1) \| \le \|\lambda f(\sum_{i=1}^{n} x_i)\|$.

1. Introduction and Preliminaries

Let $\mathcal{A}$ ba a Banach algebra and let X be a Banach $\mathcal{A}$-bimodule. Then X^*, the dual space of X, is also a Banach $\mathcal{A}$-bimodule with module multiplications defined by

$$\langle x, a \cdot x^* \rangle = \langle x \cdot a, x^* \rangle, \qquad \langle x, x^* \cdot a \rangle = \langle a \cdot x, x^* \rangle, \quad (a \in \mathcal{A},\ x \in X,\ x^* \in X^*). \tag{1.1}$$

A bounded linear operator $D : \mathcal{A} \to X$ is called a *derivation* if

$$D(ab) = a \cdot D(b) + D(a) \cdot b \quad (a, b \in \mathcal{A}). \tag{1.2}$$

Let $x \in X$. We define $\delta_x(a) = a \cdot x - x \cdot a$ for all $a \in \mathcal{A}$. δ_x is a derivation from $\mathcal{A}$ into X, which is called *inner derivation*. A Banach algebra $\mathcal{A}$ is *amenable* if every derivation from $\mathcal{A}$ into every dual $\mathcal{A}$-bimodule X^* is inner. This definition was introduced by Johnson in [1]. A Banach algebra $\mathcal{A}$ is *weakly amenable* if every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner. Bade et al. [2] have introduced the concept of weak amenability for commutative Banach algebras.

The stability problem of functional equations originated from a question of Ulam [3, 4] concerning the stability of group homomorphisms.

A famous talk presented by Ulam in 1940 triggered the study of stability problems for various functional equations.

We are given a group G_1 and a metric group G_2 with metric $\rho(\cdot,\cdot)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if $f : G_1 \to G_2$ satisfies $\rho(f(xy), f(x)f(y)) < \delta$ for all $x, y \in G_1$, then a homomorphism $h : G_1 \to G_2$ exists with $\rho(f(x), h(x)) < \epsilon$ for all $x \in G_1$?

In the following year, Hyers was able to give a partial solution to Ulam's question that was the first significant breakthrough and step toward more solutions in this area (see [5]). Since then, a large number of papers have been published in connection with various generalizations of Ulam's problem and Hyers' theorem.

Let n be a fixed integer greater than 3 and let λ be a real number with $|\lambda| \neq (n^2 - n + 4)/2$. We investigate the Hyers-Ulam stability of derivations on Banach algebras related to the following generalized Cauchy functional inequality:

$$\left\| \sum_{\substack{1\le i<j\le n \\ 1\le k_l\neq i,j\le n}} f\left(\frac{x_i + x_j}{2} + \sum_{l=1}^{n-2} x_{k_l}\right) + f\left(\sum_{i=2}^{n} x_i\right) + f(x_1) \right\| \le \left\| \lambda f\left(\sum_{i=1}^{n} x_i\right) \right\|. \tag{1.3}$$

2. Main Results

Let A be a Banach algebra and let X be a Banach A-module. From now on, the sum of $f(x)$ and $f(-x)$ will be denoted by $\tilde{f}(x)$. Also, $f(ab) - f(a)b - af(b)$ will be denoted by $\Delta f(a,b)$. In the following, we will use the Pascal formula:

$$C(r,k) = C(r-1,k) + C(r-1,k-1) \tag{2.1}$$

here, $C(r,k)$ denotes $r!/k!(r-k)!$ Moreover, we assume that $n_0 \in \mathbb{N}$ is a positive integer and suppose that $\mathbb{T}^1_{1/n_o} := \{e^{i\theta};\ 0 \le \theta \le 2\pi/n_o\}$.

Lemma 2.1. *Let $f : A \to X$ be a mapping such that*

$$\left\| \sum_{\substack{1\le i<j\le n \\ 1\le k_l\neq i,j\le n}} f\left(\frac{x_i + x_j}{2} + \sum_{l=1}^{n-2} x_{k_l}\right) + f\left(\sum_{i=2}^{n} x_i\right) + f(x_1) \right\| \le \left\| \lambda f\left(\sum_{i=1}^{n} x_i\right) \right\| \tag{2.2}$$

for all $x_1, \ldots, x_n \in A$. Then f is Cauchy additive.

Proof. Substituting $x_1, \ldots, x_n = 0$ in the functional inequality (2.2), we get

$$\| (C(n,2) + 2) f(0) \| \le \| \lambda f(0) \|. \tag{2.3}$$

Since $n \geq 3$ and $|\lambda| \neq (n^2 - n + 4)/2$, $f(0) = 0$. Letting $x_1 = x$, $x_2 = -x$ and $x_3 = \cdots = x_n = 0$ in (2.2) and using Pascal formula, we get

$$\left\| (n-2)\tilde{f}\left(\frac{x}{2}\right) + (C(n-2,2)+1)f(0) + \tilde{f}(x) \right\| \leq \|\lambda f(0)\|, \tag{2.4}$$

for all $x \in A$. Hence

$$(n-2)\tilde{f}\left(\frac{x}{2}\right) + \tilde{f}(x) = 0 \tag{2.5}$$

for all $x \in A$. Letting $x_1 = 2x$, $x_2 = -x$, $x_3 = -x$ and $x_4 = \cdots = x_n = 0$ in (2.2), we get

$$\left\| 2f\left(\frac{-x}{2}\right) + (n-3)f(-x) + f(x) + 2(n-3)f\left(\frac{x}{2}\right) + C(n-3,2)f(0) + \tilde{f}(2x) \right\| \leq \|\lambda f(0)\| \tag{2.6}$$

for all $x \in A$. Hence

$$\begin{aligned} &2f\left(\frac{-x}{2}\right) + (n-3)f(-x) + f(x) + 2(n-3)f\left(\frac{x}{2}\right) + \tilde{f}(2x) = 0, \\ &2f\left(\frac{x}{2}\right) + (n-3)f(x) + f(-x) + 2(n-3)f\left(\frac{-x}{2}\right) + \tilde{f}(-2x) = 0 \end{aligned} \tag{2.7}$$

for all $x \in A$. Since $\tilde{f}(-x) = \tilde{f}(x)$, we obtain from (2.7) and (2.4) that

$$2(n-2)\tilde{f}\left(\frac{x}{2}\right) + (n-2)\tilde{f}(x) + 2\tilde{f}(2x) = 0 \tag{2.8}$$

for all $x \in A$. It follows from (2.5) and (2.8) that

$$2\tilde{f}\left(\frac{x}{2}\right) - \tilde{f}(x) = 0 \tag{2.9}$$

for all $x \in A$. By using (2.5) and (2.9), we get $n\tilde{f}(x/2) = 0$ and so $f(-x) = -f(x)$ for all $x \in A$. Hence, we obtain from (2.7) that $f(x/2) = (1/2)f(x)$ for all $x \in A$. Letting $x_1 = x+y$, $x_2 = -x$, $x_3 = -y$ and $x_4 = \cdots = x_n = 0$ in (2.2), we get

$$\begin{aligned} &\left\| f\left(\frac{-y}{2}\right) + f\left(\frac{-x}{2}\right) + (n-3)f\left(\frac{-x-y}{2}\right) + f\left(\frac{x+y}{2}\right) + (n-3)f\left(\frac{x}{2}\right) + (n-3)f\left(\frac{y}{2}\right) \right. \\ &\quad \left. + C(n-3,2)f(0) + \tilde{f}(x+y) \right\| \leq \|\lambda f(0)\| \end{aligned} \tag{2.10}$$

for all $x, y \in A$. Next, notice that, using oddness of f and $f(x/2) = (1/2)f(x)$, we have

$$f(x+y) = f(x) + f(y) \tag{2.11}$$

for all $x, y \in A$, as desired. □

We can prove the following lemma by the same reasoning as in the proof of Theorem 2.2 of [6].

Lemma 2.2. *Let $f : A \to X$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $\mu \in T^1_{1/n_o}$ and all $x \in A$. Then the mapping f is $\mathbb{C}$-linear.*

Theorem 2.3. *Let $f : A \to X$ be a mapping satisfying $f(0) = 0$ and the inequality*

$$\left\| \sum_{\substack{1\le i<j\le n\\ 1\le k_l \ne i,j\le n}} f\left(\frac{\mu x_i + \mu x_j}{2} + \sum_{l=1}^{n-2}\mu x_{k_l}\right) + f\left(\sum_{i=2}^{n}\mu x_i\right) + \mu f(x_1) + \Delta f(a,b) \right\| \le \left\| \lambda f\left(\sum_{i=1}^{n}\mu x_i\right)\right\| + \delta \tag{2.12}$$

for some $\delta > 0$, for all $\mu \in T^1_{1/n_o}$ and all $a, b, x_1, \dots, x_n \in A$. Then there exists a unique derivation $\mathfrak{D} : A \to X$ such that

$$\| f(x) - \mathfrak{D}(x)\| \le \frac{13n-24}{n(n-4)}\delta \tag{2.13}$$

for all $x \in A$.

Proof. Letting $a = b = 0$, $x_1 = 2x$, $x_2 = -2x$, $x_3 = \cdots = x_n = 0$ and $\mu = 1$ in (2.12), we get

$$\left\|(n-2)\tilde{f}(x) + \tilde{f}(2x)\right\| \le \delta \tag{2.14}$$

for all $x \in X$. Letting $a = b = 0$, $x_1 = 2x$, $x_2 = -x$, $x_3 = -x$, $x_4 = \cdots = x_n = 0$ and $\mu = 1$ in (2.12), we get

$$\left\|2f\left(\frac{-x}{2}\right) + (n-3)f(-x) + f(x) + 2(n-3)f\left(\frac{x}{2}\right) + \tilde{f}(2x)\right\| \le \delta \tag{2.15}$$

for all $x \in X$. Letting $a = b = 0$, $x_1 = -2x$, $x_2 = x$, $x_3 = x$, $x_4 = \cdots = x_n = 0$ and $\mu = 1$ in (2.12), we get

$$\left\|2f\left(\frac{x}{2}\right) + (n-3)f(x) + f(-x) + 2(n-3)f\left(\frac{-x}{2}\right) + \tilde{f}(-2x)\right\| \le \delta \tag{2.16}$$

for all $x \in X$. It follows from (2.15) and (2.16) that

$$\left\|(n-2)\tilde{f}\left(\frac{x}{2}\right) + \frac{(n-2)}{2}\tilde{f}(x) + \tilde{f}(2x)\right\| \le \delta \tag{2.17}$$

for all $x \in X$. It follows from (2.14) and (2.17) that

$$\left\|\tilde{f}(x)\right\| \le \frac{6}{n}\delta \tag{2.18}$$

for all $x \in X$. It follows from (2.15) and (2.18) that

$$\left\| 2\tilde{f}\left(\frac{x}{2}\right) + \tilde{f}(x) + (n-4)f(-x) + 2(n-4)f\left(\frac{x}{2}\right) \right\| \le \frac{n+6}{n}\delta \tag{2.19}$$

for all $x \in X$. From the last two inequalities, we have

$$\| f(2x) + 2f(-x) \| \le \frac{n+24}{n(n-4)}\delta \tag{2.20}$$

for all $x \in X$. It follows from (2.18) and (2.20) that

$$\left\| f(x) - \frac{1}{2}f(2x) \right\| \le \frac{13n-24}{2n(n-4)}\delta \tag{2.21}$$

for all $x \in X$. Hence

$$\left\| \frac{1}{2^r}f(2^r x) - \frac{1}{2^m}f(2^m x) \right\| \le \frac{13n-24}{2n(n-4)}\sum_{k=r}^{m-1}\frac{\delta}{2^k} \tag{2.22}$$

for all $x \in X$ and integers $m > r \ge 0$. Thus it follows that a sequence $\{(1/2^m)f(2^m x)\}$ is Cauchy in Y and so it converges. Therefore we can define a mapping $\mathfrak{D} : X \to Y$ by $\mathfrak{D}(x) := \lim_{m\to\infty}(1/2^m)f(2^m x)$ for all $x \in X$. In addition it is clear from (2.12) that the following inequality:

$$\begin{aligned}
&\left\| \sum_{\substack{1\le i<j\le n \\ 1\le k_l \ne i,j\le n}} \mathfrak{D}\left(\frac{\mu x_i + \mu x_j}{2} + \sum_{l=1}^{n-2}\mu x_{k_l}\right) + \mathfrak{D}\left(\sum_{i=2}^{n}\mu x_i\right) + \mu\mathfrak{D}(x_1) \right\| \\
&\quad = \lim_{m\to\infty}\frac{1}{2^m}\left\| \sum_{\substack{1\le i<j\le n \\ 1\le k_l \ne i,j\le n}} f\left(2^{m-1}\mu(x_i+x_j) + \sum_{l=1}^{n-2}2^m\mu x_{k_l}\right) + f\left(\sum_{i=2}^{n}2^m\mu x_i\right) + \mu f(2^m x_1) \right\| \\
&\quad \le \lim_{m\to\infty}\frac{1}{2^m}\left\| \lambda f\left(\sum_{i=1}^{n}2^m\mu x_i\right) \right\| + \lim_{m\to\infty}\frac{\delta}{2^m} \\
&\quad = \left\| \lambda\mathfrak{D}\left(\sum_{i=1}^{n}\mu x_i\right) \right\|
\end{aligned} \tag{2.23}$$

holds for all $\mu \in T^1_{1/n_o}$ and all $x_1, \ldots, x_n \in X$. If we put $\mu = 1$ in the last inequality, then $\mathfrak{D}$ is additive by Lemma 2.1. Letting $x_1 = x$, $x_2 = -x$ and $x_3 = \cdots = x_n = 0$ in last inequality and using Lemma 2.1, we get

$$(n-2)\tilde{\mathfrak{D}}\left(\frac{\mu x}{2}\right) + \mathfrak{D}(-\mu x) + \mu\mathfrak{D}(x) = \mu\mathfrak{D}(x) - \mathfrak{D}(\mu x). \tag{2.24}$$

So $\mathfrak{D}(\mu x) = \mu\mathfrak{D}(x)$ for all $x \in X$ and all $\mu \in T^1_{1/n_o}$. Now by using Lemmas 2.1 and 2.2, we infer that the mapping $\mathfrak{D} : X \to Y$ is $\mathbb{C}$-linear. Taking the limit as $m \to \infty$ in (2.22) with $r = 0$, we get (2.13).

To prove the afore-mentioned uniqueness, we assume now that there is another $\mathbb{C}$-linear mapping $\mathfrak{L} : A \to X$ which satisfies the inequality (2.13). Then we get

$$\left\| \frac{1}{2^m} f(2^m x) - \mathfrak{L}(x) \right\| = \frac{1}{2^m} \left\| f(2^m x) - \mathfrak{L}(2^m x) \right\| \leq \frac{13n-24}{2^m n(n-4)} \delta \tag{2.25}$$

for all $x \in A$ and integers $m \geq 1$. Thus from $m \to \infty$, one establishes

$$\mathfrak{D}(x) - \mathfrak{L}(x) = 0 \tag{2.26}$$

for all $x \in A$, completing the proof of uniqueness.

Now, we have to show that $\mathfrak{D}$ is a derivation. To this end, let $x_1 = x_2 = \cdots = x_n = 0$ in (2.12), we get

$$\left\| f(ab) - f(a)b - af(b) \right\| \leq \delta \tag{2.27}$$

for all $a, b \in A$. It follows from linearity of $\mathfrak{D}$ and (2.27) that

$$\begin{aligned} \|\mathfrak{D}(ab) - \mathfrak{D}(a)b - a\mathfrak{D}(b)\| &= \left\| \frac{1}{2^m}\mathfrak{D}(2^m ab) - \mathfrak{D}(a)\frac{1}{2^m}(2^m b) - \frac{1}{2^m}(2^m a)\mathfrak{D}(b) \right\| \\ &= \lim_{m\to\infty} \left\| \frac{1}{4^m} f(4^m ab) - f(2^m a)\frac{1}{4^m}(2^m b) - \frac{1}{4^m}(2^m a) f(2^m b) \right\| \\ &= \lim_{m\to\infty} \frac{1}{4^m} \left\| f(2^m a 2^m b) - f(2^m a)(2^m b) - (2^m a) f(2^m b) \right\| \\ &\leq \lim_{m\to\infty} \frac{1}{4^m}\delta \\ &= 0 \end{aligned} \tag{2.28}$$

for all $a, b \in A$. This means that $\mathfrak{D}$ is a derivation from A into X. Therefore the mapping $\mathfrak{D} : A \to X$ is a unique derivation satisfying (2.13), as desired. □

Theorem 2.4. *Let A be an amenable Banach algebra and let $f : A \to X^*$ be a mapping such that $f(0) = 0$ and* (2.12). *If*

$$\sup\left\{ \left\| f(x) \right\| : \|x\| \leq 1 \right\} < \infty, \tag{2.29}$$

then there exists $x_0 \in X^*$ such that

$$\left\| f(a) - ax_0 - x_0 a \right\| \leq \frac{13n-24}{n(n-4)} \delta \tag{2.30}$$

for all $a \in A$.

Proof. Let $\sup\{\|f(x)\| : \|x\| \leq 1\} = M_f$. Then by (2.29), we have $M_f < \infty$. By Theorem 2.3, there exists a derivation $D : A \to X^*$ satisfying (2.13). Then we have

$$\sup\{\|D(x)\| : \|x\| \leq 1\} \leq M_f + \frac{13n-24}{n(n-4)}\delta. \tag{2.31}$$

This means that D is bounded, and hence D is continuous. On the other hand, A is amenable. Then every continuous derivation from A into X^* is an inner derivation. It follows that D is and an inner derivation. In the other words, there exists $x_0 \in X^*$ such that $D(a) = ax_0 - x_0a$ for all $a \in A$. This completes the proof. □

We know that every nuclear C^*-algebra is amenable (see [7]). Then we have the following result.

Corollary 2.5. *Let A be a nuclear C^*-algebra and let $f : A \to X^*$ be a mapping such that $f(0) = 0$, and* (2.12) *and* (2.29)*. Then there exists $x_0 \in X^*$ such that*

$$\|f(a) - ax_0 - x_0a\| \leq \frac{13n-24}{n(n-4)}\delta \tag{2.32}$$

for all $a \in A$.

Theorem 2.6. *Let A be a C^*-algebra and let $f : A \to A^*$ be a mapping such that $f(0) = 0$, and* (2.12) *and* (2.29)*. Then there exists $a' \in A^*$ such that*

$$\|f(a)(b) - a'(ba - ab)\| \leq \frac{13n-24}{n(n-4)}\delta\|b\| \tag{2.33}$$

for all $a, b \in A$.

Proof. We know that every C^*-algebra is weakly amenable (see, e.g., [7]). Then every continuous derivation from A into A^* is an inner derivation. By the same reasoning as in the proof of Theorem 2.4, there exists a $a' \in A^*$ such that $D(a) = aa' - a'a$ for all $a \in A$, and

$$\|f(a) - aa' - a'a\| \leq \frac{13n-24}{n(n-4)}\delta \tag{2.34}$$

for all $a \in A$. By definition of mudule actions of A on A^*, we have

$$\|f(a)(b) - a'(ba - ab)\| \leq \frac{13n-24}{n(n-4)}\delta\|b\| \tag{2.35}$$

for all $a, b \in A$. □

Corollary 2.7. *Let A be a commutative C^*-algebra and let $f : A \to A^*$ be a mapping such that $f(0) = 0$, and* (2.12) *and* (2.29). *Then*

$$\lim_{m\to\infty} \frac{1}{2^m} f(2^m a) = 0, \qquad \|f(a)\| \leq \frac{13n-24}{n(n-4)}\delta \tag{2.36}$$

for all $a \in A$.

References

[1] B. E. Johnson, "Cohomology in Banach algebras," *Memoirs of the American Mathematical Society*, vol. 127, 1972.

[2] W. G. Bade, P. G. Curtis, and H. G. Dales, "Amenability and weak amenability for Beurling and Lipschits algebra," *Proceedings London Mathematical Society*, vol. 55, no. 3, pp. 359–377, 1987.

[3] S. M. Ulam, *A Collection ofMathematical Problems*, Interscience Publisher, New York, NY, USA, 1960.

[4] S. M. Ulam, *Problems in Modern Mathematics*, vol. 6, Wiley-Interscience, New York, 1964.

[5] D.H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Academy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[6] M. E. Gordji, "Nearly involutions onBanach algebras; A fixed point approach," *Fixed Point Theory*. In press.

[7] H. G. Dales, *Banach Algebras and Automatic Continuity*, vol. 24 of *London Mathematical Society Monographs, New Series*, Oxford Science Publications, The Clarendon Press, Oxford University Press, New York, NY, USA, 2000.