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Research Article

Improved Bounds for Restricted Isometry Constants

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The purpose of this paper is to establish improved bounds for restricted isometry constants δ_k. Our results, to some extent, improve and extend the well-known bound ($\delta_k < 0.307$) in (Cai et al., 2010) to $\delta_k < 0.308$.

1. Introduction

Consider the following equation:

$$y = A\beta + z, \tag{1.1}$$

where the matrix $A \in \mathbb{R}^{n\times m}$ $(n < m)$ and $z \in \mathbb{R}^n$ is a vector of measurement errors. If $z = 0$, then (1.1) is an underdetermined linear system with fewer equations than unknowns. The task is to reconstruct the signal $\beta \in \mathbb{R}^m$ based on the matrix A and the vector y. Usually, we consider ℓ_0 minimization problem:

$$\min_{\widehat{\beta}\in\mathbb{R}^n} \left\|\widehat{\beta}\right\|_0, \quad \text{subject to } y = A\widehat{\beta} + z \text{ and } \|z\|_2 \leq \epsilon, \tag{1.2}$$

where $\|\cdot\|_0$ denotes the ℓ_0-norm of a vector, that is, the number of its nonzero components.

We need to solve this problem and find the sparsest solution among all the possible solutions. But it requires a combinatorial search and remains an NP-hard problem that cannot be solved in practice. Naturally, an alternative strategy is to find ℓ_1 minimization problem:

$$\min_{\widehat{\beta}\in\mathbb{R}^n}\left\|\widehat{\beta}\right\|_1, \quad \text{subject to } y = A\widehat{\beta} + z \text{ and } \|z\|_2 \le \epsilon, \tag{1.3}$$

and we expect to find the sparsest solution.

In order to exactly recover the sparsest solution in ℓ_1 minimization, Candès and Tao [1] introduced restricted isometry property (see Restricted Isometry Constants in Definition 2.1). So far, there are various methods [1–9] to give the sufficient conditions on δ_{2k}: Candès [3] established that $\delta_{2k} < \sqrt{2} - 1 \approx 0.4142$ is the sufficient condition of exactly recover k-sparse vectors via ℓ_1 minimization (a vector x is k-sparse if $\|x\|_0 \le k$). This sufficient condition was later improved to $\delta_{2k} < 2(3 - \sqrt{2})/7 \approx 0.4531$ in [6] and to $\delta_{2k} < 3/(4 + \sqrt{6}) \approx 0.4652$ in [5]. Later, the sufficient condition was improved to $\delta_{2k} < 1/(1 + \sqrt{1.25}) \approx 0.4721$ in [10] for the special case that k is a multiple of 4 or k is very large and to $\delta_{2k} < 4/(6 + \sqrt{6}) \approx 0.4734$ in [5]. Naturally, we want to give the sufficient condition about δ_k. To the best of our knowledge, T. T. Cai et al. [2] firstly show that the restricted isometry constant δ_k of A satisfies $\delta_k < 0.307$ for general k, then k-sparse signals are guaranteed to be recovered exactly via ℓ_1 minimization. Based on this motivation, we construct a different partition of $\{1, 2, \ldots, m\}$ and then discuss the error between original signal β and recover signal $\widehat{\beta}$ in (1.3). The main work of this paper is to improve the condition to $\delta_k < 0.308$ and to prove that the k-sparse signals can be recovered exactly via ℓ_1 minimization in no noise case and be estimated stably under the perturbation of noise.

To state our main results, we firstly give the following preliminaries.

2. Preliminaries

In 2005, Candès and Tao [1] firstly present the definition of the restricted isometry constant.

Definition 2.1 (see [1], restricted isometry constants). Let F be the matrix with finite collection of vectors $(v_j)_{j\in J} \in \mathbb{R}^n$ as columns. For every integer $1 \le S \le |J|$, the S-restricted isometry constants δ_S are defined as the smallest quantity such that F_T obeys

$$(1 - \delta_S)\|c\|_2^2 \le \|F_T c\|_2^2 \le (1 + \delta_S)\|c\|_2^2 \tag{2.1}$$

for all subsets $T \subset J$ of cardinality at most S and all real coefficients $(c_j)_{j\in T}$. Similarly, we define the S,S'-restricted orthogonality constants $\theta_{S,S'}$ for $S+S' \le |J|$ to be the smallest quantity such that

$$\left|\langle F_T c, F_T' c'\rangle\right| \le \theta_{S,S'}\|c\|_2 \cdot \left\|c'\right\|_2 \tag{2.2}$$

holds for all disjoint sets $T, T' \subseteq J$ of cardinality $|T| \le S$ and $|T'| \le S'$.

In addition, we can easily check the following monotone properties:

$$\delta_k \leq \delta_{k_1}, \quad \text{if } k \leq k_1 \leq n,$$
$$\theta_{k,k'} \leq \theta_{k_1,k_1'}, \quad \text{if } k \leq k_1, k' \leq k_1', k_1 + k_1' \leq n. \tag{2.3}$$

Apart from the above relationship, Candès and Tao [1] proved that the restricted orthogonality constant $\theta_{k,k'}$ and the restricted isometry constant δ_k are related by the following lemma.

Lemma 2.2 (see [1]). *One has $\theta_{S,S'} \leq \delta_{S+S'} \leq \theta_{S,S'} + \max(\delta_S, \delta_{S'})$ for all S, S'.*

In the sequel, a useful inequality between ℓ_1-norm and ℓ_2-norm will be introduced.

Proposition 2.3 (see [2]). *For any $x \in \mathbb{R}^n$,*

$$\|x\|_2 - \frac{\|x\|_1}{\sqrt{n}} \leq \frac{\sqrt{n}}{4}\left(\max_{1\leq i\leq n}|x_i| - \min_{1\leq i\leq n}|x_i|\right). \tag{2.4}$$

At the last of preliminaries, we introduce the square root lifting inequality [10].

Lemma 2.4 (see [10]). *For any $a \geq 1$ and positive integers k, k' such that ak' is an integer, then*

$$\theta_{k,ak'} \leq \sqrt{a}\theta_{k,k'}. \tag{2.5}$$

3. Improved Bounds for Restricted Isometry Constants

In this section, we discuss the new restricted isometry constant δ_k for sparse signal recovery via ℓ_1 minimization in (1.3).

Theorem 3.1. *Suppose β is k-sparse. Then the ℓ_1 minimizer $\widehat{\beta}$ defined in* (1.3) *satisfies*

$$\left\|\beta - \widehat{\beta}\right\|_2 \leq \frac{2\sqrt{2}\sqrt{1+\delta_k}}{1 - 13/4\delta_k}\epsilon, \tag{3.1}$$

where δ_k is the k-restricted isometry constant of A in (1.3).

Proof. Let $h = \widehat{\beta} - \beta \in \mathbb{R}^m$. Partition $\{1, 2, \ldots, m\}$ into the following sets:

$$T_0 = \{1, 2, \ldots, k\}, \qquad T_1 = \left\{k+1, \ldots, k+\frac{k}{2}\right\}, \qquad T_2 = \left\{k+\frac{k}{2}+1, \ldots, 2k\right\}, \ldots, \tag{3.2}$$

where k is an even number. And rearranging the indices if necessary, $|h(1)| \geq |h(2)| \geq \cdots$, where $|h(i)|$, $i = 1, 2, \ldots, m$ is the ith entry of the above vector by rearranging the indices. Then by Proposition 2.3, we obtain

$$\sum_{i\geq 1} \|h_{T_i}\|_2 \leq \frac{1}{\sqrt{k/2}} \sum_{i\geq 1} \|h_{T_i}\|_1 + \frac{\sqrt{k/2}}{4}\left(|h(k+1)| - \left|h\left(k+\frac{k}{2}\right)\right|\right) + \cdots . \tag{3.3}$$

By the triangle inequality for $\|\cdot\|_1$, we have

$$\left|\|\beta\|_1 - \|-h_{T_0}\|_1\right| \leq \|\beta + h_{T_0}\|_1. \tag{3.4}$$

Since $T_0 \bigcap T_0^c = \emptyset$, we have

$$\|\beta\|_1 - \|h_{T_0}\|_1 + \left\|h_{T_0^c}\right\|_1 \leq \left\|\beta + h_{T_0} + h_{T_0^c}\right\|_1 = \|\beta + h\|_1 = \left\|\widehat{\beta}\right\|_1 \leq \|\beta\|_1. \tag{3.5}$$

The last inequality holds because $\widehat{\beta}$ solves (1.3). Then the result is that

$$\left\|h_{T_0^c}\right\|_1 \leq \|h_{T_0}\|_1. \tag{3.6}$$

Substituting (3.6) into (3.3), we get

$$\begin{aligned}\sum_{i\geq 1} \|h_{T_i}\|_2 &\leq \frac{1}{\sqrt{k/2}}\left\|h_{T_0^c}\right\|_1 + \frac{\sqrt{k/2}}{4}|h(k+1)| \\ &\leq \frac{1}{\sqrt{k/2}}\|h_{T_0}\|_1 + \frac{\sqrt{k/2}}{4}\cdot\frac{\|h_{T_0}\|_2}{\sqrt{k}} \\ &\leq \frac{1}{\sqrt{k/2}}\cdot\sqrt{k}\|h_{T_0}\|_2 + \frac{1}{4\sqrt{2}}\|h_{T_0}\|_2 \\ &\leq \frac{9\sqrt{2}}{8}\|h_{T_0}\|_2.\end{aligned} \tag{3.7}$$

And note that

$$|\langle Ah, Ah_{T_0}\rangle| \geq |\langle Ah_{T_0}, Ah_{T_0}\rangle| - \sum_{i\geq 1}|\langle Ah_{T_i}, Ah_{T_0}\rangle|. \tag{3.8}$$

From (2.2) and (2.5) in Lemma 2.4, we have

$$|\langle Ah_{T_i}, Ah_{T_0}\rangle| \leq \theta_{k/2,k}\|h_{T_i}\|_2 \cdot \|h_{T_0}\|_2. \tag{3.9}$$

By Lemma 2.2, we have

$$\theta_{k/2,k} = \theta_{k/2,2\cdot k/2} \leq \sqrt{2}\delta_{k/2+k/2} = \sqrt{2}\delta_k. \tag{3.10}$$

From (3.7)–(3.10), we have

$$\begin{aligned}
|\langle Ah, Ah_{T_0}\rangle| &\geq (1-\delta_k)\|h_{T_0}\|_2^2 - \theta_{k/2,k}\|h_{T_0}\|_2\sum_{i\geq 1}\|h_{T_i}\|_2 \\
&\geq (1-\delta_k)\|h_{T_0}\|_2^2 - \sqrt{2}\delta_k\|h_{T_0}\|_2 \cdot 9\sqrt{2}/8\|h_{T_0}\|_2 \\
&\geq \left(1-\frac{13\delta_k}{4}\right)\|h_{T_0}\|_2^2.
\end{aligned} \tag{3.11}$$

From (1.3), we have

$$\|Ah\|_2 = \left\|A\left(\widehat{\beta}-\beta\right)\right\|_2 \leq \left\|A\widehat{\beta}-y\right\|_2 + \left\|A\beta - y\right\|_2 \leq 2\epsilon. \tag{3.12}$$

In addition, we obtain the following relation by simple calculation

$$\begin{aligned}
\left\|h_{T_0^c}\right\|_2^2 &= \left(|h(k+1)|^2 + |h(k+2)|^2 + \cdots\right) \\
&\leq \max_{i\geq k+1}|h(i)| \cdot (|h(k+1)| + |h(k+2)| + \cdots) \\
&= \max_{i\geq k+1}|h(i)| \cdot \left\|h_{T_0^c}\right\|_1 \\
&\leq \frac{\|h_{T_0}\|_1}{k} \cdot \left\|h_{T_0^c}\right\|_1.
\end{aligned} \tag{3.13}$$

Since $\|h_{T_0^c}\|_1 \leq \|h_{T_0}\|_1$, we have

$$\left\|h_{T_0^c}\right\|_2^2 \leq \frac{\|h_{T_0}\|_1^2}{k}. \tag{3.14}$$

By the norm inequality $\|h_{T_0}\|_1^2 \leq k\|h_{T_0}\|_2^2$ and (3.14), we have

$$\left\|h_{T_0^c}\right\|_2^2 \leq \|h_{T_0}\|_2^2. \tag{3.15}$$

From (3.7), (3.11)-(3.12), and (3.15), we have

$$\begin{aligned}
\|h\|_2 \leq \sqrt{2}\|h_{T_0}\|_2 &\leq \frac{\sqrt{2}|\langle Ah, Ah_{T_0}\rangle|}{(1-13\delta_k/4)\|h_{T_0}\|_2} \leq \frac{\sqrt{2}\|Ah\|_2 \cdot \|Ah_{T_0}\|_2}{(1-13\delta_k/4)\|h_{T_0}\|_2} \\
&\leq \frac{\sqrt{2}\cdot 2\epsilon \cdot \sqrt{1+\delta_k}\|h_{T_0}\|_2}{(1-13\delta_k/4)\|h_{T_0}\|_2} \leq \frac{2\sqrt{2}\sqrt{1+\delta_k}}{1-13\delta_k/4}\epsilon.
\end{aligned} \tag{3.16}$$

□

Remark 3.2. If $\epsilon = 0$, it is the case where the k-sparse signals are guaranteed to be recovered exactly via ℓ_1 minimization under no noise situation.

Corollary 3.3. *Let $y = A\beta + z$ with $\|z\|_2 \leq \epsilon$. Suppose β is k-sparse with $k > 1$. Then under the condition $\delta_k < 0.308$ the constrained ℓ_1 minimizer $\widehat{\beta}$ given in* (1.3) *satisfies*

$$\left\|\beta - \widehat{\beta}\right\|_2 \leq \frac{3.2344}{0.308 - \delta_k}\epsilon. \tag{3.17}$$

Proof. The proof of this corollary can be easily obtained if we put $\delta_k < 0.308$ into the inequality (3.1) in Theorem 3.1. □

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