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Research Article

Some Identities on Bernoulli and Euler Numbers

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Recently, Kim introduced the fermionic p-adic integral on $\mathbb{Z}_p$. By using the equations of the fermionic and bosonic p-adic integral on $\mathbb{Z}_p$, we give some interesting identities on Bernoulli and Euler numbers.

1. Introduction/Preliminaries

Let p be a fixed odd prime number. Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$, and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers, and the completion of algebraic closure of $\mathbb{Q}_p$, respectively. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. The p-adic absolute value $|\cdot|_p$ is normally defined by $|p|_p = 1/p$.

Let $\mathrm{UD}(\mathbb{Z}_p)$ be the space of uniformly differentiable functions on $\mathbb{Z}_p$ and $C(\mathbb{Z}_p)$ the space of continuous function on $\mathbb{Z}_p$. For $f \in C(\mathbb{Z}_p)$, the fermionic p-adic integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N \to \infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \quad \text{(see [1])}. \tag{1.1}$$

The following fermionic p-adic integral equation on $\mathbb{Z}_p$ is well known (see [1–3]):

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0), \tag{1.2}$$

where $f_1(x) = f(x+1)$.

From (1.1) and (1.2), we can derive the generating function of Euler polynomials as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_{-1}(y) = \frac{2}{e^t+1} e^{xt} = \sum_{n=0}^{\infty} E_n(x) \frac{t^n}{n!}, \tag{1.3}$$

where $E_n(x)$ is the nth ordinary Euler polynomial (see [1–4]). In the special case, $x = 0$, $E_n(0) = E_n$ is called the nth ordinary Euler number.

By (1.3), we get Witt's formula for the nth Euler polynomial as follows:

$$\int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = E_n(x), \quad \text{for } n \in \mathbb{Z}_+. \tag{1.4}$$

Thus, by (1.4), we have

$$E_n(x) = (E+x)^n = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} E_l, \tag{1.5}$$

with the usual convention about replacing E^n by E_n (see [5, 6]). From (1.3), we note that

$$(E+1)^n + E_n = 2\delta_{0,n}, \tag{1.6}$$

where $\delta_{k,n}$ is the Kronecker symbol (see [3]). By (1.2) and (1.4), we get

$$\int_{\mathbb{Z}_p} (x+y+1)^n d\mu_{-1}(y) + \int_{\mathbb{Z}_p} (x+y)^n d\mu_{-1}(y) = 2x^n. \tag{1.7}$$

Thus, by (1.4) and (1.7), we have

$$E_n(x+1) + E_n(x) = 2x^n, \quad \text{for } n \in \mathbb{Z}_+. \tag{1.8}$$

Equation (1.8) is equivalent to

$$x^n = E_n(x) + \frac{1}{2} \sum_{l=0}^{n-1} \binom{n}{l} E_l(x). \tag{1.9}$$

From (1.6), we can derive the following equation:

$$E_n(2) = 2 - E_n(1) = 2 + E_n - 2\delta_{0,n}, \quad \text{for } n \in \mathbb{Z}_+. \tag{1.10}$$

For $f \in \mathrm{UD}(\mathbb{Z}_p)$, the bosonic p-adic integral on $\mathbb{Z}_p$ is defined by

$$I_1(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \quad \text{(see [4])}. \tag{1.11}$$

From (1.11), we can easily derive the following I_1-integral equation:

$$I_1(f_1) = I(f) + f'(0), \qquad (\text{see [4, 7, 8]}), \tag{1.12}$$

where $f_1(x) = f(x+1)$ and $f'(0) = df(x)/dx|_{x=0}$.

It is well known that the Bernoulli polynomial can be represented by the bosonic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} e^{(x+y)t} d\mu_1(y) = \frac{t}{e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}, \tag{1.13}$$

where $B_n(x)$ is called the nth Bernoulli polynomial (see [4, 7–13]). In the special case, $x = 0$, $B_n(0) = B_n$ is called the nth Bernoulli number. By the definition of Bernoulli numbers and polynomials, we get

$$B_n(x) = \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y) = \sum_{l=0}^{n} \binom{n}{l} x^{n-l} B_l. \tag{1.14}$$

Thus, by (1.13) and (1.14), we see that

$$B_0 = 1, \qquad (B+1)^n - B_n = \delta_{1,n}, \tag{1.15}$$

with the usual convention about replacing B^n by B_n (see [1–22]).

By (1.11), we easily get

$$\int_{\mathbb{Z}_p} (1 - x + y)^n d\mu_1(y) = (-1)^n \int_{\mathbb{Z}_p} (x+y)^n d\mu_1(y). \tag{1.16}$$

From (1.13), (1.14), and (1.16), we have

$$B_n(1-x) = (-1)^n B_n(x) \quad \text{for } n \in \mathbb{Z}_+. \tag{1.17}$$

By (1.15), we get

$$B_n(2) = n + B_n(1) = n + B_n + \delta_{1,n}. \tag{1.18}$$

Thus, by (1.17) and (1.18), we have

$$(-1)^n B_n(-1) = B_n(2) = n + B_n + \delta_{1,n}, \qquad (\text{see [4]}). \tag{1.19}$$

From (1.12) and (1.13), we get

$$\int_{\mathbb{Z}_p} (x+1+y)^{n+1} d\mu_1(y) - \int_{\mathbb{Z}_p} (x+y)^{n+1} d\mu_1(y) = (n+1)x^n. \tag{1.20}$$

Thus, by (1.13) and (1.20), we have

$$B_{n+1}(x+1) - B_{n+1}(x) = (n+1)x^n \quad \text{for } n \in \mathbb{Z}_+. \tag{1.21}$$

Equation (1.21) is equivalent to the following equation:

$$x^n = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}B_l(x) \quad \text{for } n \in \mathbb{Z}_+. \tag{1.22}$$

In this paper we derive some interesting and new identities for the Bernoulli and Euler numbers from the p-adic integral equations on $\mathbb{Z}_p$.

2. Some Identities on Bernoulli and Euler Numbers

From (1.1), we note that

$$\int_{\mathbb{Z}_p}(1-x+y)^n d\mu_{-1}(y) = (-1)^n\int_{\mathbb{Z}_p}(x+y)^n d\mu_{-1}(y). \tag{2.1}$$

By (1.14) and (2.1), we get

$$E_n(1-x) = (-1)^n E_n(x), \qquad \text{where } n \in \mathbb{Z}_+. \tag{2.2}$$

In the special case, $x = -1$, we have

$$E_n(2) = (-1)^n E_n(-1) = 2 + E_n - 2\delta_{0,n}. \tag{2.3}$$

Let us consider the following fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\int_{\mathbb{Z}_p}B_l(x)d\mu_{-1}(x)\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k}\int_{\mathbb{Z}_p}x^k d\mu_{-1}(x)\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k}E_k.\end{aligned} \tag{2.4}$$

Therefore, by (1.4) and (2.4), we obtain the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, one has*

$$E_n = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\sum_{k=0}^{l}\binom{l}{k}B_{l-k}E_k. \tag{2.5}$$

It is known that $B_n(x) = (-1)^n B_n(1-x)$. If we take the fermionic p-adic integral on both sides of (1.22), then we have

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}\int_{\mathbb{Z}_p} B_l(x)d\mu_{-1}(x)\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\int_{\mathbb{Z}_p} B_l(1-x)d\mu_{-1}(x)\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}B_{l-k}\int_{\mathbb{Z}_p}(1-x)^k d\mu_{-1}(x)\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}B_{l-k}(-1)^k E_k(-1).\end{aligned} \tag{2.6}$$

From (2.2) and (2.6), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu_{-1}(x) &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}B_{l-k}E_k(2)\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}B_{l-k}(2+E_k-2\delta_{0,k})\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\left(2B_l(1)+\sum_{k=0}^{l}\binom{l}{k}B_{l-k}E_k-2B_l\right)\\ &= \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\left(\sum_{k=0}^{l}\binom{l}{k}B_{l-k}E_k+2\delta_{1,l}\right).\end{aligned} \tag{2.7}$$

Therefore, by (1.4) and (2.7), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, one has*

$$E_n = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\left(\sum_{k=0}^{l}\binom{l}{k}B_{l-k}E_k+2\delta_{1,l}\right). \tag{2.8}$$

Corollary 2.3. *For $n \in \mathbb{N}$, one has*

$$2+E_n = \frac{1}{n+1}\sum_{l=0}^{n}\binom{n+1}{l}(-1)^l\left(\sum_{k=0}^{l}\binom{l}{k}B_{l-k}E_k\right). \tag{2.9}$$

Let us take the bosonic p-adic integral on both sides of (1.9) as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu_1(x) &= \int_{\mathbb{Z}_p}\left(E_n(x)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(x)\right)d\mu_1(x)\\ &= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}\int_{\mathbb{Z}_p} x^l d\mu_1(x)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}\int_{\mathbb{Z}_p} x^k d\mu_1(x)\\ &= \sum_{l=0}^{n}\binom{n}{l}E_{n-l}B_l+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}B_k.\end{aligned} \tag{2.10}$$

Thus, by (1.14) and (2.10), we obtain the following theorem.

Theorem 2.4. *For $n \in \mathbb{Z}_+$, one has*

$$B_n = \sum_{l=0}^{n}\binom{n}{l}E_{n-l}B_l+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}\sum_{k=0}^{l}\binom{l}{k}E_{l-k}B_k. \tag{2.11}$$

On the other hand, by (2.2) and (2.10), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} x^n d\mu_1(x) &= (-1)^n\int_{\mathbb{Z}_p} E_n(1-x)d\mu_1(x)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l\int_{\mathbb{Z}_p} E_l(1-x)d\mu_1(x)\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}E_{n-l}\int_{\mathbb{Z}_p}(1-x)^l d\mu_1(x)\\ &\quad+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}E_{l-k}\int_{\mathbb{Z}_p}(1-x)^k d\mu_1(x)\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}E_{n-l}(-1)^l B_l(-1)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}E_{l-k}(-1)^k B_k(-1)\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}E_{n-l}B_l(2)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}E_{l-k}B_k(2)\\ &= (-1)^n\sum_{l=0}^{n}\binom{n}{l}E_{n-l}(l+B_l+\delta_{1,l})+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l\sum_{k=0}^{l}\binom{l}{k}E_{l-k}(k+B_k+\delta_{1,k})\end{aligned}$$

$$
\begin{aligned}
&= (-1)^n nE_{n-1}(1) + (-1)^n \sum_{l=0}^{n} \binom{n}{l} E_{n-l}B_l + (-1)^n nE_{n-1} + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l lE_{l-1}(1) \\
&\quad + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l \sum_{k=0}^{l}\binom{l}{k}E_{l-k}B_k + \frac{1}{2}\sum_{l=1}^{n-1}\binom{n}{l}(-1)^l lE_{l-1} \\
&= (-1)^n n(2 + E_{n-1} - 2\delta_{0,n-1}) + (-1)^n \sum_{l=0}^{n}\binom{n}{l}E_{n-l}B_l + (-1)^n nE_{n-1} \\
&\quad + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l l(2 + E_{l-1} - \delta_{0,l-1}) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}(-1)^l \sum_{k=0}^{l}\binom{l}{k}E_{l-k}B_k \\
&\quad + \frac{1}{2}\sum_{l=1}^{n-1}\binom{n}{l}(-1)^l lE_{l-1},
\end{aligned}
\tag{2.12}
$$

where $n \in \mathbb{N}$ with $n \geq 2$. Therefore, by (2.12), we obtain the following theorem.

Theorem 2.5. *For $n \in \mathbb{N}$ with $n \geq 2$, one has*

$$
\begin{aligned}
B_{2n-1} = -\frac{2n-1}{2} - (2n-1)E_{2n-2}(-1) - \sum_{l=0}^{2n-1}\binom{2n-1}{l}E_{2n-1-l}B_l \\
+ \frac{1}{2}\sum_{l=0}^{2n-2}\binom{2n-1}{l}(-1)^l \sum_{k=0}^{l}\binom{l}{k}E_{l-k}B_k.
\end{aligned}
\tag{2.13}
$$

By (1.9) and (1.22), we get

$$
\begin{aligned}
&\iint_{\mathbb{Z}_p} x^m y^n d\mu_{-1}(x)d\mu_1(y) \\
&= \iint_{\mathbb{Z}_p}\left(\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(x)\right)\left(E_n(y) + \frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(y)\right)d\mu_{-1}(x)d\mu_1(y) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}\iint_{\mathbb{Z}_p}B_k(x)E_n(y)d\mu_{-1}(x)d\mu_1(y) \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\binom{m+1}{k}\binom{n}{l}\iint_{\mathbb{Z}_p}B_k(x)E_l(y)d\mu_{-1}(x)d\mu_1(y) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\sum_{l=0}^{k}\sum_{p=0}^{n}\binom{m+1}{k}\binom{k}{l}\binom{n}{p}B_{k-l}E_{n-p}B_pE_l \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\sum_{s=0}^{k}\sum_{p=0}^{l}\binom{m+1}{k}\binom{n}{l}\binom{k}{s}\binom{l}{p}B_{k-s}E_{l-p}E_sB_p.
\end{aligned}
\tag{2.14}
$$

Therefore, by (1.4), (1.14), and (2.14), we obtain the following theorem.

Theorem 2.6. *For $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, one has*

$$
\begin{aligned}
E_m B_n &= \frac{1}{m+1}\sum_{k=0}^{m}\sum_{l=0}^{k}\sum_{p=0}^{n}\binom{m+1}{k}\binom{k}{l}\binom{n}{p}B_{k-l}E_{n-p}B_pE_l \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\sum_{s=0}^{k}\sum_{p=0}^{l}\binom{m+1}{k}\binom{n}{l}\binom{k}{s}\binom{l}{p}B_{k-s}E_{l-p}E_sB_p.
\end{aligned}
\tag{2.15}
$$

It is easy to show that

$$
\begin{aligned}
\int_{\mathbb{Z}_p} x^{m+n}d\mu_{-1}(x) &= \int_{\mathbb{Z}_p}\left(\frac{1}{m+1}\sum_{k=0}^{m}\binom{m+1}{k}B_k(x)\right)\left(E_n(x)+\frac{1}{2}\sum_{l=0}^{n-1}\binom{n}{l}E_l(x)\right)d\mu_{-1}(x) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\sum_{i=0}^{k}\sum_{j=0}^{n}\binom{m+1}{k}\binom{k}{i}\binom{n}{j}B_{k-i}E_{n-j}\int_{\mathbb{Z}_p}x^{i+j}d\mu_{-1}(x) \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\sum_{i=0}^{k}\sum_{j=0}^{l}\binom{m+1}{k}\binom{n}{l}\binom{k}{i}\binom{l}{j}B_{k-i}E_{l-j}\int_{\mathbb{Z}_p}x^{i+j}d\mu_{-1}(x) \\
&= \frac{1}{m+1}\sum_{k=0}^{m}\sum_{i=0}^{k}\sum_{j=0}^{n}\binom{m+1}{k}\binom{k}{i}\binom{n}{j}B_{k-i}E_{n-j}E_{i+j} \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\sum_{i=0}^{k}\sum_{j=0}^{l}\binom{m+1}{k}\binom{n}{l}\binom{k}{i}\binom{l}{j}B_{k-i}E_{l-j}E_{i+j}.
\end{aligned}
\tag{2.16}
$$

Therefore, by (2.16), we obtain the following corollay.

Corollary 2.7. *For $m \in \mathbb{Z}_+$ and $n \in \mathbb{N}$, one has*

$$
\begin{aligned}
E_{m+n} &= \frac{1}{m+1}\sum_{k=0}^{m}\sum_{i=0}^{k}\sum_{j=0}^{n}\binom{m+1}{k}\binom{k}{i}\binom{n}{j}B_{k-i}E_{n-j}E_{i+j} \\
&\quad + \frac{1}{2(m+1)}\sum_{k=0}^{m}\sum_{l=0}^{n-1}\sum_{i=0}^{k}\sum_{j=0}^{l}\binom{m+1}{k}\binom{n}{l}\binom{k}{i}\binom{l}{j}B_{k-i}E_{l-j}E_{i+j}.
\end{aligned}
\tag{2.17}
$$

For $f \in C(\mathbb{Z}_p)$, p-adic analogue of Bernstein operator of order n for f is given by

$$
\mathbb{B}_n(f \mid x) = \sum_{k=0}^{n} f\left(\frac{\mathrm{k}}{n}\right)\binom{n}{k}x^k(1-x)^{n-k} = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)B_{k,n}(x), \tag{2.18}
$$

where $B_{k,n}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ for $n, k \in \mathbb{Z}_+$ is called the Bernstein polynomial of degree n (see [8]). From the definition of $B_{k,n}(x)$, we note that $B_{n-k,n}(1-x) = B_{k,n}(x)$.

Let us take the fermionic p-adic integral on $\mathbb{Z}_p$ for the product of x^m and $B_{k,n}(x)$ as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} x^m B_{k,n}(x)d\mu_{-1}(x) &= \frac{1}{m+1}\sum_{l=0}^{m}\binom{m+1}{l}\int_{\mathbb{Z}_p} B_l(x)B_{k,n}(x)d\mu_{-1}(x) \\ &= \frac{\binom{n}{k}}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\binom{m+1}{l}\binom{l}{j}B_{l-j}\int_{\mathbb{Z}_p} x^{j+k}(1-x)^{n-k}d\mu_{-1}(x) \\ &= \frac{\binom{n}{k}}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{n-k}(-1)^i B_{l-j}\binom{m+1}{l}\binom{l}{j}\binom{n-k}{i}\int_{\mathbb{Z}_p} x^{i+j+k}d\mu_{-1}(x) \\ &= \frac{\binom{n}{k}}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{n-k}(-1)^i\binom{m+1}{l}\binom{l}{j}\binom{n-k}{i}B_{l-j}E_{i+j+k}.\end{aligned} \tag{2.19}$$

From (2.18), we note that

$$\begin{aligned}\int_{\mathbb{Z}_p} x^m B_{k,n}(x)d\mu_{-1}(x) &= \binom{n}{k}\int_{\mathbb{Z}_p} x^{m+k}(1-x)^{n-k}d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j\int_{\mathbb{Z}_p} x^{m+k+j}d\mu_{-1}(x) \\ &= \binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j E_{m+k+j}.\end{aligned} \tag{2.20}$$

Therefore, by (2.19) and (2.20), we obtain the following theorem.

Theorem 2.8. *For $m, n, k \in \mathbb{Z}_+$, one has*

$$\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j E_{m+k+j} = \frac{1}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{n-k}(-1)^i\binom{m+1}{l}\binom{l}{j}\binom{n-k}{i}B_{l-j}E_{i+j+k}. \tag{2.21}$$

In particular,

$$(m+1)E_{m+n} = \sum_{l=0}^{m}\sum_{j=0}^{l}\binom{m+1}{l}\binom{l}{j}B_{l-j}E_{j+n}. \tag{2.22}$$

By (1.17) and the symmetric property of $B_{k,n}(x)$, we get

$$\int_{\mathbb{Z}_p} x^m B_{k,n}(x)d\mu_{-1}(x) = \int_{\mathbb{Z}_p} x^m B_{n-k,n}(1-x)d\mu_{-1}(x)$$
$$= \frac{1}{m+1}\sum_{l=0}^{m}(-1)^l\binom{m+1}{l}\int_{\mathbb{Z}_p} B_l(1-x)B_{n-k,n}(1-x)d\mu_{-1}(x)$$
$$= \frac{\binom{n}{k}}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{k}(-1)^{i+l}\binom{m+1}{l}\binom{l}{j}\binom{k}{i}B_{l-j}\int_{\mathbb{Z}_p}(1-x)^{i+j+n-k}d\mu_{-1}(x). \tag{2.23}$$

From (1.4) and (2.2), we note that

$$\int_{\mathbb{Z}_p}(1-x)^n d\mu_{-1}(x) = (-1)^n E_n(-1) = E_n(2) = 2 + E_n - 2\delta_{0,n}. \tag{2.24}$$

By (2.23) and (2.24), we see that

$$\int_{\mathbb{Z}_p} x^m B_{k,n}(x)d\mu_{-1}(x) = \frac{\binom{n}{k}}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{k}(-1)^{i+l}\binom{m+1}{l}\binom{l}{j}\binom{k}{i}B_{l-j}\left(2 + E_{i+j+n-k} - 2\delta_{0,i+j+n-k}\right). \tag{2.25}$$

From (2.20) and (2.25), we have

$$\begin{aligned}
&\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j E_{m+k+j}\\
&= \frac{2\delta_{0,k}}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}(-1)^l\binom{m+1}{l}\binom{l}{j}B_{l-j} - \frac{2}{m+1}\sum_{l=0}^{m}(-1)^l\binom{m+1}{l}B_l\delta_{k,n}\\
&\quad + \frac{1}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{k}(-1)^{i+l}\binom{m+1}{l}\binom{l}{j}\binom{k}{i}B_{l-j}E_{i+j+n-k}\\
&= \frac{2\delta_{0,k}}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}(-1)^l\binom{m+1}{l}\binom{l}{j}B_{l-j} - \frac{2}{m+1}\left(B_{m+1}(2) + (-1)^m B_{m+1}\right)\delta_{k,n}\\
&\quad + \frac{1}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{k}(-1)^{i+l}\binom{m+1}{l}\binom{l}{j}\binom{k}{i}B_{l-j}E_{i+j+n-k}.
\end{aligned} \tag{2.26}$$

Therefore, by (1.19) and (2.26), we obtain the following theorem.

Theorem 2.9. *For $m, n, k \in \mathbb{N}$ with $n \geq k$, one has*

$$\sum_{j=0}^{n-k}\binom{n-k}{j}(-1)^j E_{m+k+j} = \frac{1}{m+1}\sum_{l=0}^{m}\sum_{j=0}^{l}\sum_{i=0}^{k}(-1)^{i+l}\binom{m+1}{l}\binom{l}{j}\binom{k}{i}B_{l-j}E_{i+j+n-k} - \frac{2}{m+1}\left(B_{m+1} + m + 1 + (-1)^m B_{m+1}\right). \tag{2.27}$$

In particular,

$$(2m+2)\left(E_{2m+n+1} + 2\right) = \sum_{l=0}^{2m+1}\sum_{j=0}^{l}\sum_{i=0}^{n}(-1)^{i+l}\binom{2m+2}{l}\binom{l}{j}\binom{n}{i}B_{l-j}E_{i+j}. \tag{2.28}$$

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References

[1] T. Kim, "Euler numbers and polynomials associated with zeta functions," *Abstract and Applied Analysis*, vol. 2008, Article ID 581582, 11 pages, 2008.
[2] T. Kim, "Note on the Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 17, no. 2, pp. 131–136, 2008.
[3] T. Kim, "Symmetry of power sum polynomials and multivariate fermionic p-adic invariant integral on $\mathbb{Z}_p$," *Russian Journal of Mathematical Physics*, vol. 16, no. 1, pp. 93–96, 2009.
[4] T. Kim, "Symmetry p-adic invariant integral on $\mathbb{Z}_p$ for Bernoulli and Euler polynomials," *Journal of Difference Equations and Applications*, vol. 14, no. 12, pp. 1267–1277, 2008.
[5] A. Bayad, "Modular properties of elliptic Bernoulli and Euler functions," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 3, pp. 389–401, 2010.
[6] D. Ding and J. Yang, "Some identities related to the Apostol-Euler and Apostol-Bernoulli polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 20, no. 1, pp. 7–21, 2010.
[7] T. Kim, "q-Volkenborn integration," *Russian Journal of Mathematical Physics*, vol. 9, no. 3, pp. 288–299, 2002.
[8] T. Kim, "A note on q-Bernstein polynomials," *Russian Journal of Mathematical Physics*, vol. 18, no. 1, pp. 73–82, 2011.
[9] A. Kudo, "A congruence of generalized Bernoulli number for the character of the first kind," *Advanced Studies in Contemporary Mathematics*, vol. 2, pp. 1–8, 2000.
[10] Q.-M. Luo and F. Qi, "Relationships between generalized Bernoulli numbers and polynomials and generalized Euler numbers and polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 7, no. 1, pp. 11–18, 2003.
[11] Q.-M. Luo, "Some recursion formulae and relations for Bernoulli numbers and Euler numbers of higher order," *Advanced Studies in Contemporary Mathematics*, vol. 10, no. 1, pp. 63–70, 2005.
[12] H. Ozden, I. N. Cangul, and Y. Simsek, "Remarks on q-Bernoulli numbers associated with Daehee numbers," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 1, pp. 41–48, 2009.
[13] Y.-H. Kim and K.-W. Hwang, "Symmetry of power sum and twisted Bernoulli polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 18, no. 2, pp. 127–133, 2009.
[14] G. Kim, B. Kim, and J. Choi, "The DC algorithm for computing sums of powers of consecutive integers and Bernoulli numbers," *Advanced Studies in Contemporary Mathematics*, vol. 17, no. 2, pp. 137–145, 2008.
[15] L. C. Jang, "A note on Kummer congruence for the Bernoulli numbers of higher order," *Proceedings of the Jangjeon Mathematical Society*, vol. 5, no. 2, pp. 141–146, 2002.

[16] L. C. Jang and H. K. Pak, "Non-Archimedean integration associated with q-Bernoulli numbers," *Proceedings of the Jangjeon Mathematical Society*, vol. 5, no. 2, pp. 125–129, 2002.

[17] S.-H. Rim, J.-H. Jin, E.-J. Moon, and S.-J. Lee, "Some identities on the q-Genocchi polynomials of higher-order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$," *International Journal of Mathematics and Mathematical Sciences*, vol. 2010, Article ID 860280, 14 pages, 2010.

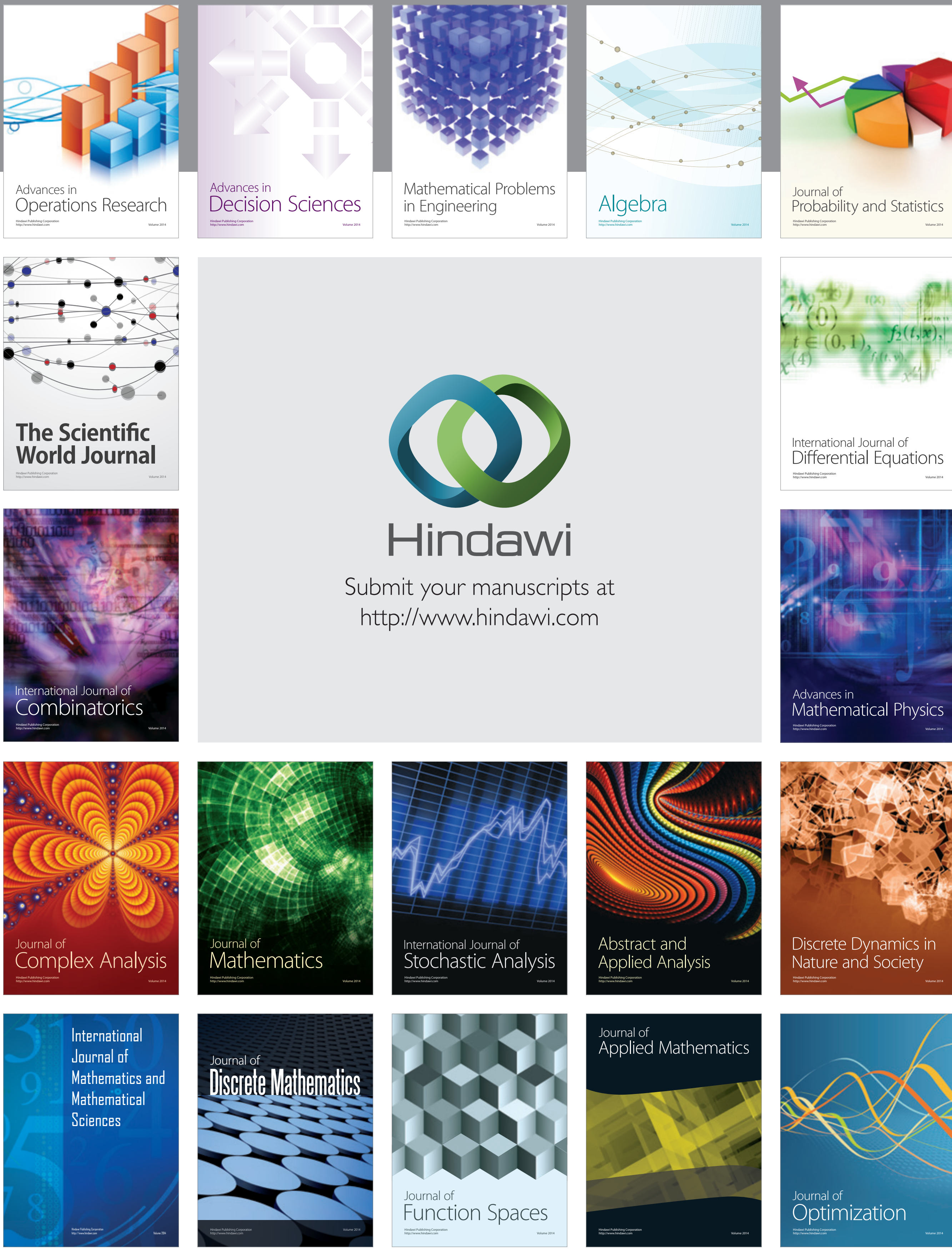
[18] C. S. Ryoo, "On the generalized Barnes type multiple q-Euler polynomials twisted by ramified roots of unity," *Proceedings of the Jangjeon Mathematical Society*, vol. 13, no. 2, pp. 255–263, 2010.

[19] C. S. Ryoo, "Some relations between twisted q-Euler numbers and Bernstein polynomials," *Advanced Studies in Contemporary Mathematics*, vol. 21, no. 2, pp. 217–223, 2011.

[20] Y. Simsek, "Generating functions of the twisted Bernoulli numbers and polynomials associated with their interpolation functions," *Advanced Studies in Contemporary Mathematics*, vol. 16, no. 2, pp. 251–278, 2008.

[21] I. Buyukyazici, "On generalized q-Bernstein polynomials," *The Global Journal of Pure and Applied Mathematics*, vol. 6, pp. 1331–1348, 2010.

[22] L.-C. Jang, W.-J. Kim, and Y. Simsek, "A study on the p-adic integral representation on $\mathbb{Z}_p$ associated with Bernstein and Bernoulli polynomials," *Advances in Difference Equations*, vol. 2010, Article ID 163217, 6 pages, 2010.



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