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Research Article

Some Notes on the Difference Equation $x_{n+1} = \alpha + (x_{n-1}/x_n^k)$

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We investigate the behavior of the solutions of the recursive sequence $x_{n+1} = \alpha + x_{n-1}/x_n^k$, $n = 0, 1, \ldots$, where $\alpha \in [0, \infty)$, $k \in (0, \infty)$, and the initial conditions x_{-1}, x_0 are arbitrary positive numbers. Included are results that considerably improve those in the recently published paper by Hamza and Morsy (2009).

1. Introduction

Our aim in this paper is to give some remarks for the positive solutions of the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n^k}, \quad n = 0, 1, \ldots, \tag{1.1}$$

where $\alpha \in [0, \infty)$, $k \in (0, \infty)$, and the initial conditions x_{-1}, x_0 are arbitrary positive numbers. Amleh et al. in [1] obtained important results for the difference equation

$$x_{n+1} = \alpha + \frac{x_{n-1}}{x_n}, \tag{1.2}$$

which guide many authors. It was proved in [1] that, when $\alpha > 1$, the equilibrium $\overline{x} = \alpha + 1$ of (1.2) is globally asymptotically stable. When $\alpha = 1$, every positive solution of (1.2) converges to a period-two solution. Every positive solution of (1.2) is bounded if and only if $\alpha \geq 1$. Finally, when $0 < \alpha < 1$, the equilibrium is an unstable saddle point. Closely related equations to (1.1) are investigated by many authors, for example, [2–10].

In [4] the authors investigated the behavior of positive solutions of (1.1). It was proved in [4] that, when $\alpha \neq 1$, every positive solution of (1.1) is bounded and when $\alpha > k^{1/k} \geq 1$, the equilibrium $\overline{x}$ of (1.1) is globally asymptotically stable. But in [4] the authors obtain some incorrect results for the boundedness character and the global stability of solutions of (1.1), and it is not shown that (1.1) has periodic solutions with conditions of α and k.

Our aim here is to improve and correct these results and extend some of the results in [4].

We say that the equilibrium point $\overline{x}$ of the equation

$$x_{n+1} = F(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots, \tag{1.3}$$

is the point that satisfies the condition

$$\overline{x} = F(\overline{x}, \overline{x}, \ldots, \overline{x}). \tag{1.4}$$

A positive semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all greater than or equal to $\overline{x}$, with $l \geq -1$ and $m \leq \infty$, such that

$$\begin{gathered} \text{either } l = -1 \text{ or } l > -1, \quad x_{l-1} < \overline{x}, \\ \text{either } m = \infty \text{ or } m < \infty, \quad x_{m+1} < \overline{x}. \end{gathered} \tag{1.5}$$

A negative semicycle of a solution $\{x_n\}_{n=-1}^{\infty}$ of (1.1) consists of a "string" of terms $\{x_l, x_{l+1}, \ldots, x_m\}$ all less than $\overline{x}$, with $l \geq -1$ and $m \leq \infty$, such that

$$\begin{gathered} \text{either } l = -1 \text{ or } l > -1, \quad x_{l-1} \geq \overline{x}, \\ \text{either } m = \infty \text{ or } m < \infty, \quad x_{m+1} \geq \overline{x}. \end{gathered} \tag{1.6}$$

A solution $\{x_n\}_{n=-k}^{\infty}$ of (1.1) is called nonoscillatory if there exists $N \geq -1$ such that either

$$x_n > \overline{x} \quad \forall n \geq N \tag{1.7}$$

or

$$x_n < \overline{x} \quad \forall n \geq N. \tag{1.8}$$

$\{x_n\}_{n=-1}^{\infty}$ of (1.1) is called oscillatory if it is not nonoscillatory. We say that a solution $\{x_n\}_{n=-1}^{\infty}$ of (1.1) is bounded and persists if there exist positive constants P and Q such that $P \leq x_n \leq Q$ for $n = -1, 0, \ldots$.

The linearized equation for (1.1) about the positive equilibrium $\overline{x}$ is

$$y_{n+1} + k\overline{x}^{-k} y_n - \overline{x}^{-k} y_{n-1} = 0, \quad n = 0, 1, \ldots. \tag{1.9}$$

We need the following lemmas, which were given in [4].

Lemma 1.1. *Let $\overline{x}$ be the equilibrium point of* (1.1).

(i) *If $k(1+k)^{(1-k)/k} < \alpha$, then the equilibrium point $\overline{x}$ of* (1.1)*, is locally asymptotically stable.*

(ii) *If $k(1+k)^{(1-k)/k} > \alpha$, then the equilibrium point $\overline{x}$ of* (1.1) *is unstable.*

Lemma 1.2. *The following statements are true.*

(i) *If $k = 1$, then* (1.1) *has a unique equilibrium point $\overline{x} = \alpha + 1$.*

(ii) *If $k \neq 1$, then* (1.1) *has a unique equilibrium point $\overline{x} > 1$.*

Lemma 1.3. *Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of* (1.1)*, which consists of at least two semicycles. Then, $\{x_n\}_{n=-1}^{\infty}$ is oscillatory and, except possibly for the first semicycle, every semicycle is of length one.*

The paper is organized as follows. In Section 2 we investigate the boundedness character of positive solutions of (1.1). We prove that if $0 \leq \alpha < 1$, then there exist unbounded solutions of (1.1) and when the cases either $\alpha \in (0, \infty)$ and $k \to 0^+$ or $\alpha > 1$ and $k \to \infty$, then every positive solution of (1.1) is unbounded. We show that when $\alpha > 1$ and $k \nrightarrow 0^+, k \nrightarrow \infty$, then every positive solution of (1.1) is bounded. Also we show that if $\alpha > 1$, $\alpha > k(1+k)^{(1-k)/k}$ and $k \nrightarrow 0^+, k \nrightarrow \infty$, then the equilibrium point $\overline{x}$ of (1.1) is globally asymptotically stable. Section 3 is devoted to the periodic character of the positive solutions of (1.1). Finally we show that a sufficient condition that every positive solution of (1.1) converges to a prime two periodic solution.

2. Boundedness and Global Stability of (1.1)

In this section, we present some results for the boundedness character of positive solutions and global stability of the equilibrium point of (1.1).

Theorem 2.1. *Consider* (1.1)*. Then, the following statements are true.*

(a) *If $\alpha \in (0, \infty)$ and $k \to 0^+$, then every positive solution of* (1.1) *is unbounded.*

(b) *If $\alpha > 1$ and $k \to \infty$, then every positive solution of* (1.1) *is unbounded.*

Proof. (a) On the contrary, we assume that $\{x_n\}_{n=-1}^{\infty}$ is a positive bounded solution of (1.1). Then, we have

$$s = \liminf x_n, \qquad S = \limsup x_n < \infty. \tag{2.1}$$

Thus, from (1.1) we get

$$s \geq \alpha + \frac{s}{S^k}. \tag{2.2}$$

Let $k \to 0^+$; then we obtain

$$\begin{aligned} s &\geq \alpha + s, \\ \alpha &\leq 0, \end{aligned} \tag{2.3}$$

which contradicts $\alpha > 0$, so the proof is complete.

(b) Again we assume that $\{x_n\}_{n=-1}^{\infty}$ is a positive bounded solution of (1.1). Then, we have

$$\alpha < s = \liminf x_n, \qquad S = \limsup x_n < \infty. \tag{2.4}$$

Thus, from (1.1), we have

$$S \leq \alpha + \frac{S}{s^k}. \tag{2.5}$$

Let $k \to \infty$; then

$$S \leq \alpha, \tag{2.6}$$

which contradicts $\alpha < s$, so the proof is complete. □

Now, we show that if $0 \leq \alpha < 1$, then there exist positive solutions of (1.1) that are unbounded.

Theorem 2.2. *One has*

$$0 \leq \alpha < 1. \tag{2.7}$$

Then there exist positive solutions of (1.1) *that are unbounded.*

Proof. Assume that $\alpha \in (0,1)$. Choose $\delta \in (0, 1-\alpha)$, and let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1) with the initial conditions such that

$$\begin{gathered} x_{-1} > \left(\left(\frac{\alpha+\delta}{\delta} \right)^{1/k} - \alpha \right)(\alpha+\delta), \\ \alpha < x_0 < \alpha + \delta. \end{gathered} \tag{2.8}$$

Then,

$$\begin{aligned} x_1 &= \alpha + \frac{x_{-1}}{x_0^k} > \alpha + \frac{x_{-1}}{(\alpha+\delta)^k}, \\ x_2 = \alpha + \frac{x_0}{x_1^k} &< \alpha + \frac{\alpha+\delta}{\left(\alpha + x_{-1}/(\alpha+\delta)^k\right)^k} < \alpha + \delta. \end{aligned} \tag{2.9}$$

Further we have

$$x_3 = \alpha + \frac{x_1}{x_2^k} > \alpha + \frac{x_1}{(\alpha+\delta)^k} > \alpha + \frac{1}{(\alpha+\delta)^k}\left(\alpha + \frac{x_{-1}}{(\alpha+\delta)^k}\right) > \alpha + \frac{x_{-1}}{(\alpha+\delta)^k},$$
$$x_4 = \alpha + \frac{x_2}{x_3^k} < \alpha + \frac{\alpha+\delta}{\left(\alpha + x_{-1}/(\alpha+\delta)^k\right)^k} < \alpha+\delta. \tag{2.10}$$

Therefore, working inductively we can prove that for $n = 0, 1, \ldots$

$$x_{2n+1} > \alpha + \frac{x_{2n-1}}{(\alpha+\delta)^k} > \alpha + \frac{x_{-1}}{(\alpha+\delta)^k},$$
$$\alpha < x_{2n} < \alpha+\delta. \tag{2.11}$$

Hence,

$$x_{2n+1} > \frac{x_{-1}}{\left((\alpha+\delta)^k\right)^{n+1}} + \alpha\left(1 + \frac{1}{(\alpha+\delta)^k} + \cdots + \frac{1}{\left((\alpha+\delta)^k\right)^n}\right). \tag{2.12}$$

Since

$$\frac{1}{(\alpha+\delta)^k} > 1, \tag{2.13}$$

which implies that

$$\lim_{n\to\infty} x_{2n+1} = \infty \tag{2.14}$$

$\{x_n\}_{n=-1}^{\infty}$ is unbounded. For $\alpha \in (0,1)$, the proof is complete.

Now, we assume that $\alpha = 0$ and choose the initial conditions such that

$$0 < x_{-1} \le 1,$$
$$x_0 > \frac{1}{(1-\varepsilon)^{1/k}} \quad \text{for some } 0 < \varepsilon < 1. \tag{2.15}$$

So, we have

$$0 < x_1 = \frac{x_{-1}}{x_0^k} \le \frac{1}{x_0^k} < 1-\varepsilon,$$
$$x_2 = \frac{x_0}{x_1^k} > \frac{x_0}{(1-\varepsilon)^k}. \tag{2.16}$$

Further we have

$$x_3 = \frac{x_1}{x_2^k} < \frac{1}{x_0^k} < 1-\varepsilon,$$
$$x_4 = \frac{x_2}{x_3^k} > \frac{x_0/(1-\varepsilon)^k}{(1-\varepsilon)^k} = \frac{x_0}{(1-\varepsilon)^{2k}}. \tag{2.17}$$

By induction we have

$$x_{2n+1} \in (0, 1-\varepsilon), \quad x_{2n} > \frac{x_0}{\left((1-\varepsilon)^k\right)^n} \quad \forall n = 0, 1, \dots. \tag{2.18}$$

Thus,

$$\lim_{n\to\infty} x_{2n} = \infty, \qquad \lim_{n\to\infty} x_{2n+1} = 0. \tag{2.19}$$

This completes the proof. □

The following theorem is given in [4].

Theorem 2.3. *Suppose that $\alpha \neq 1$; then; every positive solution of* (1.1) *is bounded.*

In [4] this result is not correct. So, we give the following theorem for the boundedness of (1.1).

Theorem 2.4. *Suppose that $\alpha > 1$, $k \nrightarrow 0^+$, and $k \nrightarrow \infty$; then every positive solution of* (1.1) *is bounded.*

Proof. From (1.1), $x_n \geq \alpha$ for $n \geq 0$. Thus, from (1.1), without loss of generality, we obtain for $n \geq 0$

$$x_{2n+1} = \alpha + \frac{x_{2n-1}}{x_{2n}^k} \leq \alpha + \frac{x_{2n-1}}{\alpha^k},$$
$$x_{2n} = \alpha + \frac{x_{2n-2}}{x_{2n-1}^k} \leq \alpha + \frac{x_{2n-2}}{\alpha^k}. \tag{2.20}$$

From (2.20) using induction, we obtain

$$x_{2n+1} \leq \alpha\left(1 + \frac{1}{\alpha^k} + \frac{1}{(\alpha^k)^2} + \cdots + \frac{1}{(\alpha^k)^n}\right) + \frac{x_{-1}}{(\alpha^k)^{n+1}}$$
$$\leq \frac{\alpha^k}{\alpha^k - 1} + \frac{x_{-1}}{(\alpha^k)^{n+1}},$$

$$x_{2n} \leq \alpha\left(1 + \frac{1}{\alpha^k} + \frac{1}{\left(\alpha^k\right)^2} + \cdots + \frac{1}{\left(\alpha^k\right)^{n-1}}\right) + \frac{x_0}{\left(\alpha^k\right)^n}$$
$$\leq \frac{\alpha^k}{\alpha^k - 1} + \frac{x_0}{\left(\alpha^k\right)^n}. \tag{2.21}$$

From which the proof follows. □

Actually, Hamza and Morsy in [4] obtained global stability of the equilibrium point of (1.1). But the result does not include the case $k \in (0,1)$ and some parts of its proof are incorrect. So, here we will obtain global stability of the equilibrium point of (1.1) when $k \in (0,1)$.

Theorem 2.5. *Consider* (1.1). *Let* $k \nrightarrow 0^+$ *and* $k \nrightarrow \infty$. *Suppose that*

$$\alpha > 1, \quad \alpha > k(1+k)^{(1-k)/k} \tag{2.22}$$

hold. Then, the unique positive equilibrium $\overline{x}$ *of* (1.1) *is globally asymptotically stable.*

Proof. By Lemma 1.1, $\overline{x}$ is locally asymptotically stable. Thus, it is enough for the proof that every positive solution of (1.1) tends to the unique positive equilibrium $\overline{x}$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1). By Theorem 2.4, $\{x_n\}_{n=-1}^{\infty}$ is bounded. Thus, we have

$$1 < s = \liminf x_n, \qquad S = \limsup x_n < \infty. \tag{2.23}$$

Then, from (2.23), we get

$$S \leq \alpha + \frac{S}{s^k}, \qquad s \geq \alpha + \frac{s}{S^k}. \tag{2.24}$$

We claim that $s = S$, otherwise $S > s$. From (2.24), we obtain

$$Ss^k \leq \alpha s^k + S, \qquad S^k s \geq \alpha S^k + s, \tag{2.25}$$

And, from (2.25),

$$\alpha S^k s^{k-1} + s^k \leq \alpha s^k S^{k-1} + S^k. \tag{2.26}$$

Thus,

$$\alpha S^{k-1} s^{k-1}(S - s) \leq S^k - s^k. \tag{2.27}$$

Assume that $k \geq 1$. We consider $f(x) = x^k$ with $x \in (0, \infty)$; then there exists $a, c \in (s, S)$ such that

$$\frac{S^k - s^k}{S - s} = kc^{k-1} \leq kS^{k-1}. \tag{2.28}$$

From (2.27) and (2.28), we obtain

$$\alpha S^{k-1} s^{k-1} \leq kS^{k-1}, \tag{2.29}$$

which is equivalent to

$$\alpha s^{k-1} \leq k. \tag{2.30}$$

Since we have $S, s \geq \alpha$, we get

$$\alpha^k \leq k. \tag{2.31}$$

Since $\alpha > 1$ and $\alpha > k(1+k)^{(1-k)/k}$, for some values α and k, (2.31) is not satisfied. This is a contradiction. Thus, we find $S = s$.

Now, assume that $k < 1$. Then, from (2.27) and arguing as above, we get

$$\alpha S^{k-1} s^{k-1} \leq ks^{k-1}. \tag{2.32}$$

Furthermore, we have

$$x_{2n+1} < \alpha + \frac{x_{2n-1}}{\alpha^k}. \tag{2.33}$$

We consider the following difference equation:

$$y_{m+1} = \alpha + \frac{y_m}{\alpha^k}. \tag{2.34}$$

Every positive solution of the previous equation converges to $\alpha^{k+1}/(\alpha^k - 1)$. It follows that $S = \limsup x_n \leq \alpha^{k+1}/(\alpha^k - 1)$. Then, we obtain that

$$\left(\frac{\alpha^{k+1}}{\alpha^k - 1}\right)^{k-1} \leq S^{k-1} \leq \frac{k}{\alpha}. \tag{2.35}$$

Thus,

$$\alpha\left(\frac{\alpha^{k+1}}{\alpha^k - 1}\right)^{k-1} \leq k. \tag{2.36}$$

Since $\alpha > 1$ and $\alpha > k(1+k)^{(1-k)/k}$, for some values α and k, (2.36) is not satisfied. So $s = S$, which implies that $\{x_n\}_{n=-1}^{\infty}$ tends to the unique positive equilibrium. From which the proof follows. □

Remark 2.6. Consider (1.1), where $0 \leq \alpha < 1$. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1). If $\{x_n\}_{n=-1}^{\infty}$ is bounded, then it is stable too.

3. Periodicity of the Solutions of (1.1)

In this section we investigate the periodicity of (1.1) when $\alpha > 1$ and $k > 1$, $k \nrightarrow \infty$.

We need the following lemma whose proof follows by simple computation and thus it will be omitted.

Lemma 3.1. *Let $\{x_n\}_{n=-k}^{\infty}$ be a solution of* (1.1)*, and let $L > \alpha$. Then, the following statements are satisfied.*

(i) $\lim_{n\to\infty} x_{2n} = L$ *if and only if* $\lim_{n\to\infty} x_{2n+1} = L^{1/k}/(L-\alpha)^{1/k}$,

(ii) $\lim_{n\to\infty} x_{2n+1} = L$ *if and only if* $\lim_{n\to\infty} x_{2n} = L^{1/k}/(L-\alpha)^{1/k}$.

Theorem 3.2. *Consider* (1.1)*, where*

$$\alpha > 1, \qquad k > 1, \qquad k \nrightarrow \infty. \tag{3.1}$$

Assume that there exists a sufficient small positive number ε_1 such that

$$(\alpha+\varepsilon_1)^{1/k} - (\alpha+\varepsilon_1)^{(1-k^2)/k} < \alpha\varepsilon_1^{1/k}, \tag{3.2}$$

$$\varepsilon_1 < (\alpha+\varepsilon_1)^{1-k}. \tag{3.3}$$

Then, (1.1) *has a periodic solution of prime period two.*

Proof. Let $\{x_n\}_{n=-1}^{\infty}$ be a solution of (1.1). It is obvious that if

$$x_{-1} = \alpha + \frac{x_{-1}}{x_0^k}, \qquad x_0 = \alpha + \frac{x_0}{x_{-1}^k} \tag{3.4}$$

hold, then $\{x_n\}_{n=-1}^{\infty}$ is a periodic solution of period two. Consider the system

$$x = \alpha + \frac{x}{y^k}, \qquad y = \alpha + \frac{y}{x^k}. \tag{3.5}$$

Then this system is equivalent to

$$y - \alpha - \frac{y}{x^k} = 0, \qquad y = \left(\frac{x}{x-\alpha}\right)^{1/k}, \tag{3.6}$$

and so we get the equation

$$F(x) = \left(\frac{x}{x-\alpha}\right)^{1/k} - \alpha - \frac{x^{(1-k^2)/k}}{(x-\alpha)^{1/k}} \tag{3.7}$$

and obtain

$$F(x) = \frac{1}{(x-\alpha)^{1/k}}\left(x^{1/k} - x^{(1-k^2)/k}\right) - \alpha. \tag{3.8}$$

Thus,

$$\lim_{x\to\alpha^+} F(x) = \infty. \tag{3.9}$$

Moreover, from (3.2) we can show that

$$F(\alpha+\varepsilon_1) < 0. \tag{3.10}$$

Therefore, the equation $F(x) = 0$ has a solution $\overline{x} = \alpha + \varepsilon_0$, where $0 < \varepsilon_0 < \varepsilon_1$, in the interval $(\alpha, \alpha+\varepsilon_1)$. So, we have

$$\overline{y} = \left(\frac{\overline{x}}{\overline{x}-\alpha}\right)^{1/k}. \tag{3.11}$$

We now consider the function

$$K(\varepsilon) = (\alpha+\varepsilon)^{1-k} - \varepsilon. \tag{3.12}$$

Since from (3.1)

$$K'(\varepsilon) = (1-k)(\alpha+\varepsilon)^{-k} - 1 < 0, \tag{3.13}$$

we have

$$K(\varepsilon_0) > K(\varepsilon_1). \tag{3.14}$$

From (3.3), we have $K(\varepsilon_1) > 0$, and thus $K(\varepsilon_0) > 0$, which implies that

$$\overline{x} = \alpha + \varepsilon_0 < \left(\frac{\alpha+\varepsilon_0}{\varepsilon_0}\right)^{1/k} = \overline{y}. \tag{3.15}$$

Hence, if $x_{-1} = \overline{x}$, $x_0 = \overline{y}$, then the solution $\{x_n\}_{n=-1}^{\infty}$ with initial values x_{-1}, x_0 is a prime 2-periodic solution. This completes the proof. □

In the following theorem we will generalize the result due to Stević [6, Theorem 2].

Theorem 3.3. *For every positive solution* $\{x_n\}$ *of* (1.1), *the sequences* $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *are eventually monotone.*

Proof. We have

$$x_{2n+1} - x_{2n-1} = \frac{x_{2n-2}^k (x_{2n-1} - x_{2n-3}) + x_{2n-3}\left(x_{2n-2}^k - x_{2n}^k\right)}{(x_{2n}x_{2n-2})^k},$$
$$x_{2n+2} - x_{2n} = \frac{x_{2n-1}^k (x_{2n} - x_{2n-2}) + x_{2n-2}\left(x_{2n-1}^k - x_{2n+1}^k\right)}{(x_{2n+1}x_{2n-1})^k}. \quad (3.16)$$

If $x_1 \geq x_{-1}$ and $x_2 \leq x_0$, we obtain from (3.16) $x_3 \geq x_1$ and consequently $x_4 \leq x_2$. By induction we obtain

$$x_0 \geq x_2 \geq \cdots \geq x_{2n} \geq \cdots, \qquad \cdots \geq x_{2n+1} \geq x_{2n-1} \geq \cdots \geq x_1 \geq x_{-1}. \quad (3.17)$$

Similarly if $x_1 \leq x_{-1}$ and $x_2 \geq x_0$, using induction we obtain from (3.16)

$$x_0 \leq x_2 \leq \cdots \leq x_{2n} \leq \cdots, \qquad \cdots \leq x_{2n+1} \leq x_{2n-1} \leq \cdots \leq x_1 \leq x_{-1}. \quad (3.18)$$

If $x_1 \geq x_{-1}$, $x_2 \geq x_0$ and $x_1 \geq x_3$, we can obtain from (3.16)

$$x_0 \leq x_2 \leq \cdots \leq x_{2n} \leq \cdots, \qquad \cdots \leq x_{2n+1} \leq x_{2n-1} \leq \cdots \leq x_1. \quad (3.19)$$

Hence, we may assume that $x_3 \geq x_1 \geq x_{-1}$ and $x_0 \leq x_2$. If further $x_2 \geq x_4$, then

$$x_2 \geq \cdots \geq x_{2n} \geq \cdots, \qquad \cdots \geq x_{2n+1} \geq x_{2n-1} \geq \cdots \geq x_1 \geq x_{-1}. \quad (3.20)$$

So we may assume that $x_3 \geq x_1 \geq x_{-1}$ and $x_0 \leq x_2 \leq x_4$. By induction we obtain the result in this case.

The cases $x_1 \leq x_{-1}$ and $x_2 \leq x_0$ can be treated similarly. □

Theorem 3.4. *Consider* (1.1) *where* (3.1), (3.2), *and* (3.3) *hold. Then, every positive solution of* (1.1) *converges to a prime two periodic solution.*

Proof. By Theorem 3.3, for every positive solution of (1.1) the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are eventually monotone. By Theorem 2.4, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are bounded. Hence, the sequences $\{x_{2n}\}$ and $\{x_{2n+1}\}$ are convergent. Using Lemmas 1.3 and 3.1 the result follows. □

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