Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 958381, 22 pages
doi:10.1155/2011/958381

Research Article

Lyapunov Function and Exponential Trichotomy on Time Scales

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Received 20 February 2011; Revised 7 May 2011; Accepted 6 July 2011

Academic Editor: Jinde Cao

We study the relation between Lyapunov function and exponential trichotomy for the linear equation $x^{\Delta} = A(t)x$ on time scales. Furthermore, as an application of these results, we give the roughness of exponential trichotomy on time scales.

1. Introduction

Exponential trichotomy is important for center manifolds theorems and bifurcation theorems. When people analyze the asymptotic behavior of dynamical systems, exponential trichotomy is a powerful tool. The conception of trichotomy was first introduced by Sacker and Sell [1]. They described SS-trichotomy for linear differential systems by linear skew-product flows. Furthermore, Elaydi and Hájek [2, 3] gave the notions of exponential trichotomy for differential systems and for nonlinear differential systems, respectively. These notions are stronger notions than SS-trichotomy. In 1991, Papaschinopoulos [4] discussed the exponential trichotomy for linear difference equations. And in 1999 Hong and his partners [5, 6] studied the relationship between exponential trichotomy and the ergodic solutions of linear differential and difference equations with ergodic perturbations. Recently, Barreira and Valls [7, 8] gave the conception of nonuniform exponential trichotomy. From their papers, we can see that the exponential trichotomy studied before is just a special case of the nonuniform exponential trichotomy. For more information about exponential trichotomy we refer the reader to papers [9–14].

Many phenomena in nature cannot be entirely described by discrete system, or by continuous system, such as insect population model, the large population in the summer, the number increases, a continuous function can be shown. And in the winter the insects freeze to death or all sleep, their number reduces to zero, until the eggs hatch in the next spring, the number increases again, this process is a jump process. Therefore, this population model is a discontinuous jump function, it cannot be expressed by a single differential equation, or by a single difference equations. Is it possible to use a unified framework to represent the above population model? In 1988, Hilger [15] first introduced the theory of time scales. From then on, there are numerous works on this area (see [16–23]). Time scales provide a method to unify and generalize theories of continuous and discrete dynamical systems.

In this paper, motivated by [8], we study the exponential trichotomy on time scales. We firstly introduce (λ, μ)-Lyapunov function on time scales. Then we study the relationship between exponential trichotomy and (λ, μ)-Lyapunov function on time scales. We obtain that the linear equation $x^{\Delta} = A(t)x$ admits exponential trichotomy, if it has two (λ, μ)-Lyapunov functions with some property; conversely, the linear equation has two (λ, μ)-Lyapunov functions, if it admits strict exponential trichotomy. At last, by using these results we investigate the roughness of exponential trichotomy on time scales. Above all, our paper gives a way to unify the analysis of continuous and discrete exponential trichotomy.

This paper is organized as follows. In Section 2, we review some useful notions and basic properties on time scales. Our main results will be stated and proved in Section 3. Finally, in Section 4, we study the roughness of exponential trichotomy on time scales.

2. Preliminaries on Time Scales

In order to make our paper independent, some preliminary definitions and theories on time scales are listed below.

Definition 2.1. Let $\mathbb{T}$ be a time scale which is an arbitrary nonempty closed subset of the real numbers. The forward jump operator is defined by

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \tag{2.1}$$

while the backward jump operator is defined by

$$\rho(t) := \sup\{s \in \mathbb{T} : s < t\}, \tag{2.2}$$

for every $t \in \mathbb{T}$. If $\sigma(t) = t$, then t is called right-dense. And if $\rho(t) = t$, then t is called left-dense. Let $\mu(t) := \sigma(t) - t$ be the graininess function.

For example, the set of real numbers $\mathbb{R}$ is a time scale with $\sigma(t) = t$ and $\mu(t) = 0$ for $t \in \mathbb{T} = \mathbb{R}$ and the set of integers $\mathbb{Z}$ is a time scale with $\sigma(t) = t + 1$ and $\mu(t) = 1$ for $t \in \mathbb{T} = \mathbb{Z}$.

Definition 2.2. A function $f : \mathbb{T} \to \mathbb{R}^n$ is called rd-continuous if it is continuous at right-dense points in $\mathbb{T}$ and left-sided limits exist at left dense points in $\mathbb{T}$.

We denote the set of rd-continuous functions by $C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$.

Definition 2.3. We say that a function $f : \mathbb{T} \to \mathbb{R}^n$ is differentiable at $t \in \mathbb{T}$, if for any $\varepsilon > 0$, there exist $\mathbb{T}$-neighborhood U of t and $f^\Delta(t) \in \mathbb{R}^n$ such that for any $s \in U$ one has

$$\left\| f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s) \right\| \leq \varepsilon(\sigma(t) - s), \tag{2.3}$$

and $f^\Delta(t)$ is called the derivative of f at t.

Relate to the differential properties, if f and g is differentiable at t, then we have the following equalities:

$$f(\sigma(t)) = f(t) + \mu(t) f^\Delta(t), \tag{2.4}$$

$$(fg)^\Delta(t) = f^\Delta(t) g(t) + f(\sigma(t)) g^\Delta(t). \tag{2.5}$$

Definition 2.4. We say that a function $p : \mathbb{T} \to \mathbb{R}$ is regressive if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}$. Furthermore, if $1+\mu(t)p(t) > 0$, p is called positive regressive. An $n \times n$ matrix valued function $A(t)$ on time scale $\mathbb{T}$ is called regressive if $\mathrm{Id} + \mu(t)A(t)$ is invertible for all $t \in \mathbb{T}$.

Let a, b be regressive. Define $(a \oplus b)(t) = a(t)+b(t)+\mu(t)a(t)b(t)$ and $(\ominus a)(t) = -a(t)/(1+\mu(t)a(t))$ for all $t \in \mathbb{T}$. Then the regressive set is a Abelian group and it is not hard to verify that the following properties hold:

(1) $a \ominus a = 0$;

(2) $\ominus(\ominus a) = a$;

(3) $a \ominus b = (a - b)/(1 + \mu(t)b)$;

(4) $\ominus(a \oplus b) = (\ominus a) \oplus (\ominus b)$, where a, b are regressive.

In order to make our paper intelligible, the exponential function on time scales which we will use in our paper is defined as following. For the more general conception of exponential function on time scales, please refer to [17].

Definition 2.5. Let p be positive regressive. We define the exponential function by

$$e_p(t, s) = \exp\left\{ \int_s^t \frac{1}{\mu(\tau)} \ln\bigl(1 + \mu(\tau)p(\tau)\bigr) \Delta\tau \right\}, \tag{2.6}$$

for all $s, t \in \mathbb{T}$. Here the integral is always understood in Lebesgue's sense and if $f \in C_{\mathrm{rd}}(\mathbb{T}, \mathbb{R}^n)$, for any $t \in \mathbb{T}$ one has

$$\int_t^{\sigma(t)} f(s) \Delta s = \mu(t) f(t). \tag{2.7}$$

From the definition of exponential function we have that if a, b are positive regressive, then

(1) $e_0(t, s) = 1$ and $e_a(t, t) = 1$;

(2) $e_a(t,s) = e_{\ominus a}(s,t)$;

(3) $e_a(t,s)e_a(s,r) = e_a(t,r)$;

(4) $e_a(t,s)e_b(t,s) = e_{a\oplus b}(t,s)$;

(5) $[e_a(c,t)]_t^{\Delta} = -a(t)e_a(c,\sigma(t))$, where $[e_a(c,t)]_t^{\Delta}$ stands for the delta derivative of $e_a(c,t)$ with respect to t.

Lemma 2.6 (L'Hôpital's Rule). *Suppose that f and g are differentiable on $\mathbb{T}$. Then if $g(t)$ satisfies $g(t) > 0$, $g^{\Delta} > 0$ and $\lim_{t\to+\infty} g(t) = \infty$, then the existence of $\lim_{t\to+\infty} f^{\Delta}(t)/g^{\Delta}(t)$ implies that $\lim_{t\to+\infty} f(t)/g(t)$ exists and $\lim_{t\to+\infty} f(t)/g(t) = \lim_{t\to+\infty} f^{\Delta}(t)/g^{\Delta}(t)$.*

3. Exponential Trichotomy and Lyapunov Function on Time Scales

From now on, we always suppose that $\mathbb{T}$ is a two-sides infinite time scale and the graininess function $\mu(t)$ is bounded, which means that there exists a $M > 0$ such that $0 \le \mu(t) \le M$. We consider the following linear equation

$$x^{\Delta} = A(t)x, \tag{3.1}$$

where $A(t)$ is an $n \times n$ matrix valued function on time scale $\mathbb{T}$, satisfying that $A(t)$ is rd-continuous and regressive. To make it simple, in this paper we always require that $A(t)$ is regressive. If $A(t)$ is not regressive, from [20] we can know that the linear evolution operator $T(t,\tau)$ associated to (3.1) is not invertible and exists only for $\tau \le t$ which will make our paper complicated. Here, $T(t,\tau)$ is called the linear evolution operator if $T(t,\tau)$ satisfies the following conditions:

(1) $T(\tau,\tau) = \mathrm{Id}$;

(2) $T(t,\tau)T(\tau,s) = T(t,s)$;

(3) the mapping $(t,\tau) \to T(t,\tau)x$ is continuous for any fixed $x \in \mathbb{R}^n$.

Equation (3.1) is said to admit an *exponential trichotomy* on time scale $\mathbb{T}$ if there exist projections $P(t)$, $Q(t)$, $R(t) : \mathbb{R}^n \to \mathbb{R}^n$ for each $t \in \mathbb{T}$ such that $P(t) + Q(t) + R(t) = \mathrm{Id}$,

$$T(t,\tau)P(\tau) = P(t)T(t,\tau), \qquad T(t,\tau)Q(\tau) = Q(t)T(t,\tau), \qquad T(t,\tau)R(\tau) = R(t)T(t,\tau) \tag{3.2}$$

for $t,\tau \in \mathbb{T}$ and there are some constants $a > b \ge 0$ and $D > 1$ such that for $t \ge \tau,\ t,\tau \in \mathbb{T}$

$$\|T(t,\tau)P(\tau)\| \le De_a(\tau,t), \qquad \|T(t,\tau)R(\tau)\| \le De_b(t,\tau), \tag{3.3}$$

and for $t \le \tau,\ t,\tau \in \mathbb{T}$

$$\|T(t,\tau)Q(\tau)\| \le De_a(t,\tau), \qquad \|T(t,\tau)R(\tau)\| \le De_b(\tau,t). \tag{3.4}$$

We say that (3.1) admits a *strong exponential trichotomy* on time scale $\mathbb{T}$ if there exist $P(t)$, $Q(t)$, $R(t) : \mathbb{R}^n \to \mathbb{R}^n$ for each $t \in \mathbb{T}$ and constants $a > b \geq 0$, $D > 1$ satisfying (3.2), (3.3), (3.4), as well as a constant c with $c \geq a$ and $c > 1$ such that

$$\|T(t,\tau)P(\tau)\| \leq De_c(\tau,t), \quad t \leq \tau,\ t,\tau \in \mathbb{T},$$
$$\|T(t,\tau)Q(\tau)\| \leq De_c(t,\tau), \quad t \geq \tau,\ t,\tau \in \mathbb{T}. \tag{3.5}$$

Obviously, if for any $t \in \mathbb{T}$ we have $\mu(t) \equiv 0$ or $\mu(t) \equiv 1$, then the notion of exponential trichotomy on time scale $\mathbb{T}$ becomes the exponential trichotomy for differential equation or difference equation, respectively.

Next, we give the notion of Lyapunov function. Consider a function $V : \mathbb{T} \times \mathbb{R}^n \to \mathbb{R}^n$. If the following two conditions are satisfied

(H1) for each $\tau \in \mathbb{T}$ set $V_\tau := V(\tau,\cdot)$ and

$$C^s(V_\tau) := \{0\} \cup V_\tau^{-1}(-\infty,0), \qquad C^u(V_\tau) := \{0\} \cup V_\tau^{-1}(0,+\infty). \tag{3.6}$$

let r_s and r_u be, respectively, the maximal dimensions of linear subspaces inside $C^s(V_\tau)$ and $C^u(V_\tau)$, then we have $r_s + r_u = n$;

(H2) for every $t \geq \tau$, $t,\tau \in \mathbb{T}$ and $x \in \mathbb{R}^n$, we have $T(\tau,t)C^s(V_t) \subset C^s(V_\tau)$ and $T(t,\tau)C^u(V_\tau) \subset C^u(V_t)$,

then we say that the function V is a *Lyapunov function*.

Let

$$H_\tau^s := \bigcap_{r\in\mathbb{T}} T(\tau,r)\overline{C^s(V_r)} \subset \overline{C^s(V_\tau)},$$
$$H_\tau^u := \bigcap_{r\in\mathbb{T}} T(\tau,r)\overline{C^u(V_r)} \subset \overline{C^u(V_\tau)}. \tag{3.7}$$

By the notion of Lyapunov function, we can see that there are subspaces $D_\tau^s(V)$ and $D_\tau^u(V)$ such that $D_\tau^s(V) \oplus D_\tau^u(V) = \mathbb{R}^n$. The function V is called a (λ,μ)-*Lyapunov function* if V is a Lyapunov function and for $t \geq \tau$, $t,\tau \in \mathbb{T}$, V satisfies the following conditions:

(L1) $|V(t,x)| \leq N\|x\|$,

(L2) for $x \in D_\tau^s(V)$, $V^2(t,T(t,\tau)x) \leq e_\lambda(t,\tau)V^2(\tau,x)$,

(L3) for $x \in D_\tau^u(V)$, $V^2(t,T(t,\tau)x) \geq e_\mu(t,\tau)V^2(\tau,x)$,

(L4) for $x \in D_\tau^s(V) \bigcup D_\tau^u(V)$, $|V(t,x)| \geq \|x\|/N$,

where $\lambda < \mu$ and N is a constant with $N > 1$.

Furthermore, let $S(t)$ for each $t \in \mathbb{T}$ be a symmetric invertible $n \times n$ matrix. Set $H(t,x) := \langle S(t)x,x\rangle$. If $V(t,x) := -\operatorname{sign} H(t,x)\sqrt{|H(t,x)|}$ is a Lyapunov function ((λ,μ)-Lyapunov function), then $V(t,x)$ is called a *quadric Lyapunov function* ((λ,μ)-*quadric Lyapunov function*). Let $\|S(t)\| := \sup_{x\in\mathbb{R}^n}|H(t,x)|/\|x\|^2$. Now, we state and prove our main results.

Theorem 3.1. *Suppose that* (3.1) *has two* (λ,μ)*-quadratic Lyapunov functions,* $V(t,x)$ *with* (r_1,r_2) *and* $W(t,x)$ *with* (l_1,l_2) *satisfying* $r_1<r_2<0<l_1<l_2$*. If the symmetric invertible* $n\times n$ *matrixes* $S(t)$ *and* $T(t)$ *for* $V(t,x)$ *and* $W(t,x)$*, respectively, satisfy* $\sup_{t\in\mathbb{T}}\|S(t)\|<\infty$ *and* $\sup_{t\in\mathbb{T}}\|T(t)\|<\infty$*, then* (3.1) *has an exponential trichotomy.*

In order to prove Theorem 3.1, we need some lemmas.

Lemma 3.2. *The subspaces* $D_\tau^s(V)$*,* $D_\tau^u(V)$*,* $D_\tau^s(W)$*, and* $D_\tau^u(W)$ *according to Lyapunov functions* V *and* W *have the following relations*

$$D_\tau^s(V)\subset D_\tau^s(W),\qquad D_\tau^u(W)\subset D_\tau^u(V). \tag{3.8}$$

Proof. By the knowledge of mathematical analysis, we can easily get that $\ln(1+lx)/x$ is a decreasing function for any $l\in\mathbb{R}$. Thus, when $0\le x\le M$, we have $\ln(1+lM)/M\le\ln(1+lx)/x\le l$. Now, we prove this lemma by contradiction. Suppose that there is $x\in D_\tau^s(V)\setminus D_\tau^s(W)$. Then there are $y\in D_\tau^s(W)$ and $z\in D_\tau^u(W)\setminus\{0\}$ such that $x=y+z$. Since $r_1<l_1<l_2$, then by Lemma 2.6 we get

$$\begin{aligned}
\lim_{t\to+\infty}\frac{1}{t}\ln|V(t,T(t,\tau)x)| &\ge \lim_{t\to+\infty}\frac{1}{t}\ln\frac{\|T(t,\tau)x\|}{N}\\
&\ge \lim_{t\to+\infty}\frac{1}{t}\ln\left(\|T(t,\tau)z\|-\left\|T(t,\tau)y\right\|\right)\\
&\ge \lim_{t\to+\infty}\frac{1}{t}\ln\left(\frac{V(t,T(t,\tau)z)}{N}-N\left|V(t,T(t,\tau)y)\right|\right)\\
&\ge \lim_{t\to+\infty}\frac{1}{t}\ln\left(\frac{\sqrt{e_{l_2}(t,\tau)}V(\tau,z)}{N}-N\sqrt{e_{l_1}(t,\tau)}\left|V(\tau,y)\right|\right)\\
&\ge \lim_{t\to+\infty}\frac{1}{t}\ln\sqrt{e_{l_1}(t,\tau)}\\
&= \lim_{t\to+\infty}\frac{1}{2t}\int_\tau^t\frac{1}{\mu(s)}\ln\left(1+\mu(s)l_1\right)\Delta s\\
&\ge \lim_{t\to+\infty}\frac{t-\tau}{2t}\frac{\ln(1+l_1M)}{M}=\frac{\ln(1+l_1M)}{2M}>0.
\end{aligned} \tag{3.9}$$

But for $x\in D_\tau^s(V)$, one has

$$\lim_{t\to+\infty}\frac{1}{t}\ln|V(t,T(t,\tau)x)|\le\lim_{t\to+\infty}\frac{1}{t}\ln\left(\sqrt{e_{r_1}(t,\tau)}|V(\tau,x)|\right)\le\frac{r_1}{2}<0. \tag{3.10}$$

It is a contradiction. So we have $D_\tau^s(V)\subset D_\tau^s(W)$.

By the concept of Lyapunov function, we get that for $t\le\tau$,

$$\begin{aligned}
\left|V'(\tau,x)\right| &\le e_\lambda(\tau,t)\left|V'(t,T(t,\tau)x)\right|, \quad x\in D_\tau^s,\\
V'(\tau,x) &\ge e_\mu(\tau,t)V'(t,T(t,\tau)x), \quad x\in D_\tau^u,
\end{aligned} \tag{3.11}$$

where $V' = V$ or W, $\lambda = r_1$ or l_1 and $\mu = r_2$ or l_2. Using above inequalities and the similar way by which we get $D^s_\tau(V) \subset D^s_\tau(W)$, we can directly obtain that $D^u_\tau(W) \subset D^u_\tau(V)$. Here we omit the detailed proofing. Thus, the proof is completed. □

Obviously, from Lemma 3.2 we have $(D^u_\tau(V) \bigcap D^s_\tau(W)) \bigcap D^u_\tau(W) = \{0\}$, $(D^u_\tau(V) \cap D^s_\tau(W)) \bigcap D^s_\tau(V) = \{0\}$ and $D^u_\tau(W) \bigcap D^s_\tau(V) = \{0\}$. Furthermore,

$$\begin{aligned} \dim\left(D^u_\tau(V) \bigcap D^s_\tau(W)\right) &= \dim D^u_\tau(V) + \dim D^s_\tau(W) - \dim(D^u_\tau(V) + D^s_\tau(W)) \\ &= n - \dim D^s_\tau(V) + n - \dim D^u_\tau(W) - \dim(D^u_\tau(V) + D^s_\tau(W)). \end{aligned} \tag{3.12}$$

Thus,

$$\dim\left(D^u_\tau(V) \bigcap D^s_\tau(W)\right) + \dim D^s_\tau(V) + \dim D^u_\tau(W) \geq 2n - \dim(D^u_\tau(V) + D^s_\tau(W)) \geq n. \tag{3.13}$$

So we obtain

$$D^u_\tau(V) \bigcap D^s_\tau(W) \oplus D^s_\tau(V) \oplus D^u_\tau(W) = \mathbb{R}^n. \tag{3.14}$$

Then from the conditions in Theorem 3.1 we have the following results.

When $t \geq \tau$, for $x \in D^s_\tau(V)$, we get

$$\|T(t,\tau)x\| \leq N|V(t,T(t,\tau)x)| \leq N\sqrt{e_{r_1}(t,\tau)}|V(\tau,x)| \leq N^2\sqrt{e_{r_1}(t,\tau)}\|x\|, \tag{3.15}$$

and for $x \in D^u_\tau(V) \bigcap D^s_\tau(W)$, we get

$$\|T(t,\tau)x\| \leq N|V(t,T(t,\tau)x)| \leq N\sqrt{e_{l_1}(t,\tau)}|V(\tau,x)| \leq N^2\sqrt{e_{l_1}(t,\tau)}\|x\|. \tag{3.16}$$

When $t \leq \tau$, for $x \in D^u_\tau(W)$, we get

$$\|T(t,\tau)x\| \leq N|V(t,T(t,\tau)x)| \leq N\sqrt{e_{l_2}(t,\tau)}|V(\tau,x)| \leq N^2\sqrt{e_{l_2}(t,\tau)}\|x\|, \tag{3.17}$$

and for $x \in D^u_\tau(V) \bigcap D^s_\tau(W)$, we get

$$\|T(t,\tau)x\| \leq N|V(t,T(t,\tau)x)| \leq N\sqrt{e_{r_2}(t,\tau)}|V(\tau,x)| \leq N^2\sqrt{e_{r_2}(t,\tau)}\|x\|. \tag{3.18}$$

Let

$$\begin{aligned} &P_V(\tau) : \mathbb{R}^n \longrightarrow D^s_\tau(V), \qquad Q_V(\tau) : \mathbb{R}^n \longrightarrow D^u_\tau(V), \\ &P_W(\tau) : \mathbb{R}^n \longrightarrow D^s_\tau(W), \qquad Q_W(\tau) : \mathbb{R}^n \longrightarrow D^u_\tau(W) \end{aligned} \tag{3.19}$$

be projections. Set $P(\tau) := P_V(\tau)$, $Q(\tau) := Q_W(\tau)$ and $R(\tau) := P_W(\tau) + Q_V(\tau)$. From Lemma 3.2, we have $Q_V(\tau)P_W(\tau) = P_W(\tau)Q_V(\tau) = 0$. Thus, by simply deducing, we get that $R(\tau)$ is also a projection. Set $L_1(\tau) = \mathrm{Id} - P(\tau)$. Next, we will give the boundedness of $P(\tau)$, $Q(\tau)$, and $R(\tau)$. By the conditions of Theorem 3.1, we suppose that there is a constant $L > 0$ such that $\sup_{t\in\mathbb{T}}\|S(t)\| \le L$ and $\sup_{t\in\mathbb{T}}\|T(t)\| \le L$. We have the following lemma.

Lemma 3.3.

$$\|P(t)x\| \le \sqrt{2}N^2\|x\|, \qquad \|Q(t)x\| \le \sqrt{2}N^2\|x\|, \qquad \|R(t)x\| \le 2\sqrt{2}N^2\|x\|. \tag{3.20}$$

Proof. Firstly, one has

$$\begin{aligned} V(t, P(t)x) &= \langle S(t)P(t)x, P(t)x\rangle \ge \frac{\|P(t)x\|^2}{N^2}, \\ V(t, L_1(t)x) &= -\langle S(t)L_1(t)x, L_1(t)x\rangle \ge \frac{\|Q(t)x\|^2}{N^2}. \end{aligned} \tag{3.21}$$

Since

$$\begin{aligned} &\frac{1}{N^2}\left\|P(t)x - \frac{N^2}{2}S(t)x\right\|^2 + \frac{1}{N^2}\left\|L_1(t)x - \frac{N^2}{2}S(t)x\right\|^2 \\ &\quad = \frac{1}{N^2}\|P(t)x\|^2 + \frac{1}{N^2}\|L_1(t)x\|^2 + \frac{N^2}{2}\|S(t)x\|^2 \\ &\qquad - \langle P(t)x, S(t)x\rangle + \langle L_1(t)x, S(t)x\rangle \\ &\quad \le \langle S(t)P(t)x, P(t)x\rangle - \langle S(t)L_1(t)x, L_1(t)x\rangle \\ &\qquad + \frac{N^2}{2}\|S(t)x\|^2 - \langle P(t)x, S(t)x\rangle + \langle L_1(t)x, S(t)x\rangle \\ &\quad = \frac{N^2}{2}\|S(t)x\|^2, \end{aligned} \tag{3.22}$$

then we obtain

$$\begin{aligned} \|P(t)x\| &\le \left\|P(t)x - \frac{N^2}{2}S(t)x\right\| + \frac{N^2}{2}\|S(t)x\| \\ &\le \frac{N^2}{\sqrt{2}}\|S(t)x\| + \frac{N^2}{2}\|S(t)x\| \\ &\le \sqrt{2}N^2\|S(t)x\| \le \sqrt{2}N^2L\|x\|. \end{aligned} \tag{3.23}$$

Similarly, we get

$$\|Q(t)x\| \le \sqrt{2}N^2L\|x\|,$$
$$\|R(t)x\| \le 2\sqrt{2}N^2L\|x\|. \tag{3.24}$$

□

Proof of Theorem 3.1. Firstly, we need to show that there are constants $r_1', r_2' < 0$ and $l_1', l_2' > 0$ such that when $t \ge \tau$, one has

$$\sqrt{e_{r_1}(t,\tau)} \le e_{r_1'}(t,\tau), \tag{3.25}$$

$$\sqrt{e_{l_1}(t,\tau)} \le e_{l_1'}(t,\tau), \tag{3.26}$$

and when $t \le \tau$, one has

$$\sqrt{e_{l_2}(t,\tau)} \le e_{l_2'}(t,\tau),$$
$$\sqrt{e_{r_2}(t,\tau)} \le e_{r_2'}(t,\tau). \tag{3.27}$$

For inequality (3.25), by the definition of exponential function, we only need to find a constant $r_1' < 0$ such that

$$1 + \mu(t)r_1 \le \left(1 + \mu(t)r_1'\right)^2. \tag{3.28}$$

That means

$$r_1 \le 2r_1' + \mu(t)r_1'^2. \tag{3.29}$$

Thus, if $r_1/2 \le r_1' < 0$, (3.25) always holds. Similarly, we get that if $l_1' \ge l_1/2$, $0 < l_2' < (\sqrt{1+Ml_2} - 1)/M$ and $(-\sqrt{1+Mr_2} - 1)/M < r_2' < 0$, then (3.26), (3.27) always hold, respectively. Here, we notice that since r_2 is positive regressive, then $\sqrt{1+Mr_2}$ is significative.

By (3.15)–(3.18) and Lemma 3.3, we get that when $t \ge \tau$,

$$\begin{aligned}
\|T(t,\tau)P(\tau)x\| &\le \left\| T(t,\tau)|_{D_\tau^s(V)} \right\| \|P(\tau)x\| \\
&\le \sqrt{2}N^2L\sqrt{e_{r_1}(t,\tau)}\|x\| \le \sqrt{2}N^2Le_{r_1'}(t,\tau)\|x\|, \\
\|T(t,\tau)R(\tau)x\| &\le \left\| T(t,\tau)|_{D_\tau^s(W)\bigcap D_\tau^u(V)} \right\| \|R(\tau)x\| \\
&\le 2\sqrt{2}N^2L\sqrt{e_{l_1}(t,\tau)}\|x\| \le 2\sqrt{2}N^2Le_{l_1'}(t,\tau)\|x\|,
\end{aligned} \tag{3.30}$$

and when $t \leq \tau$,

$$\begin{aligned}
\|T(t,\tau)Q(\tau)x\| &\leq \left\| T(t,\tau)|_{D_\tau^u(W)} \right\| \|Q(\tau)x\| \\
&\leq \sqrt{2}N^2 L\sqrt{e_{l_2}(t,\tau)}\|x\| \leq \sqrt{2}N^2 L e_{l_2'}(t,\tau)\|x\|, \\
\|T(t,\tau)R(\tau)x\| &\leq \left\| T(t,\tau)|_{D_\tau^s(W)\bigcap D_\tau^u(V)} \right\| \|R(\tau)x\| \\
&\leq 2\sqrt{2}N^2 L\sqrt{e_{r_2}(t,\tau)}\|x\| \leq 2\sqrt{2}N^2 L e_{r_2'}(t,\tau)\|x\|.
\end{aligned} \tag{3.31}$$

Set $D := 2\sqrt{2}N^2L$, $a := \min\{|r_1'|, l_2'\}$, and $b := \max\{|r_2'|, l_1'\}$. Obviously, we have $a > b$ and the above inequalities imply that for $t \geq \tau,\ t,\tau \in \mathbb{T}$ one has

$$\|T(t,\tau)P(\tau)\| \leq De_a(\tau,t), \qquad \|T(t,\tau)R(\tau)\| \leq De_b(t,\tau), \tag{3.32}$$

and for $t \leq \tau,\ t,\tau \in \mathbb{T}$ one has

$$\|T(t,\tau)Q(\tau)\| \leq De_a(t,\tau), \qquad \|T(t,\tau)R(\tau)\| \leq De_b(\tau,t). \tag{3.33}$$

Thus, from the definition of exponential trichotomy on time scales, we get that (3.1) has an exponential trichotomy on time scale $\mathbb{T}$. This completes the proof of Theorem 3.1. □

Next, we give an approximately converse result of Theorem 3.1.

Theorem 3.4. *If* (3.1) *admits a strict exponential trichotomy, then for* (3.1) *there exist two* (λ,μ)-*Lyapunov functions,* $V'(t,x)$ *with* $(\ominus r_1', \ominus r_2')$ *and* $W'(t,x)$ *with* (l_1', l_2') *satisfying* $\ominus r_1' < \ominus r_2' < 0 < l_1' < l_2'$.

Proof. Suppose that (3.1) admits a strict exponential trichotomy with projections $P(t)$, $Q(t)$, $R(t)$, and constants $c \geq a > b \geq 0$ satisfying (3.2)–(3.5). Set

$$\begin{aligned}
S(t) := &\int_t^{+\infty} (T(\sigma(v),t)P(t))^* T(\sigma(v),t)P(t) e_{c_1\oplus b}(\sigma(v),t)\Delta v \\
&- \int_{-\infty}^{t} (T(\sigma(v),t)L_1(t))^* T(\sigma(v),t)L_1(t) e_{c_2\oplus b}(\sigma(v),t)\Delta v,
\end{aligned} \tag{3.34}$$

where $L_1(t) = Q(t) + R(t)$ and $b < c_2 < c_1 < a$. Let $H(t,x) := \langle S(t)x, x\rangle$.

When $t \geq \tau,\ t, \tau \in \mathbb{T}$, we get

$$\begin{aligned}
H(t, T(t,\tau)x) &= \int_t^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c_1\oplus b}(\sigma(v),t)\Delta v \\
&\quad - \int_{-\infty}^{t} \|T(\sigma(v),\tau)L_1(\tau)x\|^2 e_{c_2\oplus b}(\sigma(v),t)\Delta v \\
&\leq \int_\tau^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c_1\oplus b}(\sigma(v),\tau)\Delta v e_{c_1\oplus b}(\tau,t) \\
&\quad - \int_{-\infty}^{\tau} \|T(\sigma(v),\tau)L_1(\tau)x\|^2 e_{c_2\oplus b}(\sigma(v),\tau)\Delta v e_{c_2\oplus b}(\tau,t) \\
&\leq e_{c_2\oplus b}(\tau,t) H(\tau,x).
\end{aligned} \tag{3.35}$$

Set $V(t,x) := -\mathrm{sign} H(t,x)\sqrt{|H(t,x)|}$. Then if $V(\tau,x) > 0$, that is, $x \in C^u(V_\tau)$, we have $H(\tau,x) < 0$. By (3.35) we obtain that $H(t,T(t,\tau)) < 0$. Then we get $V(t,T(t,\tau)x) > 0$, that is, $T(t,\tau)x \in C^u(V_t)$. That means $T(t,\tau)C^u(V_\tau) \subset C^u(V_t)$. Similarly, we have $T(\tau,t)C^s(V_t) \subset C^s(V_\tau)$. Thus, $V(t,x)$ is a Lyapunov function. By (3.3), (3.4) and the characters of exponential function on time scales, one has

$$\begin{aligned}
|H(t,x)| &\leq \int_t^{+\infty} \|T(\sigma(v),t)P(t)x\|^2 e_{c_1\oplus b}(\sigma(v),t)\Delta v \\
&\quad + \int_{-\infty}^{t} \|T(\sigma(v),t)L_1(t)x\|^2 e_{c_2\oplus b}(\sigma(v),t)\Delta v \\
&\leq \int_t^{+\infty} D^2 e_a(t,\sigma(v)) e_a(t,\sigma(v)) e_{c_1\oplus b}(\sigma(v),t)\Delta v \|x\|^2 \\
&\quad + \int_{-\infty}^{t} D^2 e_a(\sigma(v),t) e_a(\sigma(v),t) e_{c_2\oplus b}(\sigma(v),t)\Delta v \|x\|^2 \\
&\quad + \int_{-\infty}^{t} D^2 e_b(t,\sigma(v)) e_b(t,\sigma(v)) e_{c_2\oplus b}(\sigma(v),t)\Delta v \|x\|^2 \\
&\leq \int_t^{+\infty} D^2 e_{a\ominus b}(t,\sigma(v))\Delta v \|x\|^2 + \int_{-\infty}^{t} D^2 e_{a\oplus a\oplus c_2\oplus b}(\sigma(v),t)\Delta v \|x\|^2 \\
&\quad + \int_{-\infty}^{t} D^2 e_{c_2\ominus b}(\sigma(v),t)\Delta v \|x\|^2 \\
&= \int_t^{+\infty} D^2 \frac{-[e_{a\ominus b}(t,v)]_v^\Delta}{a\ominus b}\Delta v \|x\|^2 + \int_{-\infty}^{t} D^2 \frac{-[e_{\ominus(a\oplus a\oplus c_2\oplus b)}(t,v)]_v^\Delta}{\ominus(a\oplus a\oplus c_2\oplus b)}\Delta v \|x\|^2 \\
&\quad + \int_{-\infty}^{t} D^2 \frac{-[e_{b\ominus c_2}(t,v)]_v^\Delta}{b\ominus c_2}\Delta v \|x\|^2.
\end{aligned} \tag{3.36}$$

Since $\mu(t) \leq M$, then

$$\frac{1}{a\ominus b}=\frac{1+\mu(t)b}{a-b}\leq\frac{1+Mb}{a-b},$$
$$\frac{-1}{\ominus(a\oplus a\oplus c_2\oplus b)}=\frac{1+\mu(t)(a\oplus a\oplus c_2\oplus b)}{a\oplus a\oplus c_2\oplus b}\leq\frac{1}{a\oplus a\oplus c_2\oplus b}+M\leq\frac{1}{2a+c_2+b}+M, \tag{3.37}$$
$$\frac{-1}{b\ominus c_2}=\frac{1+\mu(t)c_2}{c_2-b}\leq\frac{1+Mc_2}{c_2-b}.$$

Thus,

$$|H(t,x)|\leq\left[\frac{D^2(1+Mb)}{a-b}+D^2\left(\frac{1}{2a+c_2+b}+M\right)+D^2\left(\frac{1+Mc_2}{c_2-b}\right)\right]\|x\|^2. \tag{3.38}$$

Set $K := D^2(1+Mb)/(a-b)+D^2(1/(2a+c_2+b)+M)+D^2((1+Mc_2)/(c_2-b))$.

Let $H_\tau^s(V) := P(\tau)\mathbb{R}^n$ and $H_\tau^u(V) := L_1(\tau)\mathbb{R}^n = R(\tau)\mathbb{R}^n \oplus Q(\tau)\mathbb{R}^n$. For $x \in H_\tau^s(V)$, one has

$$\begin{aligned} H(t,x) &= \int_t^{+\infty}\|T(\sigma(v),t)P(t)x\|^2 e_{c_1\oplus b}(\sigma(v),t)\Delta v \\ &= \int_t^{+\infty}\|T(\sigma(v),t)x\|^2 e_{c_1\oplus b}(\sigma(v),t)\Delta v. \end{aligned} \tag{3.39}$$

Obviously, one has

$$\|T(t,\tau)\|\leq\|T(t,\tau)P(\tau)\|+\|T(t,\tau)Q(\tau)\|+\|T(t,\tau)R(\tau)\|. \tag{3.40}$$

Then by (3.3)–(3.5), we obtain that

$$\|T(t,\tau)\|\leq De_a(\tau,t)+De_b(t,\tau)+De_c(t,\tau)\quad\text{for } t\geq\tau, \tag{3.41}$$

$$\|T(t,\tau)\|\leq De_a(t,\tau)+De_b(\tau,t)+De_c(\tau,t)\quad\text{for } t\leq\tau. \tag{3.42}$$

Thus,

$$
\begin{aligned}
|H(t,x)| &\geq \int_t^{+\infty} \frac{\|x\|^2}{\|T(t,\sigma(v))\|^2} e_{c_1\oplus b}(\sigma(v),t)\Delta v \\
&\geq \int_t^{+\infty} \frac{\|x\|^2}{\|De_a(t,\sigma(v)) + De_b(\sigma(v),t) + De_c(\sigma(v),t)\|^2} e_{c_1\oplus b}(\sigma(v),t)\Delta v \\
&\geq \frac{\|x\|^2}{9D^2}\int_t^{+\infty} e_{c_1\oplus b}(\sigma(v),t) e_{c\oplus c}(t,\sigma(v))\Delta v \\
&= \frac{\|x\|^2}{9D^2}\int_t^{+\infty} \frac{-\left(e_{c\oplus c\ominus(c_1\oplus b)}(t,v)\right)_v^{\Delta}}{c\oplus c\ominus(c_1\oplus b)}\Delta v.
\end{aligned}
\tag{3.43}
$$

Since $c\oplus c\ominus(c_1\oplus b) = (2c+\mu(s)c^2-(c_1+b+\mu(s)c_1b))/(1+\mu(s)(c_1+b+\mu(s)b)) \leq 2c-c_1-b+Mc^2-Mc_1b$, then we get

$$
|H(t,x)| \geq \frac{\|x\|^2}{9D^2(2c-c_1-b+Mc^2-Mc_1b)}.
\tag{3.44}
$$

Similarly, for $x\in H_\tau^u(V)$, we get

$$
\begin{aligned}
H(t,x) &= -\int_{-\infty}^{t} \|T(\sigma(v),t)x\|^2 e_{c_2\oplus b}(\sigma(v),t)\Delta v, \\
|H(t,x)| &\geq \frac{\|x\|^2}{9D^2(c_2+b+2c+M(2c_2c+2bc+c^2)+M^2(bc+c_2c^2+bc^2)+M^3bc^2)}.
\end{aligned}
\tag{3.45}
$$

Let $L' := \max\{c_2+b+2c+M(2c_2c+2bc+c^2)+M^2(bc+c_2c^2+bc^2)+M^3bc^2, 2c-c_1-b+Mc^2-Mc_1b\}$. Then when $x\in H_\tau^s(V)\bigcup H_\tau^u(V)$, we get $|H(t,x)|\geq \|x\|^2/9D^2L'$. Let $N^2 := \max\{9D^2L', K\}$. We get $|H(t,x)|\leq K\leq N^2\|x\|^2$ for any $t\in\mathbb{T}$, $x\in\mathbb{R}^n$, and when $x\in H_\tau^s(V)\bigcup H_\tau^u(V)$, we get $|H(t,x)|\geq \|x\|^2/N^2$.

Since $c_2<c_1$, then we have $c_2\oplus b<c_1\oplus b$ and there are constants r_1' and r_2' with $c_2\oplus b\leq r_2'<r_1'\leq c_1\oplus b$. Thus, when $x\in H_\tau^s(V)$, for any $t\geq\tau$ we get

$$
H(t,T(t,\tau)x) = \int_t^{+\infty} \|T(\sigma(v),\tau)P(\tau)x\|^2 e_{c_1\oplus b}(\sigma(v),t)\Delta v \leq e_{c_1\oplus b}(\tau,t)H(\tau,x),
\tag{3.46}
$$

that is,

$$
V^2(t,T(t,\tau)x) \leq e_{c_1\oplus b}(\tau,t)V^2(\tau,x) \leq e_{r_1'}(\tau,t)V^2(\tau,x),
\tag{3.47}
$$

and when $x\in H_\tau^u(V)$, for any $t\geq\tau$ we get

$$
H(t,T(t,\tau)x) = -\int_{-\infty}^{t} \|T(\sigma(v),\tau)L_1(\tau)x\|^2 e_{c_2\oplus b}(\sigma(v),t)\Delta v \leq e_{c_2\oplus b}(\tau,t)H(\tau,x),
\tag{3.48}
$$

that is,

$$V^2(t, T(t,\tau)x) \geq e_{c_2 \oplus b}(\tau, t) V^2(\tau, x) \geq e_{r_2'}(\tau, t) V^2(\tau, x). \tag{3.49}$$

Therefore, $V(t,x)$ is a $(\ominus r_1', \ominus r_2')$-Lyapunov function.

Set

$$\begin{aligned} T(t) := & \int_t^{+\infty} (T(\sigma(v), t) L_2(t))^* T(\sigma(v), t) L_2(t) e_{d_1 \oplus b}(t, \sigma(v)) \Delta v \\ & - \int_{-\infty}^{t} (T(\sigma(v), t) Q(t))^* T(\sigma(v), t) Q(t) e_{d_2 \oplus b}(t, \sigma(v)) \Delta v, \end{aligned} \tag{3.50}$$

where $L_2(t) = P(t) + R(t)$ and $b < d_1 < d_2 < a$. Let $G(t,x) := \langle T(t)x, x \rangle$ and $W(t,x) := -\operatorname{sign} G(t,x) \sqrt{|G(t,x)|}$. Then

$$\begin{aligned} G(t,x) = & \int_t^{+\infty} \|(T(\sigma(v), t) L_2(t))x\|^2 e_{d_1 \oplus b}(t, \sigma(v)) \Delta v \\ & - \int_{-\infty}^{t} \|(T(\sigma(v), t) Q(t))x\|^2 e_{d_2 \oplus b}(t, \sigma(v)) \Delta v. \end{aligned} \tag{3.51}$$

Let $H_\tau^s(W) := L_2(\tau)\mathbb{R}^n$ and $H_\tau^u(W) := Q(\tau)\mathbb{R}^n$. Similar to the consideration for $V(t,x)$, we can see that $G(t,x)$ is also a Lyapunov function satisfying

$$\begin{aligned} & G(t, T(t,\tau)x) \leq e_{d_2 \oplus b}(t, \tau) G(\tau, x), \\ & |G(t,x)| \leq D^2 \left(\frac{1 + Ma}{a - b} + \frac{1}{2a + d_1 + b} + M + \frac{1 + Mb}{d_1 - b} \right) \|x\|^2, \\ & |G(t,x)| \geq \frac{\|x\|^2}{9D^2 L''}, \quad \text{for } x \in H_\tau^s(W) \bigcup H_\tau^u(W), \end{aligned} \tag{3.52}$$

where $L'' = \max\{d_1 + b + 2c + M(2d_1 c + 2bc + c^2) + M^2(bc + d_1 c^2 + bc^2) + M^3 bc^2, 2c - d_2 - b + Mc^2 - Md_2 b\}$.

Since $d_1 < d_2$, then we have $d_1 \oplus b < d_2 \oplus b$ and there are constants l_1' and l_2' with $d_1 \oplus b \leq l_1' < l_2' \leq d_2 \oplus b$. Thus, for any $t \geq \tau$, we obtain that when $x \in H_\tau^s(W)$,

$$G(t, T(t,\tau)x) = \int_t^{+\infty} \|T(\sigma(v), \tau) L_2(\tau) x\|^2 e_{d_1 \oplus b}(t, \sigma(v)) \Delta v \leq e_{d_1 \oplus b}(t, \tau) G(\tau, x), \tag{3.53}$$

that is,

$$W^2(t, T(t,\tau)x) \leq e_{d_1 \oplus b}(t, \tau) W^2(\tau, x) \leq e_{l_1'}(t, \tau) W^2(\tau, x), \tag{3.54}$$

and when $x \in H_\tau^u(W)$,

$$G(t,T(t,\tau)x) = -\int_{-\infty}^{t} \|T(\sigma(v),\tau)L_1(\tau)x\|^2 e_{d_2\oplus b}(t,\sigma(v))\Delta v \le e_{d_2\oplus b}(t,\tau)G(\tau,x), \tag{3.55}$$

that is,

$$W^2(t,T(t,\tau)x) \ge e_{d_2\oplus b}(t,\tau)W^2(\tau,x) \ge e_{l_2'}(t,\tau)V^2(\tau,x). \tag{3.56}$$

Thus, $W(t,x)$ is a (l_1',l_2')-Lyapunov function.

According to all the discussions above we get that when (3.1) has a strict exponential trichotomy, there exists two (λ,μ)-Lyapunov functions, $V(t,x)$ with $(\ominus r_1',\ominus r_2')$ and $W(t,x)$ with (l_1',l_2') satisfying $\ominus r_1' < \ominus r_2' < 0 < l_1' < l_2'$. This completes the proof. □

4. Roughness of Exponential Trichotomy on Time Scales

The roughness of exponential dichotomy on time scales had been studied by Zhang and his cooperators in their paper [24] in 2010. In this section, we go further to study the roughness of exponential trichotomy on time scales, using the results which we get from Section 3. We have known that the notion of exponential trichotomy plays a central role when we study center manifolds, so it is important to understand how exponential trichotomy vary under perturbations. Here, we discuss the following linear perturbed equation

$$x^{\Delta} = A(t)x + B(t)x, \tag{4.1}$$

where $B(t)$ is an $n\times n$ matrix valued function on time scale $\mathbb{T}$ with $\|B(t)\| \le \delta$. For this linear perturbed (4.1) we get the following theorem.

Theorem 4.1. *Suppose that* (3.1) *has a strict exponential trichotomy. If* $\delta > 0$ *is sufficiently small, then the linear perturbed* (4.1) *also has an exponential trichotomy.*

Proof. Since (3.1) has a strict exponential trichotomy, then by Theorem 3.4 there are two (λ,μ)-Lyapunov functions, $V(t,x)$ with (r_1,r_2) and $W(t,x)$ with (l_1,l_2) satisfying $r_1 < r_2 < 0 < l_1 < l_2$. Here, $V(t,x)$ and $W(t,x)$ are defined in Theorem 3.4. Let $x(t)$ be a solution of (3.1) satisfying $x(\tau) = x$. Differentiating on both sides of $H(t,x(t)) = \langle S(t)x(t),x(t)\rangle$, one has

$$\begin{aligned}
H^{\Delta}(t,x(t)) &= \langle S(t)x(t),x(t)\rangle^{\Delta} \\
&= \left\langle S^{\Delta}(t)x(t),x(t)\right\rangle + \left\langle S(\sigma(t))\, x^{\Delta}(t),x(t)\right\rangle + \left\langle S(\sigma(t))x(\sigma(t)),x^{\Delta}(t)\right\rangle \\
&= \Big\langle \Big[S^{\Delta}(t) + S(\sigma(t))A(t) + A^*(t)S(\sigma(t)) \\
&\qquad + \mu(t)A^*(t)S(\sigma(t))A(t)\Big]x(t),x(t)\Big\rangle.
\end{aligned} \tag{4.2}$$

Let $T(t,\tau)$ be the linear evolution operator related to (3.1). Then we get $x(t) = T(t,\tau)x$. And we can see that $P(t)x(t) = P(t)T(t,\tau)x = T(t,\tau)P(\tau)x$ is also a solution of (3.1). By (4.2) we get

$$H^{\Delta}(t,P(t)x(t)) = \Big\langle \Big[S^{\Delta}(t) + S(\sigma(t))A(t) + A^{*}(t)S(\sigma(t)) + \mu(t)A^{*}(t)S(\sigma(t))A(t)\Big]P(t)x(t), P(t)x(t)\Big\rangle. \tag{4.3}$$

At the same time, by (2.4), (2.7), and (3.46) one has

$$\begin{aligned}
\mu(t)H^{\Delta}(t,P(t)x(t)) &= H(\sigma(t),P(\sigma(t))x(\sigma(t))) - H(t,P(t)x(t)) \\
&\le H(t,P(t)x(t))e_{c_1\oplus b}(t,\sigma(t)) - H(t,P(t)x(t)) \\
&= H(t,P(t)x(t))\left(\exp\left\{\int_t^{\sigma(t)} \frac{1}{\mu(s)}\ln\big(1+\ominus(c_1\oplus b)\mu(s)\big)\Delta s\right\} - 1\right) \\
&= \ominus(c_1\oplus b)\mu(t)H(t,P(t)x(t)).
\end{aligned} \tag{4.4}$$

Thus, we obtain that

$$H^{\Delta}(t,P(t)x(t)) \le \ominus(c_1\oplus b)H(t,P(t)x(t)). \tag{4.5}$$

Then by (4.3) and (4.5) we can see

$$S^{\Delta}(t) + S(\sigma(t))A(t) + A^{*}(t)S(\sigma(t)) + \mu(t)A^{*}(t)S(\sigma(t))A(t) \le \ominus(c_1\oplus b)S(t). \tag{4.6}$$

Now, Let $y(t)$ be a solution of equation $x^{\Delta} = A(t)x + B(t)x$ satisfying $y(\tau) = y$. Since $y(\sigma(t)) = y(t) + \mu(t)y^{\Delta}(t) = y(t) + \mu(t)(A(t)+B(t))y(t)$, one has

$$\begin{aligned}
H^{\Delta}(t,y(t)) &= \langle S(t)y(t), y(t)\rangle^{\Delta} \\
&= \Big\langle S^{\Delta}(t)y(t), y(t)\Big\rangle + \Big\langle S(\sigma(t))y^{\Delta}(t), y(t)\Big\rangle + \Big\langle S(\sigma(t))y(\sigma(t)), y^{\Delta}(t)\Big\rangle \\
&= \Big\langle S^{\Delta}(t)y(t), y(t)\Big\rangle + \langle S(\sigma(t))A(t)y(t), y(t)\rangle + \langle A^{*}(t)S(\sigma(t))y(t), y(t)\rangle \\
&\quad + \langle \mu(t)A^{*}(t)S(\sigma(t))A(t)y(t), y(t)\rangle + \langle S(\sigma(t))B(t)y(t), y(t)\rangle \\
&\quad + \langle \mu(t)S(\sigma(t))B(t)y(t), A(t)y(t)\rangle + \langle S(\sigma(t))y(t), B(t)y(t)\rangle \\
&\quad + \langle \mu(t)S(\sigma(t))A(t)y(t), B(t)y(t)\rangle + \langle \mu(t)S(\sigma(t))B(t)y(t), B(t)y(t)\rangle \\
&\le \ominus(c_1\oplus b)H(t,y(t)) + \|S(\sigma(t))\|\|B(t)\|\|y(t)\|^2 \\
&\quad + 2\|\mu(t)A(t)\|\|S(\sigma(t))\|\|B(t)\|\|y(t)\|^2 + \|S(\sigma(t))\|\|B(t)\|\|y(t)\|^2 \\
&\quad + \mu(t)\|S(\sigma(t))\|\|B(t)\|^2\|y(t)\|^2.
\end{aligned} \tag{4.7}$$

By the variation of constants formula, we have $T(t,s) = \mathrm{Id} + \int_s^t A(v)T(v,s)\Delta v$. Thus, by (2.7) we obtain

$$\begin{aligned} T(\sigma(t),t) &= I + \int_t^{\sigma(t)} A(v)T(v,t)\Delta v \\ &= \mathrm{Id} + \mu(t)A(t)T(t,t) = \mathrm{Id} + \mu(t)A(t). \end{aligned} \tag{4.8}$$

Therefore, by Definition 2.5 and (2.7), (3.41), we obtain

$$\begin{aligned} \left\|\mathrm{Id} + \mu(t)A(t)\right\| &\le De_a(t,\sigma(t)) + De_b(\sigma(t),t) + De_c(\sigma(t),t) \\ &= D\left[1 + \mu(t)(\ominus a)\right] + D\left(1 + \mu(t)b\right) + D\left(1 + \mu(t)c\right) \\ &\le 3D + DM(b + c). \end{aligned} \tag{4.9}$$

Thus, one has

$$\left\|\mu(t)A(t)\right\| \le \left\|\mathrm{Id} + \mu(t)A(t)\right\| + \|\mathrm{Id}\| \le 3D + DM(b + c) + 1. \tag{4.10}$$

From Theorem 3.4, we know

$$\begin{aligned} \|S(t)\| &= \sup_{x\in\mathbb{R}^n} \frac{H(t,x)}{\|x(t)\|^2} \le N^2, \\ \left\|y(t)\right\| &\le N\left|V\left(t,y(t)\right)\right|. \end{aligned} \tag{4.11}$$

Thus,

$$\begin{aligned} H^{\Delta}\left(t,y(t)\right) \le \ominus(c_1 \oplus b)H\left(t,y(t)\right) + \Big[&2N^2\delta + 2(3D + 1 + DMb + DMc)N^2\delta \\ &+MN^2\delta^2\Big]N^2\left|H\left(t,y(t)\right)\right|. \end{aligned} \tag{4.12}$$

Let $M(\delta) := [2N^2\delta + 2(3D + 1 + DMb + DMc)N^2\delta + MN^2\delta^2]\ N^2$. It is not hard to see that $M(\delta) \to 0$ when $\delta \to 0$. Set $E(\delta) := \ominus(c_1 \oplus b) + M(\delta)$. Then when δ is sufficiently small, we have $E(\delta) < 0$ and

$$\begin{aligned} 1 + \mu(t)E(\delta) &= 1 + \mu(t)\left[-\frac{(c_1 \oplus b)}{1 + \mu(t)(c_1 \oplus b)} + M(\delta)\right] \\ &= \frac{1}{1 + \mu(t)(c_1 \oplus b)} + \mu(t)M(\delta) > 0. \end{aligned} \tag{4.13}$$

So $E(\delta)$ is positive regressive. Then for $H(t,y(t)) > 0$, (4.12) can be written as

$$H^{\Delta}\left(t,y\right) \le E(\delta)H\left(t,y(t)\right). \tag{4.14}$$

Notice that

$$\begin{aligned}\left(H(t,y(t))e_{\ominus E(\delta)}(t,\tau)\right)^{\Delta} &= H^{\Delta}(t,y(t))e_{\ominus E(\delta)}(\sigma(t),\tau) + \ominus E(\delta)e_{\ominus E(\delta)}(t,\tau)H(t,y(t)) \\ &= H^{\Delta}(t,y(t))e_{\ominus E(\delta)}(\sigma(t),\tau) \\ &\quad + \ominus E(\delta)e_{\ominus E(\delta)}(t,\sigma(t))e_{\ominus E(\delta)}(\sigma(t),\tau)H(t,y(t)) \\ &= H^{\Delta}(t,y(t))e_{\ominus E(\delta)}(\sigma(t),\tau) \\ &\quad + \ominus E(\delta)\frac{1}{1+\ominus E(\delta)\mu(t)}e_{\ominus E(\delta)}(\sigma(t),\tau)H(t,y(t)) \\ &= \left(H^{\Delta}(t,y) - E(\delta)H(t,y(t))\right)e_{\ominus E(\delta)}(\sigma(t),\tau) \le 0.\end{aligned} \tag{4.15}$$

Thus, we get

$$H(t,y(t)) \le e_{E(\delta)}(t,\tau)H(\tau,y(\tau)). \tag{4.16}$$

Then we can see that if $y(t) \in C^s(V_t)$, we have $V(t,y(t)) < 0$ and thus $H(t,y(t)) > 0$. By (4.16) we have $H(\tau,y(\tau)) > 0$ and thus $V(\tau,y(\tau)) < 0$, that is, $y(\tau) \in C^s(V_\tau)$. Let $U(t,\tau)$ be the linear evolution operator associated to (4.1). Then $y(t) = U(t,\tau)y(\tau)$. Thus, we get

$$U(\tau,t)C^s(V_t) \subset C^s(V_\tau). \tag{4.17}$$

Let $L_1(t) = Q(t) + R(t)$. We can see that $L_1(t)x(t)$ is also a solution of (3.1). By (4.2) one has

$$\begin{aligned}H^{\Delta}(t,L_1(t)x(t)) = \Big\langle \Big[&S^{\Delta}(t) + S(\sigma(t))A(t) + A^*(t)S(\sigma(t)) \\ &+\mu(t)A^*(t)S(\sigma(t))A(t)\Big]L_1(t)x(t), L_1(t)x(t)\Big\rangle.\end{aligned} \tag{4.18}$$

Similar to the proof of (4.5), we have $H^{\Delta}(t,L_1(t)x(t)) \le \ominus(c_2 \oplus b)H(t,L_1(t)x(t))$. Then we get

$$S^{\Delta}(t) + S(\sigma(t))A(t) + A^*(t)S(\sigma(t)) + \mu(t)A^*(t)S(\sigma(t))A(t) \le \ominus(c_2 \oplus b)S(t). \tag{4.19}$$

Following the process for getting (4.12), one has

$$H^{\Delta}(t,y(t)) \le \ominus(c_2 \oplus b)H(t,y(t)) + M(\delta)\left|H(t,y(t))\right|. \tag{4.20}$$

Let $F(\delta) := \ominus(c_2 \oplus b) - M(\delta)$. When δ is sufficiently small, we obtain $F(\delta) < 0$ and

$$\begin{aligned}1 + \mu(t)F(\delta) &= 1 + \mu(t)\left[-\frac{(c_2 \oplus b)}{1+\mu(t)(c_2 \oplus b)} - M(\delta)\right] \\ &= \frac{1}{1+\mu(t)(a \oplus b)} - M(\delta).\end{aligned} \tag{4.21}$$

So $F(\delta)$ is positive regressive. Then for $H(t,y(t)) < 0$, (4.20) can be written as

$$H^{\Delta}(t,y) \le F(\delta)H(t,y(t)). \tag{4.22}$$

Following the similar process as (4.15), we get

$$H(t,y(t)) \le e_{F(\delta)}(t,\tau)H(\tau,y(\tau)). \tag{4.23}$$

Then if $y(\tau) \in C^u(V_\tau)$, we have $V(\tau,y(\tau)) > 0$ and thus $H(\tau,y(\tau)) < 0$. By (4.23) we have $H(t,y(t)) < 0$ and thus $V(t,y(t)) > 0$, that is, $y(t) \in C^u(V_t)$. So we get

$$U(t,\tau)C^u(V_\tau) \subset C^u(V_t). \tag{4.24}$$

By the notion of Lyapunov function and (4.17), (4.24) we can see that $V(t,x)$ is a Lyapunov function of (4.1).

Let

$$\begin{aligned} H_\tau^{'u}(V) &:= \bigcap_{r\in\mathbb{T}} U(\tau,r)\overline{C^u(V_r)} \subset \overline{C^u(V_\tau)}, \\ H_\tau^{'s}(V) &:= \bigcap_{r\in\mathbb{T}} U(\tau,r)\overline{C^s(V_r)} \subset \overline{C^s(V_\tau)}. \end{aligned} \tag{4.25}$$

By the notion of Lyapunov function, we know that there exist two subspaces $D_\tau^{'u}(V)$ and $D_\tau^{'s}(V)$ such that $D_\tau^{'u}(V) \subset H_\tau^{'u}(V)$, $D_\tau^{'s}(V) \subset H_\tau^{'s}(V)$, and $D_\tau^{'u}(V) \oplus D_\tau^{'s}(V) = \mathbb{R}^n$.

Now, we prove that $V(t,x)$ is a (λ,μ)-Lyapunov function. Since $c_1 > c_2$, we have $c_1 \oplus b > c_2 \oplus b$. By simple computation, we obtain $\ominus(c_1\oplus b) < \ominus(c_2 \oplus b)$. Therefore, when δ is sufficiently small, we get $E(\delta) < F(\delta)$. Then there are constants r_1, r_2 such that $E(\delta) \le r_1 < r_2 \le F(\delta)$. Thus, by inequality (4.16), for $y(\tau) \in D_\tau^{'s}$ one has

$$\begin{aligned} V^2(t,y(t)) &\le e_{E(\delta)}(t,\tau)V^2(\tau,y(\tau)) \\ &\le e_{r_1}(t,\tau)V^2(\tau,y(\tau)). \end{aligned} \tag{4.26}$$

By inequality (4.23), for $y(\tau) \in D_\tau^{'u}$, we obtain

$$\begin{aligned} V^2(t,y(t)) &\ge e_{F(\delta)}(t,\tau)V^2(\tau,y(\tau)) \\ &\ge e_{r_2}(t,\tau)V^2(\tau,y(\tau)). \end{aligned} \tag{4.27}$$

Then by (4.26) and (4.27) we can see that $V(t,y)$ is a (r_1,r_2)-Lyapunov function of (4.1).

Next, we will prove that $W(t,y)$ is also a (λ,μ)-Lyapunov function of (4.1). Similar to the consideration of $H(t,y)$, for $G(t,y)$ we have that if $y \in C^s(W_t)$, then $W(t,y(t)) < 0$, that is, $G(t,y(t)) > 0$. Thus, we get

$$G^{\Delta}(t,y(t)) \le (d_1 \oplus b + M(\delta))G(t,y(t)). \tag{4.28}$$

Let $E'(\delta) := d_1 \oplus b + M(\delta)$. By the similar deducing as (4.16), one has

$$G(t, y(t)) \le e_{E'(\delta)}(t, \tau) G(\tau, y(\tau)). \tag{4.29}$$

Thus, when $G(t, y(t)) > 0$, that is, $W(t, y(t)) < 0$, we have $G(\tau, y(\tau)) > 0$, that is, $W(\tau, y(\tau)) < 0$. That means

$$U(\tau, t) C^s(W_t) \subset C^s(W_\tau). \tag{4.30}$$

When $y \in C^u(W_t)$, we have $W(t, y(t)) > 0$, that is, $G(t, y(t)) < 0$. Then

$$G^\Delta(t, y(t)) \le (d_2 \oplus b - M(\delta)) G(t, y(t)). \tag{4.31}$$

Let $F'(\delta) = d_2 \oplus b - M(\delta)$. By the similar deducing as (4.15), one has

$$G(t, y(t)) \le e_{F'(\delta)}(t, \tau) G(\tau, y(\tau)). \tag{4.32}$$

Then when $G(\tau, y(\tau)) < 0$, that is, $W(\tau, y(\tau)) > 0$, we have $G(t, y(t)) < 0$, that is, $W(\tau, y(\tau)) > 0$. That means

$$U(t, \tau) C^u(W_\tau) \subset C^u(W_t). \tag{4.33}$$

By (4.30) and (4.33), we know that $W(t, y)$ is a Lyapunov function of (4.1).

Since $d_1 < d_2$, we have $d_1 \oplus b < d_2 \oplus b$. Then when δ is sufficiently small, we get $E'(\delta) < F'(\delta)$. Therefore, there are constants l_1, l_2 such that $E'(\delta) \le l_1 < l_2 \le F'(\delta)$.

Let

$$\begin{aligned} H_\tau^{\prime u}(W) &:= \bigcap_{r \in \mathbb{T}} U(\tau, r) \overline{C^u(W_r)} \subset \overline{C^u(W_\tau)}, \\ H_\tau^{\prime s}(W) &:= \bigcap_{r \in \mathbb{T}} U(\tau, r) \overline{C^s(W_r)} \subset \overline{C^s(W_\tau)}. \end{aligned} \tag{4.34}$$

By the notion of Lyapunov function, we can know that there exist $D_\tau^{\prime u}(W)$ and $D_\tau^{\prime s}(W)$ such that $D_\tau^{\prime u}(W) \subset H_\tau^{\prime u}(W)$, $D_\tau^{\prime s}(W) \subset H_\tau^{\prime s}(W)$, and $D_\tau^{\prime u}(W) \oplus D_\tau^{\prime s}(W) = \mathbb{R}^n$.

Furthermore, by (4.29) and (4.32), for any $t \ge \tau$, when $y(\tau) \in D_\tau^{\prime s}(W)$, one has

$$W^2(t, y(t)) \le e_{E'(\delta)}(t, \tau) W^2(\tau, y(\tau)) \le e_{l_1}(t, \tau) W^2(\tau, y(\tau)), \tag{4.35}$$

and when $y(\tau) \in D_\tau^{\prime u}(W)$, one has

$$W^2(t, y(t)) \ge e_{F'(\delta)}(t, \tau) W^2(\tau, y(\tau)) \ge e_{l_2}(t, \tau) W(\tau, y(\tau)). \tag{4.36}$$

Thus, we can see that $W(t, y)$ is a (l_1, l_2)-Lyapunov function of (4.1). Therefore, we know that (4.1) has two (λ, μ)-quadratic Lyapunov functions, $V(t, x)$ with (r_1, r_2) and $W(t, x)$ with

(l_1, l_2) satisfying $r_1 < r_2 < 0 < l_1 < l_2$. Then by Theorem 3.1 we get that (4.1) has an exponential trichotomy. The proof is completed. □

Acknowledgment

The author expresses her sincere thanks to Professors Yong Li and Zhenxin Liu for their encouragement and valuable suggestions.

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