Hindawi Publishing Corporation
Discrete Dynamics in Nature and Society
Volume 2011, Article ID 872738, 18 pages
doi:10.1155/2011/872738

Research Article

Feedback Control in a Periodic Delay Single-Species Difference System

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Received 19 October 2010; Accepted 8 January 2011

Academic Editor: Li Xian Zhang

A periodic delay single-species difference system with feedback control is established. With the help of analysis method and Lyapunov function, a good understanding of the permanence and global attractivity of the system is gained. Numerical simulations are presented to verify the validity of the proposed criteria. Our results show that feedback control has no influence on the permanence while it has influence on the global attractivity of the system.

1. Introduction

In 1978, Ludwig et al. [1] considered a single-species system which is modeled by

$$x'(t) = x(t)[a - bx(t)] - h(x), \tag{1.1}$$

where $x(t)$ is the density of species x at time t, a is the intrinsic growth rate, b is the competing rate, and $h(x)$-term represents predation. To be specific, Murray [2] took $h(x)$ in the form of $cx^2(t)/(d + x^2(t))$, and the dynamic behavior of $x(t)$ is then governed by

$$x'(t) = x(t)[a - bx(t)] - \frac{cx^2(t)}{d + x^2(t)}, \tag{1.2}$$

where $cx^2(t)/(d+x^2(\text{t}))$ is an S-shaped function and a, b, c, and d are positive constants. For the relevant ecology sense of system (1.2), we refer the readers to [1, 2] and the references cited therein.

It seems reasonable to assume that the reproduction of species x will not be instantaneous, but mediated by a delay required for gestation of species x. Thus, a revised version is

$$x'(t) = x(t)[a - bx(t-\tau)] - \frac{cx^2(t)}{d + x^2(t)}, \tag{1.3}$$

where the constant delay τ is positive. It is evident that all the coefficients of system (1.3) are assumed to be constant. However, in the real world, the coefficients are not fixed constants owing to the variation of environments. The influence of a varying environment is important for evolutionary theory as the selective forces on systems in such a fluctuating medium differ from those in a stable environment. In addition, as we know, ecosystems are often disturbed by unpredictable forces and, so, species population may experience changes. In ecology, an interesting issue is whether or not an ecosystem can withstand those unpredictable disturbances which persist for a finite period of time. In the language of control, we call the disturbance functions control variables. For more discussion on this direction, we refer the readers to [3–8].

Considering the possible effects of fluctuating environment and feedback control on system (1.3), we obtain the following periodic system:

$$\begin{aligned} x'(t) &= x(t)[a(t) - b(t)x(t-\tau)] - \frac{c(t)x^2(t)}{d(t) + x^2(t)} - g(t)x(t)u(t), \\ u'(t) &= -e(t)u(t) + h(t)x(t-\tau), \end{aligned} \tag{1.4}$$

where $u(t)$ is the control variable. We assume that coefficients $a(t)$, $b(t)$, $c(t)$, $d(t)$, $g(t)$, $e(t)$, and $h(t)$ are continuous and bounded above and below by positive constants and $e(t) \in (0,1)$.

Following the same idea and method in [9–12], one can easily derive the discrete analogue of system (1.4), which takes the form of

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{a(k) - b(k)x(k-l) - \frac{c(k)x(k)}{d(k) + x^2(k)} - g(k)u(k)\right\}, \\ \Delta u(k) &= -e(k)u(k) + h(k)x(k-l), \end{aligned} \tag{1.5}$$

where $k \in \{0,1,2,\ldots\}$, l is a positive integer, all the coefficients $a(k)$, $b(k)$, $c(k)$, $d(k)$, $g(k)$, $e(k)$, and $h(k)$ are positive bounded sequences and $e(k) \in (0,1)$. Δ is the first forward difference operator $\Delta u(k) = u(k+1) - u(k)$. For biological reasons, we only consider the solution $(x(k), u(k))$ of system (1.5) with initial value $x(-l)$, $x(-l+1), \ldots, x(-1) \geq 0$, $(x(0), u(0)) > 0$. The principle aim of this paper is to explore the permanence and global attractivity of system (1.5). To the best of our knowledge, no work has been done for system (1.5).

For the sake of simplicity and convenience, the notations and definitions below will be used through this paper: Z^+ and $[\,]$ denote the set of nonnegative integers and the greatest integer function, respectively. We denote $f^U = \sup_{k\in Z^+} f(k)$, $f^L = \inf_{k\in Z^+} f(k)$ for any bounded nonnegative sequence $\{f(k)\}$. Meanwhile, we denote the product of $f(k)$ from $k = \alpha$ to $k = \beta$ by $\prod_{k=\alpha}^{k=\beta} f(k)$ with the understanding that $\prod_{k=\alpha}^{k=\beta} f(k) = 1$ for all $\alpha > \beta$.

Definition 1.1. System (1.5) is said to be permanent if there exist positive constants $\underline{x}$, $\overline{x}$ and $\underline{u}$, $\overline{u}$ such that

$$\underline{x} \leqslant \lim_{k \to +\infty} \inf x(k) \leqslant \lim_{k \to +\infty} \sup x(k) \leqslant \overline{x},$$
$$\underline{u} \leqslant \lim_{k \to +\infty} \inf u(k) \leqslant \lim_{k \to +\infty} \sup u(k) \leqslant \overline{u}. \tag{1.6}$$

Definition 1.2. The positive solutions of system (1.5) are globally attractive if any two positive solutions $(x(k), u(k))$ and $(x^*(k), u^*(k))$ of system (1.5) satisfy

$$\lim_{k \to +\infty} |x(k) - x^*(k)| = 0, \qquad \lim_{k \to +\infty} |u(k) - u^*(k)| = 0. \tag{1.7}$$

The organization of this paper is as follows. In the next Sections 2 and 3, two main results on permanence and global attractivity of system (1.5) are given, respectively. Numerical simulations are present to illustrate the validity of our main results in Section 4, and a brief conclusion is provided to summarize the paper in the final section.

2. Permanence

This section is concerned with the permanence of system (1.5). We first introduce the following lemmas which are useful for establishing our result.

Lemma 2.1 (see [4]). *Assume the constant $A > 0$ and $y(0) > 0$, and further suppose that*

(1) *if*

$$y(k+1) \leq Ay(k) + B(k), \quad k = 1, 2, \ldots, \tag{2.1}$$

then for any integer $p \leq k$,

$$y(k) \leq A^p y(k-p) + \sum_{i=0}^{p-1} A^i B(k-i-1). \tag{2.2}$$

Especially, if $A < 1$ and $B(k)$ is bounded above with respect to M, then

$$\lim_{k \to +\infty} \sup y(k) \leq \frac{M}{1-A}. \tag{2.3}$$

(2) *If*

$$y(k+1) \geq Ay(k) + B(k), \quad k = 1, 2, \ldots, \tag{2.4}$$

then for any integer $p \leq k$*,*

$$y(k) \geq A^p y(k-p) + \sum_{i=0}^{p-1} A^i B(k-i-1). \tag{2.5}$$

Especially, if $A < 1$ *and* $B(k)$ *is bounded below with respect to* m*, then*

$$\lim_{k \to +\infty} \inf y(k) \geq \frac{m}{1-A}. \tag{2.6}$$

Lemma 2.2 (see [10]). *Assume that* $y(k)$ *satisfies* $y(n_1) > 0$ *and*

$$y(k+1) \leq y(k) \exp\left[r(k)\left(1 - \alpha y(k)\right)\right] \tag{2.7}$$

for $k \in [n_1, +\infty)$*, where* α *is a positive constant and* $n_1 \in Z^+$*. Then*

$$\lim_{k \to +\infty} \sup y(k) \leq \frac{1}{\alpha r^U} \exp\left(r^U - 1\right). \tag{2.8}$$

Lemma 2.3 (see [10]). *Assume that* $y(k)$ *satisfies* $y(n_2) > 0$ *and*

$$y(k+1) \geq y(k) \exp\left[r(k)\left(1 - \alpha y(k)\right)\right] \tag{2.9}$$

for $k \in [n_2, +\infty)$*,* $\lim_{k \to +\infty} \sup y(k) \leq M$*, where* α *is a positive constant such that* $\alpha M > 1$ *and* $n_2 \in Z^+$*. Then*

$$\lim_{k \to +\infty} \inf y(k) \geq \frac{1}{\alpha} \exp\left[r^U (1 - \alpha M)\right]. \tag{2.10}$$

Now, we state the permanence of system (1.5).

Theorem 2.4. *System* (1.5) *is permanent provided that*

$$2a^L \sqrt{d^L} > c^U. \tag{2.11}$$

Proof. It follows from the first equation of system (1.5) that

$$x(k+1) \leq x(k) \exp\{a(k)\}. \tag{2.12}$$

Let $x(k) = \exp\{y(k)\}$, then (2.12) can be rewritten as

$$y(k+1) - y(k) \leq a(k). \tag{2.13}$$

Summing both sides of (2.13) from $k-l$ to $k-1$ yields

$$\sum_{i=k-l}^{k-1}\left[y(i+1)-y(i)\right] \le \sum_{i=k-l}^{k-1} a(i) \le a^U l, \tag{2.14}$$

which implies that

$$y(k-l) \ge y(k) - a^U l, \tag{2.15}$$

and hence

$$x(k-l) \ge x(k)\exp\left(-a^U l\right). \tag{2.16}$$

Following the above result, again from the first equation of system (1.5), we have

$$x(k+1) \le x(k)\exp\left\{a(k) - b(k)\exp\left(-a^U l\right)x(k)\right\}. \tag{2.17}$$

By applying Lemma 2.2 to (2.17), we can obtain that

$$\lim_{k\to+\infty}\sup x(k) \le \frac{\exp\left[a^U(l+1)-1\right]}{b^L} \stackrel{\text{def}}{=} \overline{x}. \tag{2.18}$$

For any constant $\varepsilon > 0$, it follows from (2.18) that there exists a $n_1 \in Z^+$ large enough such that

$$x(k) \le \overline{x} + \varepsilon, \quad \text{for } k \ge n_1, \tag{2.19}$$

which, together with the second equation of system (1.5), leads to

$$u(k) \le -e(k)u(k) + h(k)(\overline{x}+\varepsilon), \quad \text{for } k \ge n_1 + l, \tag{2.20}$$

that is,

$$u(k+1) \le \left(1-e^L\right)u(k) + h^U(\overline{x}+\varepsilon), \quad \text{for } k \ge n_1 + l. \tag{2.21}$$

Combing (2.21) with Lemma 2.1(1) and setting $\varepsilon \to 0$ in (2.21), one has

$$\lim_{k\to+\infty}\sup u(k) \le \frac{h^U\overline{x}}{e^L} \stackrel{\text{def}}{=} \overline{u}. \tag{2.22}$$

For any sufficient small constant $\varepsilon > 0$, it follows from (2.18) and (2.22) that there exists $n_2 > n_1 + l$ such that

$$x(k) \le \overline{x} + \varepsilon, \quad u(k) \le \overline{u} + \varepsilon, \quad \text{for } k \ge n_2. \tag{2.23}$$

Thus, by (2.23) and the first equation of system (1.5), we have

$$\begin{aligned} x(k+1) &= x(k)\exp\left\{a(k) - b(k)x(k-l) - \frac{c(k)x(k)}{d(k)+x^2(k)} - g(k)u(k)\right\} \\ &\geq x(k)\exp\left\{a(k) - b(k)\overline{x} - \frac{c(k)}{d(k)/x(k)+x(k)} - g(k)\overline{u}\right\} \\ &\geq x(k)\exp\left\{a(k) - b(k)\overline{x} - \frac{c(k)}{2\sqrt{d(k)}} - g(k)\overline{u}\right\} \end{aligned} \tag{2.24}$$

for $k \geq n_2$. Let $x(k) = \exp\{y(k)\}$, then (2.24) is equivalent to

$$y(k+1) - y(k) \geq \left\{a(k) - b(k)\overline{x} - \frac{c(k)}{2\sqrt{d(k)}} - g(k)\overline{u}\right\}. \tag{2.25}$$

For convenience of exposition, we set $R(k) = \{a(k) - b(k)\overline{x} - c(k)/2\sqrt{d(k)} - g(k)\overline{u}\}$. Summing both sides of (2.25) from $k-l$ to $k-1$ leads to

$$\sum_{i=k-l}^{k-1}\big(y(i+1) - y(i)\big) \geq \sum_{i=k-l}^{k-1} R(i). \tag{2.26}$$

This implies that

$$y(k-l) \leq y(k) - \sum_{i=k-l}^{k-1} R(i), \tag{2.27}$$

and hence,

$$x(k-l) \leq x(k)\exp\left\{-\sum_{i=k-l}^{k-1} R(i)\right\}. \tag{2.28}$$

From the second equation of system (1.5), we have

$$\begin{aligned} u(k+1) &= (1-e(k))u(k) + h(k)x(k-l) \\ &\leq \left(1-e^L\right)u(k) + h(k)x(k-l) \\ &\overset{\text{def}}{=} Au(k) + B(k), \end{aligned} \tag{2.29}$$

where $A = 1 - e^L$ and $B(k) = h(k)x(k-l)$. Then, for any integer $p \leq k - l$, combing (2.28), (2.29) with Lemma 2.1 (1), we have

$$\begin{aligned} u(k) &\leq A^p u(k-p) + \sum_{i=0}^{p-1} A^i B(k-i-1) \\ &= A^p u(k-p) + \sum_{i=0}^{p-1} A^i h(k-i-1)x(k-i-1-l) \\ &\leq A^p u(k-p) + \sum_{i=0}^{p-1} A^i h^U x(k) \exp\left\{ - \sum_{j=k-(i+1+l)}^{k-1} R(j) \right\}. \end{aligned} \tag{2.30}$$

Since $e(k) \in (0,1)$, we obtain that $0 < A < 1$. So

$$0 \leq A^p u(k-p) \leq \overline{u} A^p \longrightarrow 0, \quad \text{for } p \longrightarrow +\infty. \tag{2.31}$$

By the assumption of Theorem 2.4, for any solution $(x(k), u(k))$ of system (1.5), there exists an integer $q > 0$ such that $g^U A^p u(k-p) < (1/2)(a^L - c^U/2\sqrt{d^L})$ for $p > q$. In fact, we can choose $q = \max\{1, [\log_A((a^L - c^U)/(2\sqrt{d^L})/2g^U\overline{u})] + 1\}$. Then

$$\begin{aligned} u(k) &\leq A^q u(k-q) + \sum_{i=0}^{q-1} A^i h^U x(k) \exp\left\{ - \sum_{j=k-(i+1+l)}^{k-1} R(j) \right\} \\ &\leq \overline{u} A^q + \left\{ \sum_{i=0}^{q-1} A^i h^U \exp\left[-(i+1+l) R^L \right] \right\} x(k), \quad \text{for } k > q. \end{aligned} \tag{2.32}$$

Since $\sum_{i=0}^{q-1} A^i h^U \exp[-(i+1+l)R^L]$ is bounded above, let $K_1 = \{\sum_{i=0}^{q-1} A^i h^U \exp[-(i+1+l)R^L]\}$, then we have

$$u(k) \leq \overline{u} A^q + K_1 x(k), \quad \text{for } k > q, \tag{2.33}$$

which, together with (2.24) and (2.28), leads to

$$\begin{aligned} x(k+1) &= x(k) \exp\left\{ a(k) - b(k)x(k-l) - \frac{c(k)x(k)}{d(k) + x^2(k)} - g(k)u(k) \right\} \\ &\geq x(k) \exp\left\{ \left[a(k) - \frac{c(k)}{\left(2\sqrt{d(k)}\right)} - \overline{u} A^q g(k) \right] - \left[K_1 g(k) + \exp\left(-lR^L\right) b(k) \right] x(k) \right\} \\ &\geq x(k) \exp\left\{ S_1(k) \left[1 - \left(\frac{S_2(k)}{S_1(k)} \right) x(k) \right] \right\}, \end{aligned} \tag{2.34}$$

where $S_1(k) = a(k) - c(k)/(2\sqrt{d(k)}) - \overline{u}A^q g(k)$, $S_2(k) = K_1 g(k) + \exp(-lR^L)b(k)$. Note that

$$\begin{aligned}\frac{S_2^U}{S_1^L}\overline{x} &= \frac{g^U K_1 + b^U \exp(-lR^L)}{a^L - c^U/\left(2\sqrt{d^L}\right) - \overline{u}A^q g^U} \times \overline{x} \\ &\geq \frac{b^U \exp(-lR^L)}{a^L} \times \frac{\exp(a^U(l+1)-1)}{b^L} \\ &\geq \frac{b^U \exp(-la^L)}{a^L} \times \frac{\exp(a^L(l+1)-1)}{b^U} \\ &\geq \frac{\exp(a^L - 1)}{a^L} \\ &> 1,\end{aligned} \tag{2.35}$$

where we use the inequality $\exp(x-1) > x$ for $x > 0$. Therefore, by Lemma 2.3, it follows that

$$\liminf_{k\to+\infty} x(k) \geq \frac{S_1^L}{S_2^U}\exp\left[S_1^U\left(1 - \frac{S_2^U}{S_1^L}\overline{x}\right)\right] \stackrel{\text{def}}{=} \underline{x}. \tag{2.36}$$

For any constant $\varepsilon > 0$, by (2.36) we know that there exists a sufficiently large integer $n_3 > n_2 + l$ such that

$$x(k) \geq \underline{x} - \varepsilon, \quad \text{for } k \geq n_3. \tag{2.37}$$

Hence, it follows from (2.37) and the second equation of system (1.5) that

$$u(k+1) \geq \left(1 - e^U\right)u(k) + h^L(\underline{x} - \varepsilon), \quad \text{for } k \geq n_3. \tag{2.38}$$

Applying Lemma 2.1(2) and setting $\varepsilon \to 0$ in (2.38), we have

$$\liminf_{k\to+\infty} u(k) \geq \frac{h^L}{e^U}\underline{x} \stackrel{\text{def}}{=} \underline{u}. \tag{2.39}$$

Consequently, combing (2.18), (2.22), and (2.36) with (2.39), system (1.5) is permanent. This completes the proof. □

3. Global Attractivity

On the basis of permanence, in this section we further provide sufficient conditions that guarantee the positive solutions of system (1.5) are globally attractive. To do so, we first give the following lemma.

Lemma 3.1. *For any two positive solutions* $(x(k), u(k))$ *and* $(x^*(k), u^*(k))$ *of system* (1.5), *one has*

$$\begin{aligned}
\ln \frac{x(k+1)}{x^*(k+1)} = \ln \frac{x(k)}{x^*(k)} &- b(k)[x(k) - x^*(k)] - c(k)J(k) - g(k)[u(k) - u^*(k)] \\
&+ b(k) \sum_{s=k-l}^{k-1} \left\{ P(s) \left\{ a(s) - b(s)x^*(s-l) - \frac{c(s)x^*(s)}{d(s) + x^*(s)^2} - g(s)u^*(s) \right\} \right. \\
&\qquad \times [x(s) - x^*(s)] - Q(s)x(s)\{b(s)[x(s-l) - x^*(s-l)] \\
&\qquad \left. + \; c(s)J(s) + g(s)[u(s) - u^*(s)]\} \right\},
\end{aligned} \tag{3.1}$$

where

$$\begin{aligned}
J(s) &= \frac{\{[d(s) - x(s)x^*(s)] \times [x(s) - x^*(s)]\}}{\left\{\left[d(s) + x(s)^2\right] \times \left[d(s) + x^*(s)^2\right]\right\}}, \\
P(s) &= \exp\left\{ \eta(s) \left[a(s) - b(s)x^*(s-l) - \frac{c(s)x^*(s)}{d(s) + x^*(s)^2} - g(s)u^*(s) \right] \right\} \\
Q(s) &= \exp\left\{ \xi(s) \left[a(s) - b(s)x(s-l) - \frac{c(s)x(s)}{d(s) + x(s)^2} - g(s)u(s) \right] \right. \\
&\qquad \left. + \; (1 - \xi(s)) \left[a(s) - b(s)x^*(s-l) - \frac{c(s)x^*(s)}{d(s) + x^*(s)^2} - g(s)u^*(s) \right] \right\} \\
&\qquad \eta(s), \; \xi(s) \in (0, 1).
\end{aligned} \tag{3.2}$$

Proof. It follows from system (1.5) that we have

$$\begin{aligned}
\ln \frac{x(k+1)}{x^*(k+1)} - \ln \frac{x(k)}{x^*(k)} &= \ln \frac{x(k+1)}{x(k)} - \ln \frac{x^*(k+1)}{x^*(k)} \\
&= \left[a(k) - b(k)x(k-l) - \frac{c(k)x(k)}{d(k) + x(k)^2} - g(k)u(k) \right] \\
&\quad - \left[a(k) - b(k)x^*(k-l) - \frac{c(k)x^*(k)}{d(k) + x^*(k)^2} - g(k)u^*(k) \right] \\
&= -b(k)[x(k-l) - x^*(k-l)] - c(k)J(k) - g(k)[u(k) - u^*(k)].
\end{aligned} \tag{3.3}$$

Hence,

$$\begin{aligned}\ln\frac{x(k+1)}{x^*(k+1)} &= \ln\frac{x(k)}{x^*(k)} - c(k)J(k) - g(k)[u(k)-u^*(k)] \\ &\quad - b(k)\{[x(k)-x^*(k)] - [x(k)-x(k-l)] + [x^*(k)-x^*(k-l)]\} \\ &= \ln\frac{x(k)}{x^*(k)} - c(k)J(k) - g(k)[u(k)-u^*(k)] \\ &\quad - b(k)[x(k)-x^*(k)] + b(k)\{[x(k)-x(k-l)] - [x^*(k)-x^*(k-l)]\}.\end{aligned} \tag{3.4}$$

Note that

$$\begin{aligned}[x(k)-x(k-l)] - [x^*(k)-x^*(k-l)] &= \sum_{s=k-l}^{k-1}[x(s+1)-x(s)] - \sum_{s=k-l}^{k-1}[x^*(s+1)-x^*(s)] \\ &= \sum_{s=k-l}^{k-1}\{[x(s+1)-x^*(s+1)] - [x(s)-x^*(s)]\},\end{aligned} \tag{3.5}$$

$$\begin{aligned}&[x(s+1)-x^*(s+1)] - [x(s)-x^*(s)] \\ &\quad = x(s)\exp\left[a(s) - b(s)x(s-l) - \frac{c(s)x(s)}{d(s)+x(s)^2} - g(s)u(s)\right] \\ &\qquad - x^*(s)\exp\left[a(s) - b(s)x^*(s-l) - \frac{c(s)x^*(s)}{d(s)+x^*(s)^2} - g(s)u^*(s)\right] - [x(s)-x^*(s)] \\ &\quad = x(s)\left\{\exp\left[a(s) - b(s)x(s-l) - \frac{c(s)x(s)}{d(s)+x(s)^2} - g(s)u(s)\right]\right. \\ &\qquad\qquad \left. - \exp\left[a(s) - b(s)x^*(s-l) - \frac{c(s)x^*(s)}{d(s)+x^*(s)^2} - g(s)u^*(s)\right]\right\} \\ &\qquad + [x(s)-x^*(s)]\left\{\exp\left[a(s) - b(s)x^*(s-l) - \frac{c(s)x^*(s)}{d(s)+x^*(s)^2} - g(s)u^*(s)\right] - 1\right\}.\end{aligned} \tag{3.6}$$

By the mean value theorem, one has

$$\begin{aligned}&[x(s+1)-x^*(s+1)] - [x(s)-x^*(s)] \\ &\quad = P(s)\left\{a(s) - b(s)x^*(s-l) - \frac{c(s)x^*(s)}{d(s)+x^*(s)^2} - g(s)u^*(s)\right\}[x(s)-x^*(s)] \\ &\qquad - Q(s)x(s)\{b(s)[x(s-l)-x^*(s-l)] + c(s)J(s) + g(s)[u(s)-u^*(s)]\}.\end{aligned} \tag{3.7}$$

Then we can easily obtain (3.1) by substituting (3.5) and (3.7) into (3.4). The proof is complete. □

Now, we state our main result on the global attractivity of system (1.5).

Theorem 3.2. *If the assumption of Theorem 2.4 holds and, further, suppose there exist positive constants ρ and δ such that*

$$\lambda \overset{\text{def}}{=} \min\left\{\rho\phi - \delta h^{U}, \delta e^{L} - \rho\varphi\right\} > 0, \tag{3.8}$$

where ϕ, φ are defined by (3.27) *and* (3.28)*, respectively. Then the positive solutions of system* (1.5) *are globally attractive.*

Proof. Let $(x(k), u(k))$ and $(x^*(k), u^*(k))$ be any two positive solutions of system (1.5). To prove Theorem 3.2, we first consider the following three steps for the first equation of system (1.5).

Step 1. Let $V_{11}(k) = |\ln x(k) - \ln x^*(k)|$. It follows from (3.1) that

$$\begin{aligned}
\left|\ln\frac{x(k+1)}{x^*(k+1)}\right| \le{}& \left|\ln\frac{x(k)}{x^*(k)} - b(k)[x(k)-x^*(k)]\right| + c(k)|J(k)| + g(k)|u(k)-u^*(k)| \\
&+ b(k)\sum_{s=k-l}^{k-1}\{P(s)\beta(s)|x(s)-x^*(s)| + c(s)Q(s)x(s)|J(s)| \\
&\qquad + Q(s)x(s)\left[b(s)|x(s-l)-x^*(s-l)| + g(s)|u(s)-u^*(s)|\right]\} \\
\le{}& \left|\ln\frac{x(k)}{x^*(k)} - b(k)[x(k)-x^*(k)]\right| + \gamma(k)|x(k)-x^*(k)| + g(k)|u(k)-u^*(k)| \\
&+ b(k)\sum_{s=k-l}^{k-1}\{\left[P(s)\beta(s)+\gamma(s)Q(s)x(s)\right]|x(s)-x^*(s)| \\
&\qquad + Q(s)x(s)\left[b(s)|x(s-l)-x^*(s-l)| + g(s)|u(s)-u^*(s)|\right]\},
\end{aligned} \tag{3.9}$$

where

$$\begin{aligned}
\beta(s) &= a(s) + b(s)x^*(s-l) + \frac{c(s)}{2\sqrt{d(s)}} + g(s)u^*(s), \\
\gamma(s) &= \frac{c(s)(d(s)+x(s)x^*(s))}{d(s)^2}.
\end{aligned} \tag{3.10}$$

By the mean value theorem, we get

$$x(k) - x^*(k) = \exp[\ln x(k)] - \exp[\ln x^*(k)] = \theta(k)\ln\frac{x(k)}{x^*(k)}, \tag{3.11}$$

that is,

$$\ln \frac{x(k)}{x^*(k)} = \frac{1}{\theta(k)}[x(k) - x^*(k)], \tag{3.12}$$

where $\theta(k)$ lies between $x(k)$ and $x^*(k)$, then

$$\begin{aligned}
&\left|\ln \frac{x(k)}{x^*(k)} - b(k)[x(k) - x^*(k)]\right| \\
&\quad = \left|\ln \frac{x(k)}{x^*(k)}\right| - \left|\ln \frac{x(k)}{x^*(k)}\right| + \left|\ln \frac{x(k)}{x^*(k)} - b(k)[x(k) - x^*(k)]\right| \\
&\quad = \left|\ln \frac{x(k)}{x^*(k)}\right| - \frac{1}{\theta(k)}|x(k) - x^*(k)| + \left|\frac{1}{\theta(k)}[x(k) - x^*(k)] - b(k)[x(k) - x^*(k)]\right| \\
&\quad = \left|\ln \frac{x(k)}{x^*(k)}\right| - \frac{1}{\theta(k)}|x(k) - x^*(k)| + \left|\frac{1}{\theta(k)} - b(k)\right||x(k) - x^*(k)| \\
&\quad = \left|\ln \frac{x(k)}{x^*(k)}\right| - \left[\frac{1}{\theta(k)} - \left|\frac{1}{\theta(k)} - b(k)\right|\right]|x(k) - x^*(k)|.
\end{aligned} \tag{3.13}$$

According to Theorem 2.4, there exists a $n_0 \in Z^+$ such that $\{x(k), x^*(k)\} \leq \overline{x}$ and $\{u(k), u^*(k)\} \leq \overline{u}$ for all $k \geq n_0$. Therefore, for all $k \geq n_0$, we can obtain that

$$\begin{aligned}
\Delta V_{11} &= V_{11}(k+1) - V_{11}(k) \\
&\leq -\left[\frac{1}{\theta(k)} - \left|\frac{1}{\theta(k)} - b(k)\right|\right]|x(k) - x^*(k)| + \gamma(k)|x(k) - x^*(k)| \\
&\quad + g(k)|u(k) - u^*(k)| \\
&\quad + b(k)\sum_{s=k-l}^{k-1}\big\{\big[P(s)\beta(s) + \gamma(s)Q(s)\overline{x}\big]|x(s) - x^*(s)| \\
&\qquad + b(s)Q(s)\overline{x}|x(s-l) - x^*(s-l)| + g(s)Q(s)\overline{x}|u(s) - u^*(s)|\big\}.
\end{aligned} \tag{3.14}$$

Step 2. Let

$$\begin{aligned}
V_{12}(k) = \sum_{s=k}^{k-1+l} b(s) \sum_{i=s-l}^{k-1}&\big\{\big[P(i)\beta(i) + \gamma(i)Q(i)\overline{x}\big]|x(i) - x^*(i)| \\
&+ b(i)Q(i)\overline{x}|x(i-l) - x^*(i-l)| + g(i)Q(i)\overline{x}|u(i) - u^*(i)|\big\}.
\end{aligned} \tag{3.15}$$

Then

$$\begin{aligned}\Delta V_{12} &= V_{12}(k+1) - V_{12}(k)\\ &= \sum_{s=k+1}^{k+l} b(s)\Big\{\big[P(k)\beta(k)+\gamma(k)Q(k)\overline{x}\big]|x(k)-x^*(k)|\\ &\qquad + b(k)Q(k)\overline{x}|x(k-l)-x^*(k-l)| + g(k)Q(k)\overline{x}|u(k)-u^*(k)|\Big\}\\ &\quad - b(k)\sum_{i=k-l}^{k-1}\Big\{\big[P(i)\beta(i)+\gamma(i)Q(i)\overline{x}\big]|x(i)-x^*(i)|\\ &\qquad + b(i)Q(i)\overline{x}|x(i-l)-x^*(i-l)| + g(i)Q(i)\overline{x}|u(i)-u^*(i)|\Big\}.\end{aligned} \tag{3.16}$$

Step 3. Let

$$V_{13}(k) = \overline{x}\sum_{w=k-l}^{k-1} b(w+l)Q(w+l)|x(w)-x^*(w)|\sum_{s=w+l+1}^{w+2l} b(s). \tag{3.17}$$

By a simple calculation, it derives that

$$\begin{aligned}V_{13} &= V_{13}(k+1) - V_{13}(k)\\ &= \sum_{s=k+l+1}^{k+2l} b(s)b(k+l)Q(k+l)\overline{x}|x(k)-x^*(k)|\\ &\quad - \sum_{s=k+1}^{k+l} b(s)b(k)Q(k)\overline{x}|x(k-l)-x^*(k-l)|.\end{aligned} \tag{3.18}$$

Now, we can define

$$V_1(k) = V_{11}(k) + V_{12}(k) + V_{13}(k). \tag{3.19}$$

Then for all $k \ge n_0$, it follows from (3.14)–(3.18) that

$$\begin{aligned}V_1 &= V_1(k+1) - V_1(k)\\ &\le -\left[\frac{1}{\theta(k)} - \left|\frac{1}{\theta(k)} - b(k)\right|\right]|x(k)-x^*(k)| + \gamma(k)|x(k)-x^*(k)| + g(k)|u(k)-u^*(k)|\\ &\quad + \sum_{s=k+1}^{k+l} b(s)g(k)Q(k)\overline{x}|u(k)-u^*(k)| + \sum_{s=k+l+1}^{k+2l} b(s)b(k+l)Q(k+l)\overline{x}|x(k)-x^*(k)|\\ &\quad + \sum_{s=k+1}^{k+l} b(s)\Big\{\big[P(k)\beta(k)+\gamma(k)Q(k)\overline{x}\big]|x(k)-x^*(k)|\Big\}.\end{aligned} \tag{3.20}$$

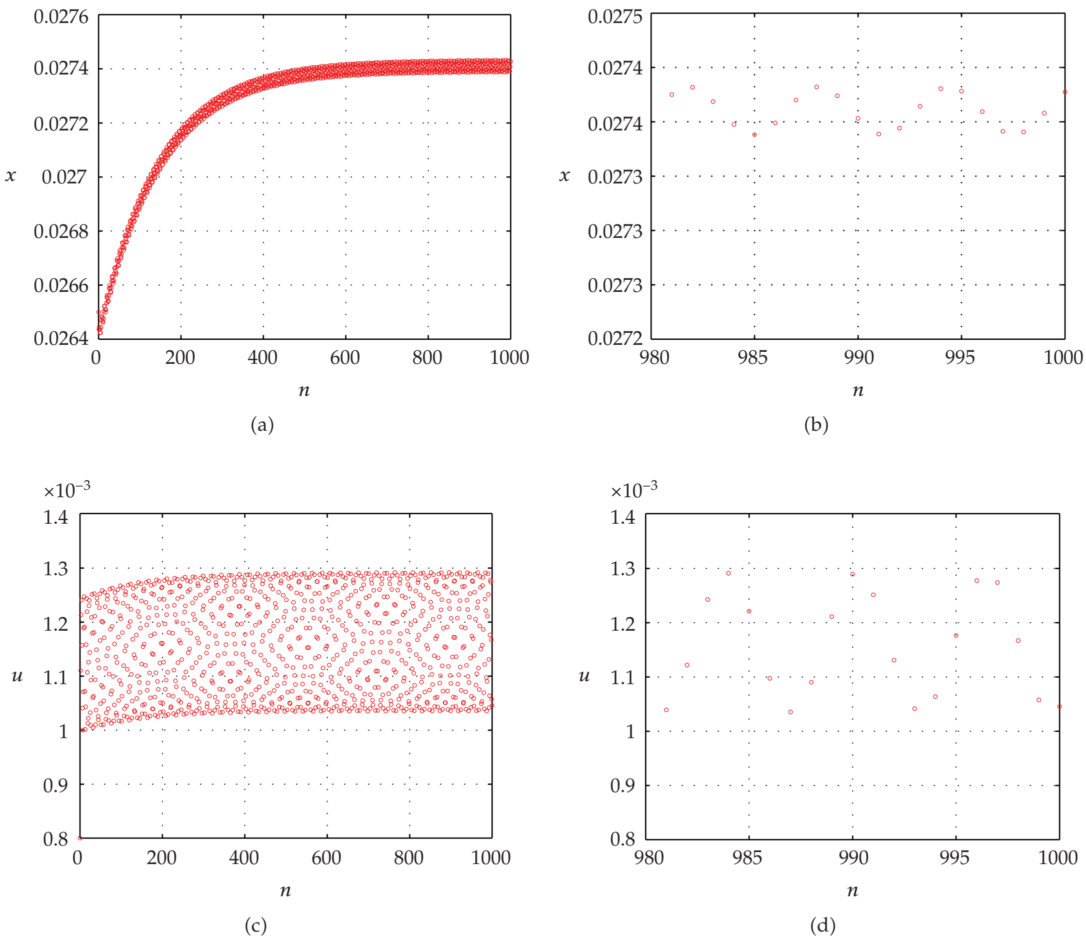

Figure 1: Permanence of system (4.1) with $x(-1) = 0.04$ and $\{x(0), u(0)\} = \{0.0265, 0.0008\}$. (a) and (b) show x for $k \in [0, 1000]$ and x for $k \in [980, 1000]$, respectively. (c) and (d) show u for $k \in [0, 1000]$ and u for $k \in [980, 1000]$, respectively.

For the second equation of system (1.5), we will consider the following two steps.

Step 1. Let $V_{21}(k) = |u(k) - u^*(k)|$. Then

$$\begin{aligned}
V_{21} &= V_{21}(k+1) - V_{21}(k) \\
&= |u(k+1) - u^*(k+1)| - |u(k) - u^*(k)| \\
&= |(1 - e(k))[u(k) - u^*(k)] + h(k)[x(k-l) - x^*(k-l)]| - |u(k) - u^*(k)| \\
&\le (1 - e(k))|u(k) - u^*(k)| - |u(k) - u^*(k)| + h(k)|x(k-l) - x^*(k-l)| \\
&\le -e(k)|u(k) - u^*(k)| + h(k)|x(k-l) - x^*(k-l)|.
\end{aligned} \tag{3.21}$$

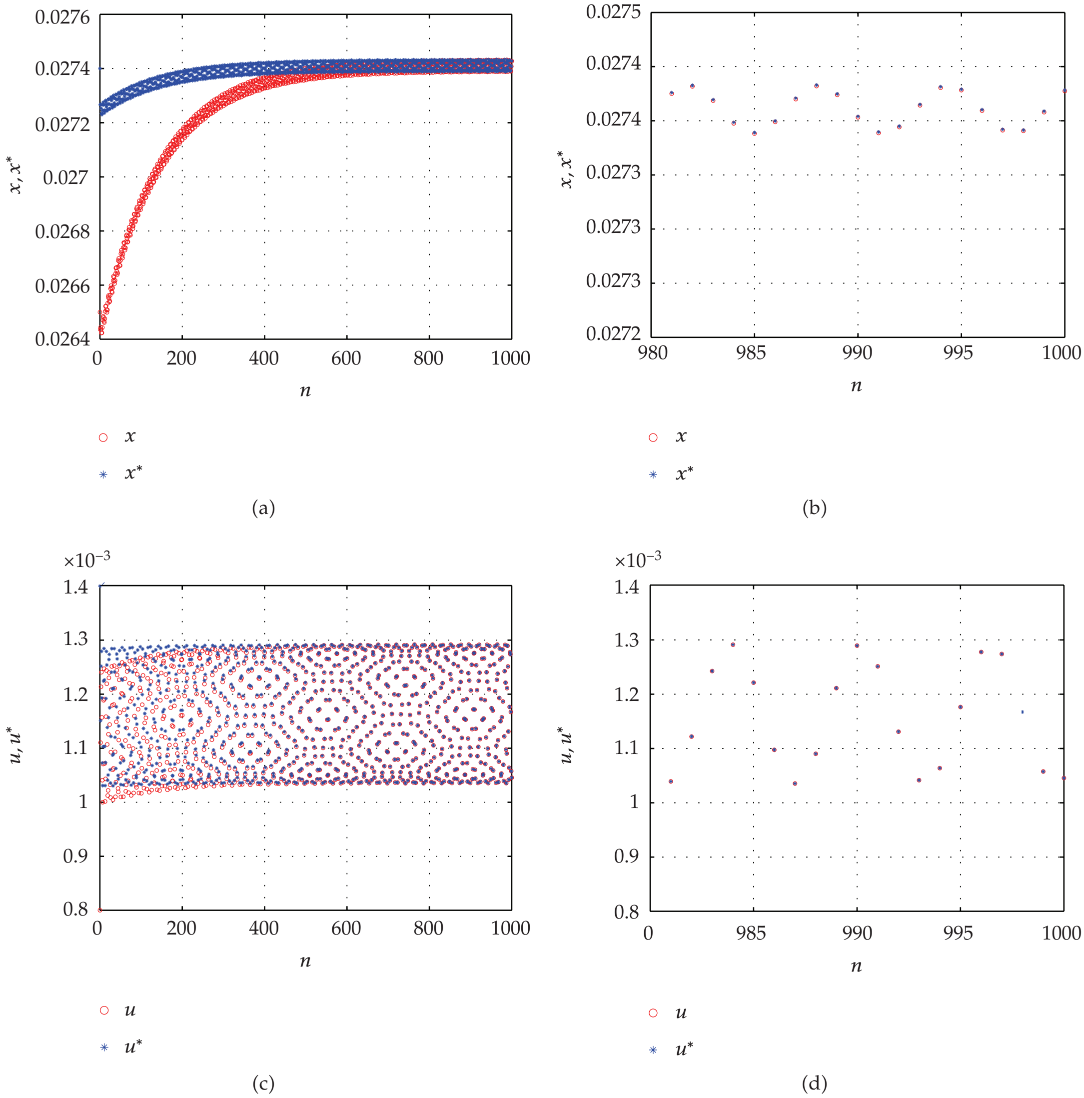

Figure 2: Global attractivity of system (4.1) with different initial values. x with $x(-1) = 0.04$, $\{x(0), u(0)\} = \{0.0265, 0.0008\}$ and x^* with $x^*(-1) = 0.05$, $\{x^*(0), u^*(0)\} = \{0.0274, 0.0014\}$. (a) and (b) show (x, x^*) for $k \in [0, 1000]$ and for $k \in [980, 1000]$, respectively. (c) and (d) show $(u,\ u^*)$ for $k \in [0, 1000]$ and for $k \in [980, 1000]$, respectively.

Step 2. Let $V_{22}(k) = \sum_{w=k-l}^{k-1} h(w+l)|x(w) - x^*(w)|$. Then

$$\begin{aligned} V_{22} &= V_{22}(k+1) - V_{22}(k) \\ &= \sum_{w=k+1-l}^{k} h(w+l)|x(w) - x^*(w)| - \sum_{w=k-l}^{k-1} h(w+l)|x(w) - x^*(w)| \\ &= h(k+l)|x(k) - x^*(k)| - h(k)|x(k-l) - x^*(k-l)|. \end{aligned} \tag{3.22}$$

So we can define

$$V_2(k) = V_{21}(k) + V_{22}(k). \tag{3.23}$$

Then it follows from (3.21) and (3.22) that

$$\begin{aligned} V_2 &= V_2(k+1) - V_2(k) \\ &\leq h(k+l)|x(k) - x^*(k)| - e(k)|u(k) - u^*(k)|. \end{aligned} \tag{3.24}$$

Now, we could define

$$V(k) = \rho V_1(k) + \delta V_2(k), \tag{3.25}$$

where ρ, δ are mentioned in (3.8).

Obviously, $V(k) \geq 0$ for all $k \in Z^+$ and $V(n_0 + l) < +\infty$. Therefore, combining (3.20) and (3.24), for all $k \geq n_0 + l$, we have

$$\begin{aligned} \Delta V &= \rho \Delta V_1 + \delta \Delta V_2 \\ &\leq \rho \left\{ -\left[\frac{1}{\theta(k)} - \left| \frac{1}{\theta(k)} - b(k) \right| \right] + \gamma(k) + \sum_{s=k+l+1}^{k+2l} b(s) b(k+l) Q(k+l) \overline{x} \right. \\ &\quad \left. + \sum_{s=k+1}^{k+l} b(s) \left[P(k)\beta(k) + \gamma(k) Q(k) \overline{x} \right] \right\} |x(k) - x^*(k)| + \delta h(k+l) |x(k) - x^*(k)| \\ &\quad + \rho \left\{ g(k) + \sum_{s=k+1}^{k+l} b(s) g(k) Q(k) \overline{x} \right\} |u(k) - u^*(k)| - \delta e(k) |u(k) - u^*(k)| \\ &\leq -\rho \left\{ \min\left(b^L, \frac{2}{\overline{\overline{x}}} - b^U \right) - \gamma^U - l b^U F \right\} |x(k) - x^*(k)| + \delta h^U |x(k) - x^*(k)| \\ &\quad - \left\{ \delta e^L - \rho \left(g^U + l b^U g^U Q^U \overline{x} \right) \right\} |u(k) - u^*(k)| \\ &\leq -\left(\rho\phi - \delta h^U \right) |x(k) - x^*(k)| - \left(\delta e^L - \rho\varphi \right) |u(k) - u^*(k)| \\ &\leq -\lambda \{ |x(k) - x^*(k)| + |\mathrm{u}(k) - u^*(k)| \}, \end{aligned} \tag{3.26}$$

where

$$\begin{aligned} F &= b^U Q^U \overline{x} + P^U \beta^U + \gamma^U Q^U \overline{x}, \\ \phi &= \min\left(b^L, \frac{2}{\overline{\overline{x}}} - b^U \right) - \gamma^U - l b^U F, \end{aligned} \tag{3.27}$$

$$\varphi = g^U + lb^U g^U Q^U \overline{x}. \tag{3.28}$$

Summing both sides of (3.26) from $n_0 + l$ to k, it derives that

$$\sum_{s=n_0+l}^{k} [V(s+1) - V(s)] \leq -\lambda \sum_{s=n_0+l}^{k} \{|x(s) - x^*(s)| + |u(s) - u^*(s)|\}, \tag{3.29}$$

which implies

$$V(k+1) + \lambda \sum_{s=n_0+l}^{k} \{|x(s) - x^*(s)| + |u(s) - u^*(s)|\} \leq V(n_0 + l) \leq V(0). \tag{3.30}$$

It follows from the above inequality that

$$\sum_{s=n_0+l}^{+\infty} \{|x(s) - x^*(s)| + |u(s) - u^*(s)|\} \leq \frac{V(0)}{\lambda} < +\infty, \tag{3.31}$$

that is,

$$\lim_{k \to +\infty} \{|x(k) - x^*(k)| + |u(k) - u^*(k)|\} = 0, \tag{3.32}$$

and we can easily obtain that

$$\lim_{k \to +\infty} |x(k) - x^*(k)| = 0, \qquad \lim_{k \to +\infty} |u(k) - u^*(k)| = 0, \tag{3.33}$$

which implies that the positive solutions of system (1.5) are globally attractive, this completes the proof. □

4. Numerical Simulations

To verify the feasibilities of our main results, we consider a specific example:

$$\begin{aligned} x(k+1) &= x(k) \exp\Big\{ [0.0069 + 0.00069 \sin(k)] - [0.25 - 0.0025 \sin(k)] x(k-1) \\ &\quad - \frac{[0.0011 + 0.00011 \sin(k)] x(k)}{[4.02 + 0.402 \sin(k)] + x^2(k)} - [0.034 - 0.0034 \sin(k)] u(k) \Big\}, \\ u(k+1) &= \{1 - [0.99 + 0.0099 \sin(k)]\} u(k) + [0.042 - 0.0042 \sin(k)] x(k-1). \end{aligned} \tag{4.1}$$

Obviously, $2a^L\sqrt{d^L} \approx 0.0236$, $c^U \approx 0.0012$, that is, $2a^L\sqrt{d^L} > c^U$. Then the assumption of Theorem 2.4 is satisfied, which indicates that system (4.1) is permanent (see Figure 1).

We assume $\rho = 20$ and $\delta = 1.5$, by a simple computation, we have $\phi \approx 0.0049$, $\varphi \approx 0.0584$, $\lambda \approx 0.0286$, then the sufficient conditions of Theorem 3.2 are satisfied. Thus, the positive solutions of system (4.1) are globally attractive. From Figure 2, we can see that $x(k)$ and $u(k)$ tend to $x^*(k)$ and $u^*(k)$, respectively.

5. Conclusion

We conclude with a brief discussion of our results. The sufficient condition required for the result of Theorem 2.4 does not depend on the size of the feedback control. However, the sufficient conditions required for the result of Theorem 3.2 show that the feedback control has an effect on the global attractivity of system (1.5).

Acknowledgments

The paper is supported by the Key Project of Chinese Ministry of Education(no. #210134), the Innovation Term of Educational Department of Hubei Province of China (no. #T200804), and the Scientific Research Foundation of the Education Department of Hubei Province of China (no. #D20101902).

References

[1] D. Ludwig, D. D. Jones, and C. S. Holling, "Qualitative analysis of insect outbreak systems:the spruce budworm and forest," *Journal of Animal Ecology*, vol. 47, no. 1, pp. 315–332, 1978.

[2] J. D. Murray, *Mathematical Biology I: An Introduction*, Springer, Berlin, Germany, 3rd edition, 2002.

[3] X. Liao, S. Zhou, and Y. Chen, "Permanence and global stability in a discrete n-species competition system with feedback controls," *Nonlinear Analysis: Real World Applications*, vol. 9, no. 4, pp. 1661–1671, 2008.

[4] Y.-H. Fan and L.-L. Wang, "Permanence for a discrete model with feedback control and delay," *Discrete Dynamics in Nature and Society*, vol. 2008, Article ID 945109, 8 pages, 2008.

[5] K. Gopalsamy and P. Weng, "Global attractivity in a competition system with feedback controls," *Computers & Mathematics with Applications*, vol. 45, no. 4-5, pp. 665–676, 2003.

[6] F. Chen, "Permanence of a single species discrete model with feedback control and delay," *Applied Mathematics Letters*, vol. 20, no. 7, pp. 729–733, 2007.

[7] X. Feng and F. Zhang, "The permanence in a single species nonautonomous system with delays and feedback control," *Discrete Dynamics in Nature and Society*, vol. 2010, Article ID 195983, 11 pages, 2010.

[8] Z.-j. Liu, "Persistence and periodic solution in two species competitive system with feedback controls," *Journal of Biomathematics*, vol. 17, no. 2, pp. 251–255, 2002.

[9] M. Fan and K. Wang, "Periodic solutions of a discrete time nonautonomous ratio-dependent predator-prey system," *Mathematical and Computer Modelling*, vol. 35, no. 9-10, pp. 951–961, 2002.

[10] X. Yang, "Uniform persistence and periodic solutions for a discrete predator-prey system with delays," *Journal of Mathematical Analysis and Applications*, vol. 316, no. 1, pp. 161–177, 2006.

[11] W. J. Qin and Z. J. Liu, "Asymptotic behaviors of a delay difference system of plankton allelopathy," *Journal of Mathematical Chemistry*, vol. 48, no. 3, pp. 653–675, 2010.

[12] W. J. Qin and Z. J. Liu, "Permanence and positive periodic solutions of a discrete delay competitive system," *Discrete Dynamics in Nature and Society*, vol. 2010, no. 1, Article ID 381750, 22 pages, 2010.